UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
FACULTAD DE INGENIERÍA MECÁNICA Y ELÉCTRICA

# A STOCHASTIC LOCATION-INVENTORY PROBLEM: COMPLEXITY AND MATHEMATICAL FORMULATIONS

POR
NELLY MONSERRAT HERNÁNDEZ GONZÁLEZ

COMO REQUISITO PARCIAL PARA OBTENER EL GRADO DE DOCTOR EN INGENIERÍA CON ESPECIALIDAD EN INGENIERÍA DE SISTEMAS

ENERO 2017

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SUBDIRECCIÓN DE ESTUDIOS DE POSGRADO

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Los miembros del Comité de Tesis recomendamos que la tesis titulada "A Stochastic Location-Inventory Problem: Complexity and Mathematical Formulations", realizada por la alumna Nelly Monserrat Hernández González, con número de matrícula 1541903, sea aceptada para su defensa como requisito parcial para obtener el grado de Doctor en Ingeniería con especialidad en Ingeniería de Sistemas.

El Comité de Tesis

Dra. Ada Margarita Álvarez Socarrás
Directora

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Dr. Fernando López Irarragorri
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Subdirección de Estudios de Posgrado

San Nicolás de los Garza, Nuevo León, enero 2017

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# Abstract

Nelly Monserrat Hernández González.

Candidato para obtener el grado de Doctor en Ingeniería con especialidad en Ingeniería de Sistemas.

Universidad Autónoma de Nuevo León.

Facultad de Ingeniería Mecánica y Eléctrica.

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Objective and methodology of the study: The objective of this research is the study and the optimization of the Location-Inventory Problem (Lip). This problem defines how many facilities to locate, where to locate them, which retailers they serve, how manage their inventory, so as to minimize the total cost, while ensuring a specified service level.

Because of the nature of the problem, an extensive variety of models can be generated. The problem addressed in this thesis consists on a mono-product supply chain of two echelons formed by a set of plants, a set of distribution centers, and a set of retailers. The most distinctive features are the stochastic demand, the consideration of multiple plants and the inventory measurement in each selected facility. The inclusion of several plants involves the presence of allocation decisions. In our case they are limited as single source, which means that each demand point should be serviced by a single supply point.

This study provides the computational complexity proof of LIP, the mathematical formulation of the problem and some exact techniques for solving it. In the complexity proof, we demonstrate that the decision version of the Location-Inventory Problem is NP-complete by building a reduction from the Bin Packing Problem, revealing that LIP is at least as difficult as the Bin Packing Problem. As consequence the Location-Inventory Problem is NP-hard.

Regarding to the mathematical formulation, we present a Mixed Integer Nonlinear Problem (MINLP), which is demonstrated to be nonconvex. Therefore, two reformulations are developed, one, still, a Mixed Integer Nonlinear Problem (MINLP2) and one as a Mixed Integer Linear Problem (MILP). The nonlinear terms in MINLP2 are treated by approximations made through the secant and the pieciwese method. Concerning the MILP, it is solved by a column generation technique.

Finally, we conduct an analysis for evaluating the behavior of the supply chain, and also the robustness of the solution against the uncertainty. To do that we performed a simulation that generates different sizes of demand and evaluates the feasibility of the optimal solutions to the new demand values.

CONTRIBUTIONS AND CONCLUSIONS: The importance of the presented work consists on the integration of elements that, as far as we know, have not been addressed together in the literature. This integration allows to study more practical situations, but resulted in a much more complex problem, mainly due to the consideration of several plants with different parameters, the stochastic demands and the capacitated locations. Moreover, inventory management involve necessarily nonlinear terms, which make the problem even harder to solve.

Some papers involve multiple plants, but they assume that the value of their parameters such as delivery time, shipping cost or production capacity are the same for all the plants. This assumption make the allocation decisions indistinct and therefore, greatly simplifies the problem

In addition, none of the published works formally demonstrated the computational complexity of the general problem. It is also an important contribution of this research. The result that LIP is NP-hard suggests that any exact algorithm may fail to solve large instances, nevertheless, the strategic nature of the problem indicates that an optimal solution is valuable, since the savings between an optimal and other feasible solution can represent a large amount of money. For that reason, we decided to explore exact solution methods.

The mathematical models are also a contribution. On one side, the reformulations allow to solve the problem, to optimality or near to optimality, directly with an solver without the need of decomposing the problem. On the other side, the column generation takes advantages of the structure of the problem.

Computational experiments evaluate the techniques, showing that the approximation by piecisewise method generates the best results, i.e., good solutions in a reasonable time. When the instances to be solved are not too large and when the CPU time is not limited, MILP model developed in this thesis can be used.

Firma del asesor: ________________________________________

Dra. Ada Margarita Álvarez Socarrás

CHAPTER 1

# INTRODUCTION

In a competitive world, companies must ensure an efficient use of their resources. This can be achieved by the integration of the productive actors along with the decision-making coordination at all levels in the supply chain. In this work we study the interactions between tactical and strategic decisions; concretely location and inventory decisions in a mono-product supply chain.

*Location decisions* involve determining the number, location, and size of the facilities to be used. They are typically classified as strategic decisions because they involve large monetary investments and long-lasting effect on supply-chain performance. Several studies (Gilmore, 2014) have been conducted to conclude that around 80% of the cost of a supply chain is locked in its initial design.

The *inventory decisions*, on the other hand, hold the key to success of physical distribution, so they affect the finances and the competitive advantage of a company. In Mexican companies, by instance, according to data from the company A.T. Kearney [1] cited by the Secretary of Economy [2], the logistic costs represent the 10.3% of their sales, of which the 40% corresponds to transportation cost and the 60% to inventories, order processing, storage and distribution management.

Location and inventory decisions are correlated in the costs. For example, many

[1] http://www.atkearney.com.mx/

[2] http://www.gob.mx/se/

distribution centers reduce the cost of transporting product to customers and it will provide better service, but having few distribution centers reduces the cost of holding inventory via pooling effects, and reduces the fixed costs associated with operating distribution centers via economies of scale (Erlebacher and Meller, 2000). Integrating both decisions into the Location-Inventory Problem (LIP), the goal is to determine how many facilities to locate, where to locate them, which clients to assign to each facility, how often to reorder the product, and what level of stock to maintain, so as to minimize the total cost, while ensuring a specified service level. The problem applies in a company when the following questions arise:

- Is the number of depots ideal?
- Should some facilities be relocated, opened or closed?
- Is the product moving in the best way?
- Is it necessary to adjust the capacity of some facilities?

These questions are more relevant in presence of high variability demand, since that produces high levels of inventories, therefore, the location selection is affected by the assignment of clients and the demand variability.

Because of the nature of the problem, an extensive variety of models can be generated. They may vary substantially based upon the assumptions considered such as: location area (discrete or continuous), facilities features (capacitated or uncapacitated), costs (variable, fixed, economies of scale), product demand (stochastic or deterministic), inventory review (continuous or periodic), etc. Therefore, the literature on Location-Inventory Problems is extensive. Farahani et al. (2015) present a wide survey of recent works.

We study a specific problem that arises in a mono-product supply chain of two level networks, consisting of a set of plants, a set of candidate distribution centers and a set of retailers. The aim is to select suitable distribution centers in order to meet the demand at the lowest possible cost. The most distinctive features are:

- Optimization by cost, ignoring the geographic location of facilities.
- Stochastic demands in retailers and distribution centers. This type of demand is

caused by ups and downs in demand without a clear tendency. It is a crucial determinant of supply chain performance.

- Single source constraint. Each selected distribution center must be supplied by a single plant (first echelon) and each retailer should be serviced by a single distribution center (second echelon). This consideration plays a significant role for several computational methods, nevertheless, it is a common policy that offers advantages as reduced managerial coordination complexity, decreased need for information systems integration between source facilities, and guaranteed customer service.
- The inventory management in each selected facility: we use the variant of the Economic Order Quantity EOQ model developed by Axsäter (1996), in which the stochastic demand is represented by its mean value. Define the reorder point and the quantity of order, which affect in having excessive or insufficient stock to be able to meet the demand.
- Limited capacity in the distribution centers, which is the reason for considering the highest possible level of inventory, i.e., estimated inventory at the moment a new order of the product arrives.
- The presence of multiple plants producing allocation decisions, in our case as single source constraints.

An example is shown in the Figure 1.1. There, two distribution centers (hereinafter referred to as DC) are selected, each retailer is assigned to one of them, and each DC is supplied by only a single plant. Notice that neither the full set of the plants nor the full set of DC is used.

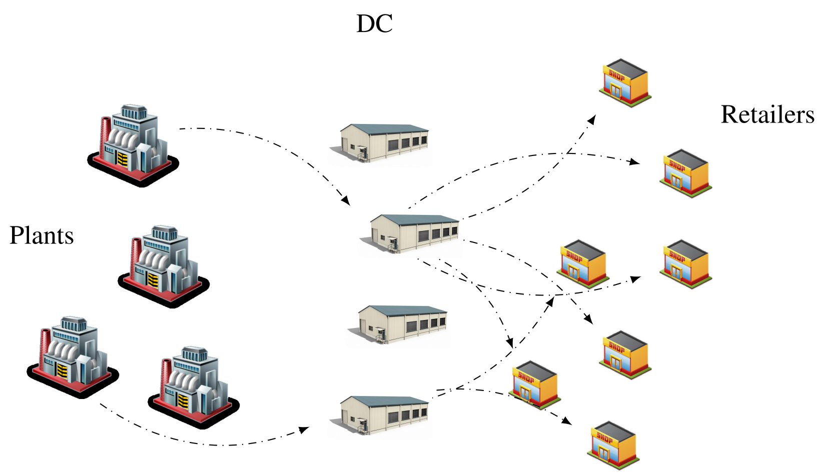



Figure 1.1: Elements in the supply chain

## 1.1 OBJECTIVES

The main objective of this research is to contribute to the state of the art of the the Location-Inventory Problem, to prove its computational complexity and to develop an exact algorithm to solve it. The specific objectives are the following:

- To prove the computational complexity of the problem.
- To formulate a mathematical model and validate it.
- To develop exact algorithms and evaluate them.

## 1.2 DISSERTATION STRUCTURE

The remainder of this dissertation is organized as follows. Chapter 2 recalls the main concepts for the understanding of the research, these are: principles of inventory management, mixed integer nonlinear programming, and column generation method. The state of art is presented in Chapter 3, meanwhile Chapter 4 is dedicated to the mathematical formulation of the problem. The computational complexity is studied in Chapter 5, the methodology proposed is explained in Chapter 6 and 7, showing the approximation developed and the column generation technique, respectively. The computational experiments are reported in Chapter 8 followed by the conclusions in Chapter 9.

Chapter 2

# Background

In this chapter, we present the theory for the understanding of the work. The topics are exposed in three categories: Inventory management, Mixed Integer Nonlinear Programming, and Column Generation.

## 2.1 Inventory theory

The main goal of inventory management is to meet the demand at the lowest possible cost. Even for high standards of demand attention, the inventory can not be excessive, since there are monetary and capacity limitations. In this section, inventory management is discussed, focusing on two aspects: the definition of an inventory policy and the compliance with the storage capacity.

### 2.1.1 Inventory policy

Inventory policy states two basic questions that must be answered in order to satisfy the demand: how much to order (known as *order quantity*) and when to order (known as *reorder point*). The most common inventory model is the Economic Order Quantity Model (EOQ) developed by F. W. Harris of General Electric. The EOQ consists of a formula for

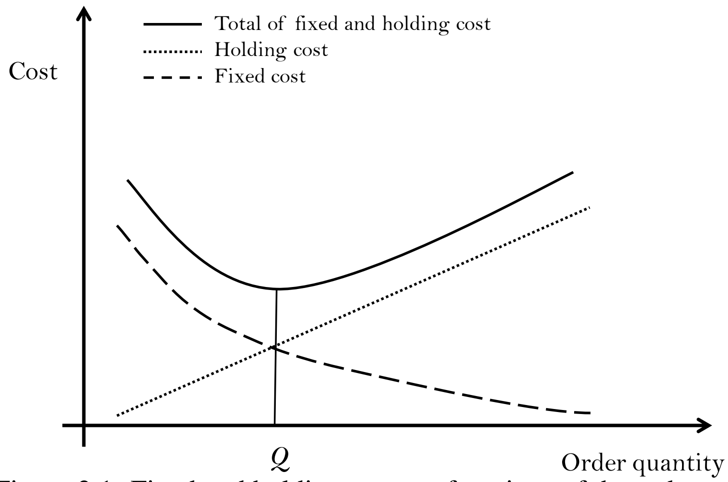


Figure 2.1: Fixed and holding costs as functions of the order quantity.

calculating the optimal order quantity that minimizes the sum of holding cost and ordering cost with the following assumptions (Muller, 2011):

- the demand rate is constant and known,
- the carrying cost and ordering cost are independent of the quantity ordered (no discounts),
- the *lead time*, the time that elapses between placing an order and actually receiving it, is constant and known,
- orders arrive in a single batch (no vendor stockouts or backorders).

Within an irregular market behavior, namely in the presence of uncertainty, these assumptions do not reflect practical cases. Hence, complicated variations of the basic model have been developed. Nonetheless, Zheng (1992) and Axsäter (1996) demonstrated that EOQ model produces a very good approximation for working inventory costs of systems under uncertainty of demand, so the EOQ is still widely used (Alhaj et al., 2016; Chuang and Chiang, 2016; Muriana, 2016; Roy et al., 2016; Sana, 2015).

*Order quantity* ($Q$) is the amount of product to order from the supplier. The goal is to optimize the compromise between ordering and holding costs. The average annual fixed order cost decreases as $Q$ increases because fewer orders are placed. On the other hand, the average annual holding cost increases as $Q$ increases since units remain in inventory longer. Thus, the order quantity affects the two types of costs in opposite ways. The annual fixed ordering cost is minimized by making $Q$ as large as possible, but the holding cost is minimized by having $Q$ as small as possible (Muckstadt and Sapra, 2010), as shown in Figure 2.1.

*Reorder point* ($R$) is the inventory level at which an order should be placed. It has to be high enough to satisfy the demand until the new order arrives. It depends on the demand and the lead time; when these parameters are unknown, meeting the demand can only be achieved with some probability. To analyze that, we study the inventory by cycles, which is defined as the time between two successive-order arrivals. Observe Figure 2.2, when the reorder point and the order quantity ($Q$) are fixed but the demand is uncertain, stockout and excess inventory may occur in the inventory cycles ($IC_1, IC_2, \cdots, IC_i$), even when the lead time ($LT$) is known and constant. In the figure, during the first inventory cycle ($IC_1$), the reorder point is less than the demand causing stockout, but it is not the case for the other cycle.

The desired probability of fulfilling the demand during the lead time is called *service level* and it is used for calculating the appropriate stock to protect against the variance in lead-time demand (to mitigate risk of stockout). This stock, called *safety stock* ($S$), is defined as a function of the service level in such a way that the higher the desired service level is, the more safety stock needs to be held and consequently, the involved costs (maintenance, transportation, and holding costs) are also higher. So, the reorder point ($R$) is then the sum of the expected lead-time demand ($D^{'}$) plus the safety stock:

$$R = D^{'} + S. \tag{2.1}$$

### 2.1.2 Storage capacity limitation

Because of the uncertainty on the demand, the product in stock may not be entirely consumed, in some cases, the excess of the storage capacity may be present. As a conse-

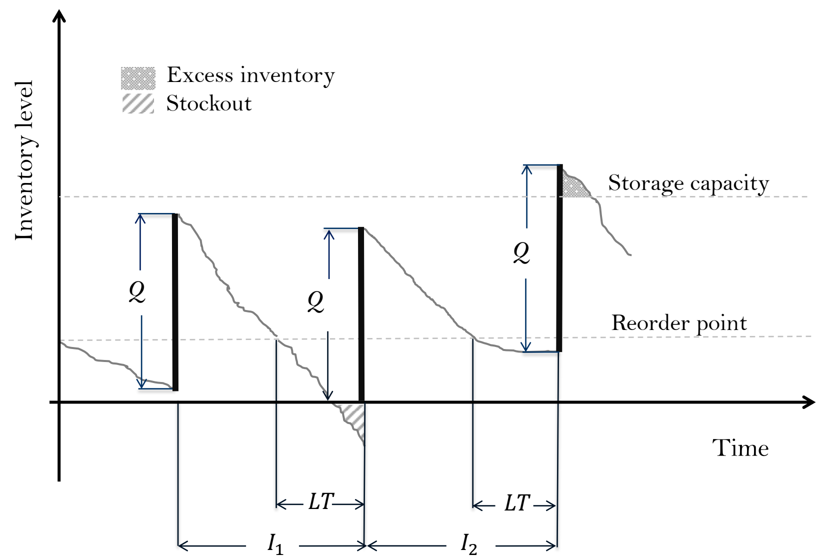



Figure 2.2: Evolution of inventory level at the distribution center. Adapted from Miranda et al. (2009).

quence, unnecessary costs are incurred in storage, insure, interests, taxes, etc. Depending on the specific industry and the actions taken to deal with this matter, other effects are produced. For example, in technology industries, usually the prices rapidly drop and having excess inventory implies losses; in other industries, the excess inventory may violate safety regulations resulting in fines. In either case, enforcing the storage limitations is a good business practice, both for keeping costs low and for facilitating inventory management, particularly for high-value items.

In summary, the inventory policy should consider not only stockout, but also the possibility of exceeding the storage capacity. This may happen upon the arrival of a new order, as shown in Figure 2.2. The key decision is the reorder-point magnitude: if the demand during lead time is lower than the expected, the arrival of the new order may exceed the capacity. We capture this consideration in the capacity constraint, formalized in Section 4.2.2.

## 2.2 MIXED INTEGER NONLINEAR PROGRAMMING

Typically, models for the Location-Inventory Problem involve nonlinearity associated to the safety stock cost and integer variables associated to location decisions, causing mixed integer nonlinear programming models (MINLP). Geometrically, nonlinear programs can behave much differently from linear programs, mainly the optimal solution can occur:

(a) at an interior point of the feasible region;

(b) on the boundary of the feasible region, but no an extreme point; or

(c) at an extreme point of the feasible region.

Consequently, algorithms such as the simplex method, that search only extreme points may not determine an optimal solution.

The tractability of these problems depends significantly on whether the objective function and the constraints are convex or not and if the model has a combinatorial and integer domain. Generally, nonlinear problems are non convex and it is difficult to reach optimality (Bussieck and Vigerske, 2010). Moreover, it is known that MINLP models are NP-hard (Kannan and Monma, 1978), it is so often, that linear programming models are preferable over non linear programming models.

### 2.2.1 MINLP SOLVER SOFTWARE

Algorithms for solving MINLP models are often built by combining algorithms from Linear Programming, Integer Programming, and Nonlinear Programming, e.g., branch and bound, outer approximation, local search, global optimization. Most of the solvers implement one (or several) of three algorithmic ideas to tackle MINLP models (Bussieck and Vigerske, 2010). In case of a nonconvex MINLP, these solvers can still be used as heuristic. Especially branch and bound techniques that use nonlinear programming for bounding often find good solutions also for nonconvex problems, while pure outer approximation based algorithms may easily run into infeasible linear programming or MIP relaxation due to wrong cutting planes.

The basic purpose of the solvers is to find a solution – that is, values for the decision variables in the model – that satisfies all of the constraints and maximizes or minimizes the value of the objective function (if there is one). The kind of solution one can expect, and how much computing time may be needed to find a solution, depends primarily on the next characteristics of the model:

- Model size, number of decision variables and constraints,
- type of variables,
- type of functions,
- other issues as scaling, convexity, definition of the parameters, etc.

According to Bussieck and Vigerske (2010) the earliest commercial software package that could solve MINLP problems was SCICONIC in the mid 1970s. It links Special-Ordered-Set variables provided a mechanism to represent low dimensional nonlinear terms by a piecewise linear approximation and thus allowed to use mixed-integer linear programming (MIP) to obtain solutions to an approximation of the MINLP. In 1980 Grossmann and Kocis developed DICOPT, a general purpose algorithm for convex MINLP based on the outer approximation method. Since then, a number of academic and commercial codes for convex MINLP have emerged, either based on outer approximation using MIP relaxations, an integration of outer approximation into a linear programming (LP) relaxation based branch and cut, or nonlinear programming (NLP) relaxation based branch and bound algorithms. For the global solution of nonconvex MINLP, the first general purpose solvers were alphaBB, BARON, and GLOP, all based on convexification techniques for nonconvex constraints.

While state of art MIP solvers typically implement advanced automatic reformulation and preprocessing algorithms, such techniques are less commonly available in MINLP solvers, and in a limited form. Therefore, the modelers choice of problem formulation is still very important when solving an MINLP.

Table 2.1: MINLP solvers

| Solver | Optimality scope | | Algorithms | | | Other characteristics of the algorithms | Developer |
|---|---|---|---|---|---|---|---|
| | Global | Local | B* | OAA | Other | | |
| AlphaECP | CM & PM | | | | Extended cutting plane | MIP is redefined by linearizing non-linear constraint at solutions of the MIP outer approximation | R. G of T. Westerlund in Abo Akademic University, Finland |
| Dicopt | CM | | | ✓ | | Used NLP relaxation & MIP | Carnegie Mellon University |
| SBB | CM | | ✓ | | | Handle discrete variable (NLP & MIP) | ARKI Consulting & Development A/S. |
| AlphaBB* | CM & NM | | ✓ | | | | R.G. of C. Fludas in Princenton University |
| Baron* | CM & NM | | ✓ | | CTNF | | Carnegie Mellon University & Purdue University |
| GLOP* | ✓ | | | | | | |
| Couenne | CM & NM | | | | CTNF | Use MINLP reformulations | |
| Bonmin | CM | | ✓ | ✓ | GBL | LP NLPr and B&B | Carnegie Mellon University |
| Fmincoset | ✓ | | | | | | |
| Cplex | CM | | ✓ | ✓ | CTNF | Inputs: NC objective function, NCBV, SOC, QCSOC. | |
| Mosek | CM (MIQCPs) | | ✓ | | | Input: SOC | Mosek Aps |
| FICO xpress optimizar | CM | | ✓ | | | Input: SOC, NCBV | Dash Optimization |
| SCIP | CM & NM | | | | CTNF | | Zuse Institute Berlin |
| Gurobi | CM | | ✓ | ✓ | | Input: NCBV, SOC, QCSOC | Gurobi optimzation |
| Lindo API | CM & NM | | | | CTNF | B&C that utilizes LPs for bounding | |
| FMINCONSET | ✓ | | ✓ | | | | |
| Knitro | ✓ | | ✓ | | | | |
| MILANO | CM | | ✓ | | | | H. Y. Benson in Drexel University. |
| MINLP-BB | CM | | | | | | R. Fletcher & S. Leyffer at the University of Dundee |
| MISQP | CM & NM | | | ✓ | SQP | Restriction integrality in solution of NLP sequential quadratic program . | K. Schittkowki at the University of Bayreuth |
| OQNLP | | ✓ | | | Multistart scartter search algorthm. | Randomized approach by sampling starting points. Use nonlinear relaxation . | |
| BNB | CM | | ✓ | | | Uses nonlinear relaxation for the bounding. | K. Kuipers in the University of Groningen. |
| ANTIGONE | CM & NM | | | | CTNF | It implements a spacitial B&B algorithm that utilizes MIPs for bounding. | R. Misener & C. Floudas in Princeton University |
| AOA | CM | | | | OAA / GBL | Constructs an MIP outer approx. Of the feasible region of the MINLP. | Paragon Decision Technology |
| LAGO | CM & NM (MIQCPs) | | | | CTNF | | I. Nowak in Humboldt University |
| MIDACO | CM & NM | | | | | Extended ant colony based on an Oracle penalty function. | Schuller in Hokkaido University. |
| MINOTAUR | CM | | ✓ | | | Offers to replace | S. Leyffer, J. Linderoth, J. Luedtke… at the University of Winsconsin Madison. |
| Filmint | CM | | | | | | Lehight University |

## 2.3 COLUMN GENERATION

Column generation is an efficient algorithm for solving programs that are too large to consider all the variables explicitly. The process operates by splitting the problem being solved into two problems: the master problem and the pricing problem (Lasdon, 2013, cap.3). The master problem is the original problem with only a subset of variables being considered; as this problem does not contain all of the columns (variables), sometimes it is called *restricted master problem* (Barnhart et al., 1998).

The pricing problem is a new problem created to find the most profitable column (variable) for being added to the master problem. If the optimal value of the pricing problem is negative (assuming without loss of generality that the problem is a minimization problem), a variable with negative reduced cost has been identified, this variable is added to the master problem, and the master problem is solved again. Then, a new set of dual values is obtained, and the process is repeated until no negative reduced cost variables are identified, when the solution to the master problem is optimal (Desrosiers and Lübbecke, 2005). These ideas can be observed in Figure 2.3.

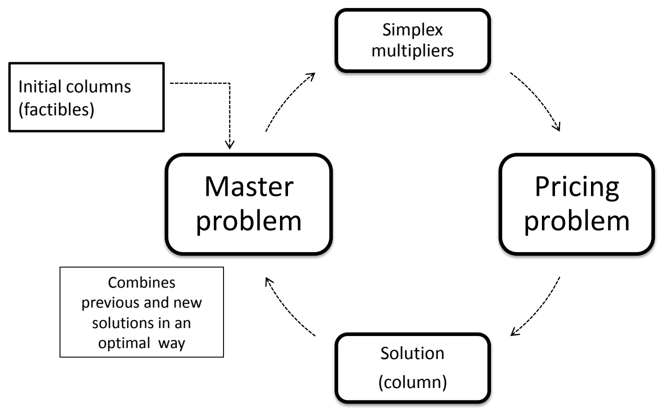



Figure 2.3: General idea of column generation method.

## Chapter 3

# Related Literature

The Location-Inventory Problem was first studied by Daskin et al. (2002), since then the problem has been increasingly recognized, generating an extensive variety of models. Farahani et al. (2015) present a recent review of the general problem and its variants. In this section we analyze # selected referred journal articles with the closest features to the problem addressed in this thesis. They were published in the period from 2000 to 2015. Three issues are examined, i) the supply chain features, ii) the mathematical formulation and iii) the methodology of solution.

Erlebacher and Meller (2000) formulate a non-linear location-inventory model. They use a continuous approximation and bounding heuristics. For problems with 16 customers, they obtained solutions that were between 3.78% and nearly 36% of a lower bound. An exchange heuristic improved the solution considerably

Daskin et al. (2002) present a Location-Inventory Problem considering a single uncapacitated supplier and multiple retailers, modelled as a nonlinear integer programming model which is restructured as a set-covering integer programming model. The inventory is estimated and just considered into the objective function, being ignored in the constraints, which only are of single source and retailers allocation. They solve cases where the variance-to-mean ratio at each retailer is identical for all retailers, using a Lagrangian relaxation combined with branch and bound. Shen et al. (2003) study the same problem and the same cases using a column generation technique, that allows optimal solution up to 150 retailers. Jia Shu (2005) propose, for the general case, a column generation algorithm exploiting special structures of the model. Their approach is able to solve larger problems (up to 500 retailers).

Ozsen et al. (2009) consider a centralized logistics system of uncapacitated warehouses and retailers, formulated as a nonlinear integer-programming problem and solved by a Lagrangian heuristic algorithm.

Tancrez et al. (2012) propose a problem of two levels (factories, distribution centers and customers) with multiple sourcing in every layer. They use a local search with several descent techniques, that allows get out of a local minimum. It gets solutions for large networks (1000 customers).

Atamtürk et al. (2012) present several Facility Location-Inventory Problems of two levels (capacitated or uncapacitated, correlated retailer demand, stochastic lead times and multicomodities). They modelled the problems as conic quadratic mixed integer problems. The approach leads to similar or better computational solution times than the reported by other publications.

Nyberg et al. (2013) study a three-stage multiechelon inventory system with specific exact linearizations without decomposition. An MILP underestimation of the problem can be solved as part of a sequential piecewise approximation scheme to solve the problem within a desired optimality gap.

Diabat and Theodorou (2015) study a two-echelon inventory management problem with multiple warehouses and retailers,formulated as a mixed integer non-linear program such that its continuous relaxation is non-convex. A piecewise linearization is used to transform the model.

Table 3.1: LIP literature

| References | Supply chain specifications | | | | | Model features | | | Solution technique |
|---|---|---|---|---|---|---|---|---|---|
| | Levels | Risk pooling | Single source | Capacitated facility | Economies of scale | Type | Inventory model | Demand | |
| Present work | 2 | ✓ | ✓ | ✓ | T | MINLP, BLP | EOQ | Normal | Approximation, Column generation |
| Erlebacher and Meller (2000) | 2 | – | ✓ | ✓ | – | NLP | EOQ | Uniform | Clarke&Wright vehicle routing (heuristic) |
| Nozick and Turnquist (2001) | 1 | ✓ | ✓ | – | – | BLP | Average | Poisson | Weighting method |
| Daskin et al. (2002) | 1 | ✓ | ✓ | – | T | MINLP | EOQ | Normal | LR, B&B |
| Shen et al. (2003) | 1 | ✓ | ✓ | – | – | MINLP, MILP | EOQ | Normal | Column generation |
| Miranda and Garrido (2004) | 1 | ✓ | ✓ | ✓ | – | MINLP | | Normal | |
| Elhedhli and Goffin (2005) | 2 | – | ✓ | ✓ | – | MILP | – | – | Hybrid: B&P, Interior Point |
| Shen and Qi (2007) | 1 | ✓ | ✓ | – | T | MINLP | EOQ | Normal | Hybrid: B&B, greedy |
| Snyder et al. (2007) | 1 | – | ✓ | – | T | MINLP | EOQ | Normal | Hybrid: B&B, LR |
| You and Grossmann (2008) | 1 | ✓ | ✓ | – | T | MINLP | EOQ | Normal | ES-LR |
| Miranda et al. (2009) | 1 | ✓ | ✓ | ✓ | – | MINLP | (Q,R) | Normal | e-Work based collaborative opt (heuristic) |
| Qin et al. (2009) | 1 | ✓ | ✓ | ✓ | – | MINLP | (Q,R) | Normal | Dividing decisions: SA |
| Ozsen et al. (2009) | 1 | ✓ | ✓ | ✓ | – | MINLP | EOQ | Poisson | ES - LR |
| You and Grossman (2010) | 2 | ✓ | ✓ | – | – | MINLP | Basic Stock Level | Normal | Hybrid: LR & piecewise linear approx |
| Sadjady and Davoudpour (2012) | 2 | – | – | ✓ | – | MILP | Average | – | HS - LR |
| Tancrez et al. (2012) | 2 | – | – | – | – | NLP | EOQ | – | |
| Brahimi and Khan (2012) | 2 | – | – | ✓ | – | MILP | Average | – | – |
| Nyberg et al. (2013) | 2 | ✓ | ✓ | – | – | MINLP, MILP | EOQ | Normal | Linearizations |
| Diabat and Theodorou (2015) | 1 | – | – | – | – | MINLP, MILP | Average | – | Linearizations |

In the Table 3.2 we go through the LIP literature in terms of its applications.

Table 3.2: LIP applications

| Reference | Application | Instances |
|---|---|---|
| Daskin et al. (2002) | Platelets distribution of a blood bank | 150 hospitals |
| Tancrez et al. (2012) | Reverse logistic of glass panes in a European manufacturer | 10 factories, 500 customers |

CHAPTER 4

# MATHEMATICAL FORMULATION

In this section we formally describe the addressed problem. First, the used notation is specified and then we explain how the inventory has been modeled and how it is included in the location problem. Finally, the developed models are described.

## 4.1 NOTATION

The main elements of the problem are denoted as follows: a set of $p$ plants denoted by $I = 1,2,...,p$, a set of $n$ candidate distribution centers denoted by $J = 1,2,...,n$ and a set of $m$ retailers denoted by $K = 1,2,...,m$. Each distribution center has a specific storage capacity $q_j$ as well as the plants has a specific production capacity $p_i$. Regarding to the retailers, there is a mean $\mu_k$ and a variance $\sigma_k$ of the daily demand for each one. The involved costs are:

$h$: Annual holding cost per item.

$u_j$: Fixed annual cost for locating the distribution center $j$.

$f_j$: Fixed cost for placing an order from distribution center $j$.

$g_{ij}$: Variable cost of shipping an order from plant $i$ to distribution center $j$.

$a_{ij}$: Unit shipment cost from plant $i$ to distribution center $j$.

$c_{jk}$: Unit shipment cost from distribution center $j$ to retailer $k$.

Other important elements of the problem are:

$r$: Number of working days in a year.

$\beta$: Weight factor associated with the shipment cost.

$\theta$: Weight factor associated with the inventory cost.

$\ell_{ij}$: Lead time from plant $i$ to distribution center $j$.

$\alpha$: Probability of meeting the demand during lead time.

$z_\alpha$: Standard normal random variable corresponding to cumulative probability of $\alpha$.

## 4.2 INVENTORY MANAGEMENT

Inventory is considered only in open distribution centers. Inventory in plants and retailers is ignored because it does not affect the main decision regarding to which distribution centers should be open. We next explain how the inventory is defined and evaluated in a specific distribution center, assuming for the moment that we know which retailers are assigned to this distribution center, and also which plant supplies it. For defining the inventory management, we assume that the demand is uncertain, independent, and can be described by a normal distribution. The stochastic EOQ model proposed by Axsäter (1996) is used for this matter. We focus the analysis on defining the inventory policy and obeying the storage capacity.

### 4.2.1 INVENTORY POLICY

Unlike the classical inventory problem, for our model the inventory policy has to consider the network configuration since the costs are different depending on the plant and the distribution center involved. Thus, the order quantity and the reorder point are defined as a function of the allocation between the elements of the problem.

The *order quantity* $(Q_j)$ in distribution center $j$ is defined as a ratio between its annual demand $(D_j)$ and the number of orders $(O_j)$ during the planning horizon,

$$Q_j = \frac{D_j}{O_j}. \tag{4.1}$$

On the other hand, the total number of orders is defined based on the model proposed by Daskin et al. (2002) and is calculated as:

$$O_j = \sqrt{\frac{\theta h D_j}{2\left(f_j + \beta \sum_{i \in I} g_{ij} Z_{ij}\right)}}, \tag{4.2}$$

where $Z_{ij}$ is a binary variable that takes value one if the plant $i$ serves the distribution center $j$ and zero otherwise. Note that Equation (4.2) is considering not only the annual demand but also the ordering $(f_j)$ and transportation cost $(g_{ij})$, being both dependent on the network configuration. Substituting the value of the total number of orders into Equation (4.1) and simplifying, the order quantity can be expressed as:

$$Q_j = \sqrt{\omega D_j \left(f_j + \beta \sum_{i \in I} g_{ij} Z_{ij}\right)}. \tag{4.3}$$

Due to the uncertain demand the *reorder point* $(R_j)$ must be sufficiently high to satisfy the current demand during lead time $(C_j)$. This is achieved with some probability, as follows:

$$P\left(C_j \leq R_j\right) = \alpha, \tag{4.4}$$

where the knowledge of the current demand is assumed, and the desired probability to avoid stockout is denoted by $\alpha$. Now, for calculating the reorder point, we normalize Equation (4.4),

$$P\left(\frac{C_j - \acute{\mu}_j}{\acute{\sigma}_j} \leq \frac{R_j - \acute{\mu}_j}{\acute{\sigma}_j}\right) = \alpha, \tag{4.5}$$

$$P(z \leq z_\alpha) = \alpha, \tag{4.6}$$

where the variable $R_j$ may be deduced as a function of known parameters, as shown in

Equation (4.7). Reminding Equation (2.1), the reorder point is the sum of the expected lead-time demand plus the safety stock, here the safety stock is denoted as $z_\alpha \acute{\sigma}_j$.

$$R_j = \acute{\mu}_j + z_\alpha \acute{\sigma}_j. \tag{4.7}$$

### 4.2.2 STORAGE CAPACITY LIMITATION

Storage capacity limitation is considered because of the possibility of slow sales during the lead time, causing over inventory at the moment an order arrives. This is controlled through a constraint that measures the stock in the distribution centers. It involves the order size plus the reorder point minus the product consumed during lead time, the elements in an inventory (see Figure 2.2). Since the current demand is unknown, we use a minimum probable demand ($M_j$) that generates a large but acceptable quantity of product in the distribution center $j$. The worst case is when there is no demand at all, causing that all the product remains in the distribution center. However, it is very unlikely, so it has not been considered. Therefore, the constraint for storage-capacity limitation is:

$$Q_j + R_j - M_j \leq q_j. \tag{4.8}$$

It is expected that the current demand will be at least as equal as the defined minimum demand with probability of $1-\gamma$, which is the same as:

$$P(C_j \leq M_j) = \gamma. \tag{4.9}$$

We normalize Equation (4.9):

$$P\left(\frac{C_j - \acute{\mu}_j}{\acute{\sigma}_j} \leq \frac{M_j - \acute{\mu}_j}{\acute{\sigma}_j}\right) = \gamma, \tag{4.10}$$

$$P(z \leq z_\gamma) = \gamma, \tag{4.11}$$

and isolate the variable $M_j$ to compute its value in terms of known parameters, as follows:

$$M_j = \acute{\mu}_j + z_\gamma \acute{\sigma}_j, \tag{4.12}$$

where $z_\gamma$ is the quantile of the standard normal distribution of demand that contains the probability $\gamma$ of not exceeding capacity. Substituting Equations (4.7) and (4.12) into Equation (4.8) and denoting $z_e = z_\alpha - z_\gamma$, the capacity constraint becomes:

$$Q_j + z_e \acute{\sigma}_j \leq q_j, \tag{4.13}$$

This estimates the inventory size as the order quantity plus $z_e$ times the standard deviation of demand during lead time at the distribution center $j$.

## 4.3 NONLINEAR MODEL

Before describing the first model developed for the problem, let us introduce some additional notation. The parameters are shown in Table 4.1, meanwhile the variables are shown in Table 4.2.

Table 4.1: Calculated parameters in LIP

| Notation | Interpretation | Calculation |
|---|---|---|
| $D$ | Total mean daily demand of all set of retailers | $\sum_k \mu_k$ |
| $\rho_{ij}$ | Cost parameter for sending product from the plant $i$ to the distribution center $j$ | $f_j + \beta g_{ij}$ |
| $d^u_{ij}$ | Upper bound on the total amount of product sending from the plant $i$ to the distribution center $j$ | $\min\{\sum_k \mu_k, p_i, q_j\}$ |
| $\acute{c}_{jk}$ | Weighted annual shipment cost per item from the distribution center $j$ to the retailer $k$ | $\beta r c_{jk}$ |
| $\acute{a}_{ij}$ | Weighted annual shipment cost per item from the plant $i$ to the distribution center $j$ | $\beta r a_{ij}$ |
| $\phi$ | Inventory factor | $\theta h r z_\alpha$ |
| $\eta$ | Inventory factor | $\sqrt{2r\theta h}$ |
| $\tau$ | Inventory factor | $\eta(\theta h)^{-1}$ |

Table 4.2: Variables in LIP model

| Notation | Interpretation | Type |
|---|---|---|
| $X_j$ | Binary variable that takes the value of one if the distribution center $j$ is selected and zero otherwise | Decision variables for defining the supply chain network |
| $Y_{jk}$ | Binary variable that takes value one if the distribution center $j$ supplies retailer $k$ and zero otherwise | |
| $Z_{ij}$ | Binary variable that takes value one if the plant $i$ serves the distribution center $j$ and zero otherwise | |
| $D_{ij}$ | Mean daily demand in the distribution center $j$ that is served by the plant $i$ | Auxiliary variables for inventory management |
| $S_j$ | Variance of the daily demand for each distribution center $j$ | |
| $T_j$ | Lead time in days for deliveries at each distribution center $j$ | |

Now, the problem is modeled as a Mixed Integer Nonlinear Problem (MINLP).

$$\min \quad \sum_{j\in J}\left(u_jX_j+\sum_{k\in K}\mu_k\acute{c}_{jk}Y_{jk}+\sum_{i\in I}\acute{a}_{ij}D_{ij}+\eta\sqrt{\sum_{i\in I}\rho_{ij}D_{ij}}+\phi\sqrt{S_jT_j}\right) \tag{4.14}$$

s.t:

$$\sum_{j\in J}Y_{jk}=1 \qquad \forall k\in K \tag{4.15}$$

$$\sum_{i\in I}Z_{ij}=X_j \qquad \forall j\in J \tag{4.16}$$

$$D_{ij}\leq d^u_{ij}Z_{ij} \qquad \forall i\in I, j\in J \tag{4.17}$$

$$\sum_{i\in I}D_{ij}\geq\sum_{k\in K}\mu_kY_{jk} \qquad \forall j\in J \tag{4.18}$$

$$\sum_{j\in J}D_{ij}\leq p_i \qquad \forall i\in I \tag{4.19}$$

$$\tau\sqrt{\sum_{i\in I}\rho_{ij}D_{ij}}+z_e\sqrt{S_jT_j}\leq q_jX_j \qquad \forall j\in J \tag{4.20}$$

$$T_j=\sum_{i\in I}\ell_{ij}Z_{ij} \qquad \forall j\in J \tag{4.21}$$

$$S_j = \sum_{k \in K} \sigma_k Y_{jk} \qquad \forall j \in J \tag{4.22}$$

$$X_j, Y_{jk}, Z_{ij} \in \{0,1\} \qquad \forall k \in K, i \in I, j \in J \tag{4.23}$$

$$T_j, D_{ij} \in \mathbb{Z}^+ \qquad \forall j \in J \tag{4.24}$$

$$S_j \in \mathbb{R} \qquad \forall j \in J \tag{4.25}$$

The objective function minimizes the total weighted cost of the distribution network. The first term in Equation (4.14) calculates the cost for locating distribution centers, while the transportation costs are simplified in the second and third term. The weighted cost for holding inventory, ordering cost and variable cost for sending orders to the selected centers are shown simplified in the fourth term. The last sum corresponds to the weighted cost for holding safety stock.

Regarding to constraints, Equations (4.15) ensure that each retailer is assigned to a single distribution center, while Equations (4.16) ensure that each open distribution center is assigned to a single plant. Equations (4.17, 4.18) are bounds for the value of distribution-center demand when it is attended by plant $i$, which has to be at least as equal to the served demand, but no greater than the production capacity or the storage capacity. Equations (4.19) express the capacity production of each plant. Inequality expressed in Equation (4.20) indicates the capacity constraints for distribution centers. These constraints indicate the same as Equation (4.13) but expressed in terms of the variables defining the supply chain network. Expressions (4.21) define the lead time of delivering an order in the distribution centers. The constraints in Equation (4.22) define the variance of the daily demand for each distribution center. Finally, Equation (4.23) and Equation (4.25) establishe the nature of the variables.

Since the solvers for MINLP usually are made under the assumption of convexity, their performance is depended on that property. We study the model convexity in the next section.

### 4.3.1 NON-CONVEXITY PROOF

In this section we study some convexity properties to prove that MINLP, expressed in Equations (4.14–4.25), is not a convex problem, nevertheless it is a model formed by concave functions.

In an optimization problem, convexity of the objective function and constraints are crucial. Problems with this property have important theoretical properties (e.g., the local necessary optimality conditions for these problems are sufficient for global optimality), and what is much more important, convex problems can be efficiently solved numerically, which is not, the case for general nonconvex problems.

**Definition 1.** A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex if the domain in $f$ is a convex set and if for all $x, y \in$ domain $f$, and $\theta$ with $0 \leq \theta \leq 1$, we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y). \tag{4.26}$$

**Definition 2.** A convex optimization problem is one of the form:

$$\begin{aligned} \min \quad & f_0(x) && \qquad (4.27) \\ \text{s.t:} \quad & f_i(x) \leq 0, && i = 1, \ldots m, \\ & a_i^T x = b_i, && i = 1, \ldots p, \end{aligned}$$

where the functions $f_0, \ldots f_m : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex, and the equality constraint functions $h_i(x) = a_i^T x - b_i$ must be affine (Boyd and Vandenberghe, 2004, chap. 4).

#### 4.3.1.0 CALCULUS RULES FOR CONVEXITY

We apply the next calculus rules of convexity (see Boyd and Vandenberghe, 2004, chap. 3).

**Rule 1** [Nonnegative weighted sums]: Let $f$ be a convex function and $\alpha \geq 0$, then the function of $\alpha f$ is convex. If $f_1$ and $f_2$ are both convex function, then so is their sum

$f_1 + f_2$. Combining nonnegative scaling and addition, the set of convex functions is itself a convex cone: a nonnegative weighted sum of convex functions,

$$f = w_1 f_1 + \ldots + w_m f_m,$$

is convex. Similarly, a nonnegative weighted sum of concave functions is concave.

**Rule 2** [Vector composition]: When the function $h$ is convex (concave), $h$ is nondecreasing in each argument, and $g_i$ are convex (concave) then $f$ is convex (concave) in $f(x) = h(g(x)) = h(g_1(x), \ldots g_k(x))$, with no assumption of differentiability of $h$ or $g$, and general domains.

**Rule 3** [Concave function]: If $x^a$ is convex on $\mathbb{R}^+$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$.

**Rule 4** [Affine function]: A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is affine if it is a sum of a linear function and a constant. It always keep the equality in Equation 1, so all affine (and therefore also linear) functions are both convex and concave.

Returning to our model, we first examine the objective function of MINLP. It was established as:

$$\min \quad \sum_{j \in J} \left( u_j X_j + \sum_{k \in K} \mu_k \acute{c}_{jk} Y_{jk} + \sum_{i \in I} \acute{a}_{ij} D_{ij} + \eta \sqrt{\sum_{i \in I} \rho_{ij} D_{ij}} + \phi \sqrt{S_j T_j} \right)$$

Since it is a nonnegative weighted sum, which preserves convexity, the focus will be on each of its terms, which are linear functions except two of them. Taking into account the previous rules, in order to prove the convexity, we only need to verify the convexity of the radicands in each square root. The radicands are concave, so using Rule 2, it can be concluded that the complete functions are concave. Since all the summands in the objective function are concave, the complete function is concave.

Now, the aim is to know if the feasible set is a convex set. Constraints defined by

Equations (4.15–4.19) and (4.21–4.22) are linear, so they are convex. However, constraint expressed in Equation 4.20 is a non negative sum of the functions evaluated before but with different scale. We already know that they are concave functions, so the composition is also concave.

#### 4.3.1.0 ALTERNATIVE PROOF

Other way to prove convexity is using the Hessian Matrix. For simplicity we consider the specific case of three variables, setting the cardinality of set $I$ in 2 and the cardinality of set $J$ in 1. A function of more than two variables, $f(x_1, x_2, \ldots, x_n)$ is said convex (concave) if and only if its Hessian matrix $(\nabla^2 f(\mathbf{x}))$ of $nxn$ is positive (negative) semi-definite for all the possible values of $(x_1, x_2, \ldots, x_n)$. In the term $\eta\sqrt{\sum_{i\in I}\rho_{ij}D_{ij}}$ the scalar does not affect the convexity, for the rest, the matrix of Hessian is:

$$H = \begin{bmatrix} -\rho_{\acute{\sigma}_j 11}^{\acute{\sigma}_j 2}\varepsilon_{\acute{\sigma}_j 1} & -\rho_{\acute{\sigma}_j 11}\rho_{\acute{\sigma}_j 21}\varepsilon_{\acute{\sigma}_j 1} \\ -\rho_{\acute{\sigma}_j 11}\rho_{\acute{\sigma}_j 21}\varepsilon_{\acute{\sigma}_j 1} & -\rho_{\acute{\sigma}_j 21}^{\acute{\sigma}_j 2}\varepsilon_{\acute{\sigma}_j 1} \end{bmatrix}, \tag{4.28a}$$

where $\varepsilon_1 = \frac{1}{4}\left(\sum_{i\in I}\rho_{i1}D_{i1}\right)^{-\frac{3}{2}}$. A $2\times 2$ matrix can be classified as positive (negative) semi-definite if and only if its diagonal entries are both non-negative (non-positive) and its determinant is non-negative. For this function, the entries are negative and the determinant is zero, so this function is not convex, it is concave.

Now, for the term $\sqrt{S_j T_j}$ the Hessian matrix is:

$$H = \begin{bmatrix} -\varphi T_{\acute{\sigma}_j 1}^{\acute{\sigma}_j 2} & -\varphi T_{\acute{\sigma}_j 1}S_{\acute{\sigma}_j 1} \\ -\varphi T_{\acute{\sigma}_j 1}S_{\acute{\sigma}_j 1} & -\varphi S_{\acute{\sigma}_j 1}^{\acute{\sigma}_j 2} \end{bmatrix}, \tag{4.29a}$$

where $\varphi = \frac{1}{4}\left(S_1 T_1\right)^{-\frac{3}{2}}$.The function is clearly not convex, accordingly the objective function neither is.

Summarizing, the model MINLP is not a convex optimization problem, but the objective function and constraints are concave.

Solving the MINLP model, we could not get optimal solutions, even after preprocessing and trying with several solvers, moreover, the feasible solutions found were not good. Because it was not possible to find optimal solutions, we decided to look for a new formulation that is exposed in the next section.

## 4.4 REFORMULATION

### 4.4.1 NON LINEAR REFORMULATION

For the reformulation, a new decision variable is required for expressing the connection between the sets. We denoted the link between the plant $i$, distribution center $j$ and the retailer $k$ by the variable $W_{ijk}$, which takes the value of 1 if the connection exists, and 0 is not. Then, the new model is denoted by (MINLP2), it is expressed as follows:

$$\min \quad \sum_{j\in J}\left(u_jX_j + \sum_{k\in K}\mu_k \acute{c}_{jk}Y_{jk} + \sum_{i\in I}\acute{a}_{ij}D_{ij} + \eta\sum_{i\in I}\hat{\rho}_{ij}\hat{D}_{ij} + \phi\sum_{i\in I}\hat{\ell}_{ij}\hat{S}_{ij}\right) \tag{4.30}$$

s.t:

$$W_{ijk} \leq Y_{jk} \qquad \forall i\in I, j\in J, k\in K \tag{4.31}$$

$$W_{ijk} \leq Z_{ij} \qquad \forall i\in I, j\in J, k\in K \tag{4.32}$$

$$W_{ijk} \geq Y_{jk} + Z_{ij} - 1 \qquad \forall i\in I, j\in J, k\in K \tag{4.33}$$

$$D_{ij} = \sum_{k\in K}\mu_k W_{ijk} \qquad \forall i\in I, j\in J \tag{4.34}$$

$$S_{ij} = \sum_{k\in K}\sigma_k W_{ijk} \qquad \forall i\in I, j\in J \tag{4.35}$$

$$\sum_{k\in K}Y_{jk} \leq mX_j \qquad \forall j\in J \tag{4.36}$$

$$\sum_{j \in J} Y_{jk} = 1 \qquad \forall k \in K \qquad (4.37)$$

$$\sum_{i \in I} Z_{ij} = X_j \qquad \forall j \in J \qquad (4.38)$$

$$\sum_{j \in J} D_{ij} \leq p_i \qquad \forall i \in I \qquad (4.39)$$

$$\tau \sum_{i \in I} \hat{\rho}_{ij} \hat{D}_{ij} + z_e \sum_{i \in I} \hat{\ell}_{ij} \hat{S}_{ij} \leq q_j X_j \qquad \forall j \in J \qquad (4.40)$$

$$\hat{D}_{ij} = \sqrt{D_{ij}} \qquad \forall i \in I, j \in J \qquad (4.41)$$

$$\hat{S}_{ij} = \sqrt{S_{ij}} \qquad \forall i \in I, j \in J \qquad (4.42)$$

$$X_j, Y_{jk}, Z_{ij}, W_{ijk} \in \{0,1\} \qquad \forall k \in K, i \in I, j \in J \qquad (4.43)$$

$$D_{ij}, S_{ij}, \hat{D}_{ij}, \hat{S}_{ij} \in \mathbb{R} \qquad \forall i \in I, j \in J \qquad (4.44)$$

where $\forall i \in I, j \in J$ :

$$\hat{\rho}_{ij} = \sqrt{\rho_{ij}}, \qquad \hat{\ell}_{ij} = \sqrt{\ell_{ij}}.$$

### 4.4.2 LINEAR REFORMULATION

It is not difficult to see that the problem could be visualized in another way. Specifically, a subset of retailers could be preassigned to a distribution center enable to meet its demand, and then, the selected distribution center could be assigned to a plant, minimizing cost at the same time. With this idea in mind, let us define *additionally* the following set and variables which are necessary to establish a new formulation:

$B$: The collection of nonempty subsets ($b$) of the retailer set ($K$). Note that there are $2^m subsets, m = |K|$.

$E_{bij}$: Binary variable that specifies if the assignment formed by subset $b$, distribution center $j$ and plant $i$ is selected.

$\delta_{kb}$: Parameter that takes value of one if the retailer $k$ belongs to set $b$.

$\omega_{bij}$: Cost for ordering, holding inventory and transporting product regarding plant $i$, distribution center $j$ and subset $b$ of retailers.

$$\omega_{bij} = \sum_k (\acute{c}_{jk} + \acute{a}_{ij})\mu_k \delta_{kb} + \phi \sqrt{\sum_k \sigma_k \delta_{kb} \ell_{ij}} + \eta \sqrt{\sum_k \mu_k \delta_{kb} \rho_{ij}} \tag{4.45}$$

$s_{bij}$: Capacity used in the distribution center $j$ to satisfy demand of $b$ when it is supplied by $q_i$.

$$s_{bij} = \tau \sqrt{\sum_k \mu_k \delta_{kb} \rho_{ij}} + z_e \sqrt{\sum_k \sigma_k \delta_{kb} \ell_{ij}} \tag{4.46}$$

Now the problem can mathematically be stated as a Mixed Integer Linear Problem (MILP):

$$\min \quad \sum_j u_j X_j + \sum_b \sum_i \sum_j \omega_{bij} E_{bij} \tag{4.47}$$

s.t:

$$\sum_b \sum_i \sum_j \delta_{kb} E_{bij} = 1 \qquad \forall k \in K \tag{4.48}$$

$$\sum_b \sum_j \sum_k \mu_k \delta_{kb} E_{bij} \leq p_i \qquad \forall i \in I \tag{4.49}$$

$$\sum_b \sum_i s_{bij} E_{bij} \leq q_j X_j \qquad \forall j \in J \tag{4.50}$$

$$X_j, E_{bij} \in \{0,1\} \qquad \forall i \in I,\ j \in J, b \in B \tag{4.51}$$

The objective function minimizes the location cost and the cost of the distribution network, namely the transportation cost in all levels and the cost of ordering and holding inventory (working inventory and safety stock). Regarding to constraints, EquationS (4.48) ensure that each retailer is served, Equations (4.49) and (4.50) are the capacity constraints for plants and distribution centers respectively and finally, constraints expressed through Equations (4.51) establish the nature of the variables.

It is important to emphasize that this formulation has the same search space than MINLP expressed through Equations (4.14–4.25); here, all possible subsets of retailers are beforehand considered and it is possible to calculate the inventory and transportation costs before optimizing, as opposed to formulation MILP where these costs are variables.Nevertheless, the optimal solutions are the same in both formulations.

This model has two important advantages, one theoretical and one practical. The first one is that the model gets rid of the nonlinearity as function of the variables, which means that now the model is a linear problem and a local optimum is also a global optimum. The practical advantage is related to the cost. When the retailers are grouped, significant inventory–cost savings can be achieved in a phoneme called *risk pooling* (Eppen, 1979). The disadvantage is that the number of constraints and variables grows exponentially, so it is only possible to solve small and medium instances.

CHAPTER 5

# COMPUTATIONAL COMPLEXITY ANALYSIS

In this chapter, we formally establish the computational tractability of a LIP, even though numerous studies have addressed LIP and its variants, to the best of our knowledge, none formally demonstrated its computational complexity. Variants of LIP have been considered as NP-hard; some of them because they extend the Uncapacitated Facility Location Problem (UFLP), others because they use the Economic Order Quantity Model (EOQ) (Harris, 1990) for managing the inventory. Both, UFLP and EOQ, are NP-hard; the former is proved by Krarup and Pruzan (1983), the latter by Gallego et al. (1992).

The complexity proof is made through the theory of NP-completeness, which provides techniques for proving that a given problem is as hard as other problems that are recognized as being difficult. The knowledge of the complexity of a problem gives information about what solution techniques can be considered. For complex problems, exact algorithms are often impractically inefficient for all but the smallest problem instances, and instead techniques that require less computational resources even though they do not guarantee an optimal solution are preferred: approximation algorithms (including randomized approaches) and heuristics (Papadimitriou, 1994).

## 5.1 COMPUTATIONAL COMPLEXITY THEORY

Without addressing the depths of theoretical computer science and avoiding topics such as Turing machines, we first give an introduction to NP-completeness and then the complexity proof for LIP.

The term NP stands for *nondeterministic polynomial time* and refers the ability to, given an input, guess a correct solution, if it exists, and verify it, all in polynomial time. An optimization problem is called NP-hard if its corresponding decision problem is NP-complete (Papadimitriou, 1994). While the optimization problems seek the best solution according to some criteria, the decision problems seek to determine whether at least one solution fulfills the criteria. An optimization problem can be transformed into a decision problem by replacing each objective function with a fixed bound (i.e., requiring the value to be at most a given constant for minimized objectives and at least a given constant for maximized objectives), and then asking whether a solution satisfying all the original constraints of the optimization problem as well as the newly placed bounds exist, which is now a yes-or-no question.

According to Garey and Johnson (1979) and Papadimitriou and Steiglitz (1982), the steps to prove that a decision problem is NP-complete are the following:

1. Prove that the decision problem under study belongs in the class NP.
2. Prove that every problem in the class NP is can be reduced in polynomial time to the problem under study.

In the next sections, we describe how we performed each of these steps for proving that the Location-Inventory Problem is NP-hard.

## 5.2 DECISION PROBLEM

The decision problem associated to the Location-Inventory Problem is denoted here by LIP-D. It is stated as follows:

*Instance:* Given the following inputs:

- a set $K$ of $m$ retailers indexed by $k$, each with a mean daily demand $\mu_k$ and a variance of daily demand $\sigma_k$,
- a set $J$ of $n$ potential distribution centers indexed by $j$, each with a fixed opening cost $u_j$, a fixed cost of placing an order $f_j$, and a storage capacity $q_j$,
- a set $I$ of $p$ plants indexed by $i$, each with a production capacity $b_i$, a specific time of delivering products $\ell_{ij}$ to distribution center, and transportation costs $(g_{ij}, a_{ij})$,
- an annual budget $H \in \mathbb{Z}^+$.

*Question:* Does an assignment of retailers to distribution centers and of distribution centers to plants exist such that neither production capacities of plants nor storage capacities of distribution centers are violated (capacity constraints), each open distribution center is supplied by a single plant and each retailer is supplied by a single distribution center (single-source constraints), the number of open distribution centers is at most $n$ (activation constraint), and the total cost (objective function converted into a constraint) is less or equal to $H$?

## 5.3 MEMBERSHIP TO NP-CLASS

**Theorem 5.3.1.** LIP-D $\in$ NP.

*Proof.* We can assert that LIP-D is in NP, since a nondeterministic algorithm only needs to select elements of the set $I$ and set $J$ and assign the retailers in such way that each retailer is attended by one open distribution center which is attended by a single plant and neither the production capacity of the plants nor the storage capacity of the distribution center is exceeded. To verify the feasibility of the solution, an algorithm, called *witness* or *certificate* is needed. In our case the algorithm must check the next requirements:

- The allocation of each retailer. This task is bounded by $\mathcal{O}(m)$.
- The fulfillment of the storage capacity. The inventory is calculated and evaluated in each distribution center open, since it is a function of the assigned retailer and the assigned plant, this step has the order of $\mathcal{O}(m \cdot n \cdot p)$ regardless the inventory model used. In our problem this is evaluated in Equation (4.13).
- The fulfillment of the production capacity $b_i$ in plants. The demand satisfied by a plant is calculated and compared to the capacity. This task is bound by $\mathcal{O}(m \cdot p)$.
- The number of open distribution centers is at most $n$. This task is bound by $\mathcal{O}(n)$.
- The fulfillment of the annual budget. This requires to evaluate the total cost, the argument in Function (4.14), and compare it to the budget. This task is bound by $\mathcal{O}(m \cdot n \cdot p)$.

If the constraints are satisfied, the algorithm returns "yes". The certificate for LIP-D runs in polynomial time, it is bounded by $\mathcal{O}(m(1+2np+p)+n)$. □

## 5.4 REDUCTION

The second step to establish that a particular problem is NP-complete entails the use of a reduction algorithm ($F$). This is a polynomial algorithm, which, given any input $x$ to problem $P_1$, transforms it into an equivalent input $F(x)$ to problem $P_2$ (Papadimitriou, 1994). This is denoted by $P_1 \prec P_2$.

In practice, it is not necessary to reduce every problem in the class NP to the problem that is being tested, it is enough to reduce a problem that has been previously proven as NP-complete. The first compendium of NP-complete problems was given by Garey and Johnson (1979) and it was updated by Crescenzi and Kann (2005).

To achieve this step, we study the problem in three basic cases: a deterministic version with only one plant (1-LIP-D), a deterministic version with multiple plants (M-LIP-D) and a stochastic version with multiple plants (S-LIP-D). For the first one, we use a reduction from Bin Packing Problem, once we have proven this case, that is, 1-LIP-D is NP-complete, we reduce it to M-LIP-D (the second case), at last this second case is reduced to the third one.

### 5.4.1 REDUCTION FROM BIN PACKING PROBLEM (CASE 1-LIP-D)

For the 1-LIP-D complexity proof, we chose the Bin Packing Problem in its decision version, denoted here by BP-D. The idea is to demonstrate that an algorithm capable of solving 1-LIP-D can also solve BP-D. Due to these problems have not exactly the same inputs, a reduction $F$ of the input $x$ is needed, as illustrated in Figure 5.1.

The proof that BP-D is NP-complete is made through the reduction from the Tripartite Matching Problem and the Partition Problem. The demonstration can be found in Papadimitriou and Steiglitz (1982) and Garey and Johnson (1979). BP-D is stated as follows:

*Instance:* Given a finite set $U$ of items, a size $s_u \in \mathbb{Z}^+$ for each $u \in U$, a bin capacity $c \in \mathbb{Z}^+$, and a bin count $w \in \mathbb{Z}^+$.

*Question:* Is there a partition $U_1, U_2, \ldots, U_w$ with $U = \bigcup_{i=1}^{w} U_i$ and $\forall i \neq j : U_i \cap U_j = \emptyset$ such that $\forall U_i : \sum_{u \in U_i} s_u \leq c$?

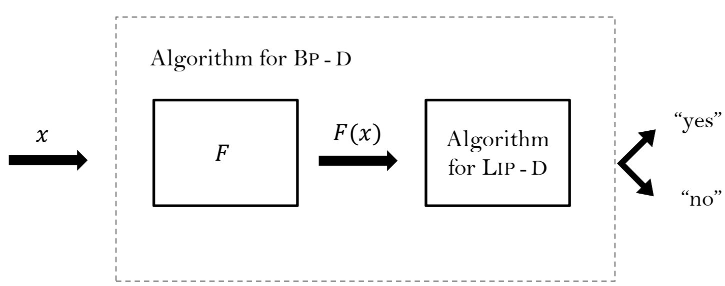



Figure 5.1: Reduction from BP-D to 1-LIP-D.

Table 5.1: Analogy between input parameters

| BP-D | | 1-LIP-D | |
|---|---|---|---|
| Notation | Input | Notation | Input |
| $U$ | Set of items | $K$ | Set of retailers |
| $s_u$ | Size of items | $\mu_k$ | Demand of retailers |
| $c$ | Bin capacity | $q$ | Distribution center capacity |
| $w$ | Number of bins | $n$ | Number of distribution centers |

**Theorem 5.4.1.** BP-D $\prec$ 1-LIP-D.

*Proof.* Some inputs of BP-D can be associated to inputs of LIP-D as indicated in Table 5.1, the missing inputs can be set to a constant value or to a parameter from BP-D as shown in Table 5.2. Such assignments produce the case 1-LIP-D, which consists in having known demands, without uncertainty, and having also a single plant of unlimited capacity of production, i.e., the plant is able to supply any quantity of demand. Note that the distribution centers have the same capacity, by other hand, there is now, one type of allocation (retailers to distribution centers) and, since the variance of the demand is null, there is no need to keep inventory, so we have that:

- the reorder point (from Equation 4.7) and minimum probable demand (from Equation 4.12) are equal to zero, that is:

$$\forall j: \; R_j = M_j = 0, \tag{5.1}$$

- the order quantity (from Equation 4.3) depends only on the demand in the distribution center,

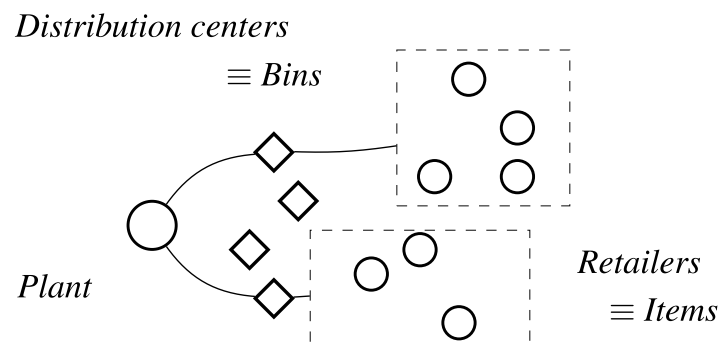
$$\forall j: \; Q_j = \sqrt{D_j} = \sqrt{\sum_k \mu_k Y_{kj}}, \tag{5.2}$$

where $Y_{kj}$ is a binary variable that specifies whether the retailer $k$ is assigned to the distribution center $j$. □

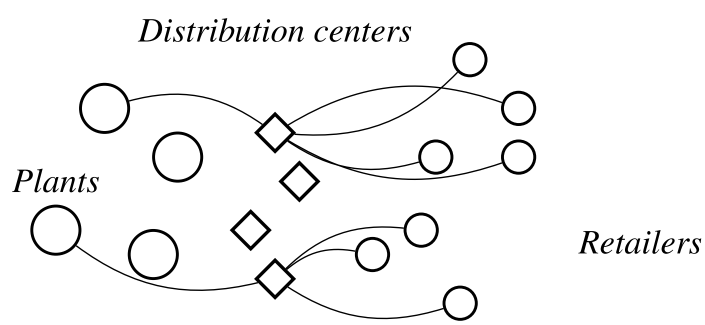
An instance of the 1-LIP-D case is shown in Figure 5.2(a): two distribution centers are open, each retailer is assigned to one of them and each open distribution center is supplied by the unique plant. In Figure 5.2(b), an instance for the general case is shown, considering several plants, which is more complex.

Table 5.2: Assignment of the remaining parameters

| Inputs that do not depend on the instance | | Inputs that depend on the instance | |
|---|---|---|---|
| $\forall i,j$ | $g_{ij}=0, a_{ij}=0, \ell_{ij}=0$ | $\forall j$ | $q_j=\sqrt{c}$ |
| $\forall j,k$ | $c_{jk}=0$ | $\forall i$ | $b_i \geq \sum_{u\in U} s_u$ |
| $\forall j$ | $u_j=0, f_j=0$ | | |
| $\forall k$ | $\sigma_k=0$ | | |
| | $\alpha=0, z_\alpha=0, \beta=0, \theta=2, h=1$ | | |




**(a) *The simplest case:*** $p=1$




**(b) *General case:*** $p \geq 1$

Figure 5.2: Two instances of LIP and their corresponding solutions. The difference between them is the number of plants and the assignment of retailers. In Figure 5.2(a) the allocation of the retailers is made by groups: each distribution center represents a bin and each retailer represents an item of the Bin Packing Problem.

### 5.4.2 SOLUTION EQUIVALENCE

To prove the equivalence between the solution problems, it should be shown that both have the same answer with the same input. We introduce a BP-D input in the reduction to be transformed into a 1-LIP-D input. Then, the BP-D instance is solved through an algorithm for 1-LIP-D (Figure 5.1). We get the answer "yes" (or "no") if and only if, the answer of the related instance of problem BP-D is "yes" (or "no"), i.e., we must show that the set of items can be divided in $w$ subsets or less if and only if there is a feasible solution to 1-LIP-D problem with $n$ or less distribution centers open:

- "yes" answer in BP-D $\Rightarrow$ "yes" answer in 1-LIP-D: the feasibility in BP-D consists in assigning the items while respecting the bin capacity, that is,

$$\sum_{u} s_u V_u \leq c, \tag{5.3}$$

  where $V_u$ is a binary variable that specifies whether the item is assigned to the bin. Meanwhile, from Equations (4.13) and (5.2), the feasibility in each distribution center consists in the allocation of all the retailers without exceeding the storage capacity. Therefore, in 1-LIP-D,

$$\sqrt{\sum_{k} \mu_k Y_{kj}} \leq q_j = \sqrt{c}, \tag{5.4}$$

  which is equivalent to Equation (5.3). So, if there is a nondeterministic algorithm that decides "yes" the input of the BP-D then there is a partition of items that does no exceed the bin capacity, which means that there is a partition of retailers that does not exceed the capacity of the distribution centers in LIP-D,

- "no" answer in BP-D $\Rightarrow$ "no" answer in 1-LIP-D: if an instance of BP-D has answer "no", that means that all possible partitions of items are unable to respect the capacity of the bin, which implies that all possible assignments of retailers exceed the storage capacity of at least one of the open distribution centers.

### 5.4.3 EFFICIENCY

An important aspect of a reduction is its efficiency, meaning that the resources used are limited in such a manner that they do not absorb the complexity of the problem and affect the interpretation of the demonstration (Papadimitriou, 1994); specifically, we limit the amount of memory used by the algorithm to transformed one input $(x)$ of BP-D into an equivalent input $F(x)$ to problem 1-LIP-D. In our case, the reduction requires the number of items $(|U|)$, the number of bins $(w)$, the bin capacity $(c)$ and the list of items size $(s_u)$ and generates the inputs expressed in Table 5.2 as indicated in Algorithm 1. Note that for each bin a distribution center is defined (lines 9–18) and likewise for each item a retailer is defined (lines 19–22); no additional memory is needed to accomplish this when reading the input. The transformation computes the storage capacity $(\sqrt{c})$ in $\mathscr{O}(1)$ time (and outputs it $n = w$ times, once per each distribution center). It also computes the production capacity $(\sum_{k \in K} \mu_k)$ for the plant, iterating over the set of items $(m = |U| = |K|$, as the items are represented as retailers), which is carried out in $\mathscr{O}(m)$ time with a single scalar variable and is, therefore, efficient.

### 5.4.4 GENERALIZATIONS (CASES M-LIP-D & S-LIP-D)

The reduction was carried out considering the simplest case of the LIP, that is, the deterministic version of the problem with just one plant (1-LIP-D). The presented proof establishes that also variants of LIP with more than one plant and/or non-zero variances of retailer demand are NP-complete: firstly, it can be established that such variants belong to NP-class with polynomially verifiable certificates (applying the same certificate shown in Section 5.3) and secondly, the simplest case of LIP used in the present proof can be trivially and efficiently reduced to such variants.

**Theorem 5.4.2.** 1-LIP-D $\prec$ M-LIP-D.

*Proof.* Both problems have the same parameters, so there is no need to transform or define

**Algorithm 1** Reduction from BP-D to LIP-D

**Input:** $|U|, s_u, c, w$
**Output:** $m, n, p, \alpha, z_\alpha, \beta, h, u_k, g_{1j}, a_{1j}, \ell_{1j}, c_{jk}, \mu_k, \sigma_k, b_1, q_j$
1: $m \leftarrow |U|$ ▷ Fix the cardinality of the sets $(I,J,K)$
2: $n \leftarrow w$
3: $p \leftarrow 1$
4: $\beta \leftarrow 0$ ▷ Define inventory parameters
5: $\alpha \leftarrow 0$ ▷ The safety stock is no required
6: $z_\alpha \leftarrow 0$
7: $\theta \leftarrow 2$
8: $h \leftarrow 1$
9: **for** $x \leftarrow 1$ to $n$ **do** ▷ Define the distribution centers parameters
10: $u_x \leftarrow 0$
11: $f_x \leftarrow 0$
12: $g_{1x} \leftarrow 0$
13: $a_{1x} \leftarrow 0$ ▷ There is a unique plant
14: $\ell_{1x} \leftarrow 0$
15: **for** $k \leftarrow 1$ to $w$ **do**
16: $c_{xk} \leftarrow 0$
17: **end for**
18: **end for**
19: **for all** $u \in U$ **do** ▷ Define the retailers parameters
20: $\mu_u \leftarrow s_u$
21: $\sigma_u \leftarrow 0$ ▷ Deterministic version
22: **end for**
23: $b_1 \leftarrow 0$
24: **for all** $k \in K$ **do** ▷ Assert feasibility of the plant capacity
25: $b_1 \leftarrow b_1 + \mu_k$
26: **end for**
27: **for** $x \leftarrow 1$ to $n$ **do** ▷ Assert feasibility in distribution centers
28: $q_x \leftarrow \sqrt{c}$ ▷ Distribution centers have the same capacity
29: **end for**

the inputs, except the production capacity which is set as the value of at least the sum of the retailers demand. This step is just a sum and an assignment which is an efficient step. Then LIP-D with multiple plants can solve a version with an unique plant, which is able to fully supply the retailers set, this is:

$$b_i \geq \sum_{k \in K} \mu_k. \tag{5.5}$$

□

**Theorem 5.4.3.** M-LIP-D $\prec$ S-LIP-D.

*Proof.* These problems also have the same parameters, no transformation is required. The stochastic version is perfectly able to solve a deterministic version, just set the value of demand variance to zero, which is already done in previous reductions (see Algorithm 1 in line 21). This step is also efficient. □

CHAPTER 6

# APPROXIMATIONS

In this chapter, we work with the MINLP2 model expressed in Equations(4.30–4.44). Since it involves nonlinear terms and the used solvers did not report good solutions, we decided to replace these terms. We did it in two ways: by a secant approximation and by a piecewise-linear curve. These two approximations will be evaluated in order to determine which one performs better.

## 6.1 APPROXIMATION THROUGH A SECANT

We substitute the Equations (4.41) and (4.42) for an approximation made by the secant, resulting on the next constraints:

$$\hat{D}_{ij} \geq \frac{D_{ij}}{\sqrt{D}}, \tag{6.1}$$

$$\hat{S}_{ij} \geq \frac{S_{ij}}{\sqrt{S}}. \tag{6.2}$$

This new model, denoted by MINLP-R, is a relaxation of MINLP2, which means that the problem is easier than the original one. It optimizes on a larger feasible region, allowing more candidates to be the optimum. The optimum value of the relaxed problem, that is the best over the expanded feasible region, must then equal or improve the optimum value of the original model. Being MINLP2 a minimization problem, the relaxation provides the same or an smaller value for all feasible solutions to MINLP2. Formally:

**Definition** Relaxation [Wolsey (1998)]: A problem $(RP)$ $z^R = \min\{f(x) : x \in T \subseteq R^n\}$ is a *relaxation* of $(IP)$ $z = \min\{c(x) : x \in X \subset R^n\}$ if :

- $X \subseteq T$, and
- $f(x) \leq c(x)$ for all $x \in X$.

To show that MINLP-R is a relaxation of MINLP2, observe that the objective function remains unchanged, so the second condition is fulfilled. Regarding to the first condition, we need to demonstrate that the feasible set of MINLP2 (denoted as $Fo$) is contained inside the feasible set of MINLP-R (denoted by $Fr$. Considering that only two constraints have been modified, we simply need to demonstrate that the original constraints can be transformed into the new ones,that is a solution that satisfies these two original constraints also satisfied the modified constraints.

**Theorem 6.1.1.** $Fo \subseteq Fr$

*Proof.* Suppose that $x$ is a solution of MINLP2, $x \in Fo$ and it satisfies the Equation (4.41). Let us prove that $x$ also satisfies the constraint that replaces it (Equation 6.1).

$$\forall i,j: \qquad \hat{D}_{ij} = \sqrt{D_{ij}} \qquad \Rightarrow \qquad \hat{D}_{ij} \geq \frac{D_{ij}}{\sqrt{D}}$$

$$\text{We know that} \qquad D_{ij} \leq D, \tag{6.3}$$

$$\text{applying the square root in both sides} \qquad \sqrt{D_{ij}} \leq \sqrt{D}, \tag{6.4}$$

$$\text{when } D_{ij} = 0 \qquad \sqrt{D_{ij}} = 0, \tag{6.5}$$

$$\text{when } D_{ij} > 0$$

$$\text{we can divide by } D_{ij} \qquad \frac{\sqrt{D_{ij}}}{D_{ij}} \leq \frac{\sqrt{D}}{D_{ij}}, \tag{6.6}$$

$$\text{and simplifying and rearranging terms} \qquad \sqrt{D_{ij}} \geq \frac{D_{ij}}{\sqrt{D}}. \tag{6.7}$$

In a similar way, we proceed to demonstrate that every solution that satisfies Equation (4.42) also satisfies Equation (6.2).

$$\forall i,j: \qquad \hat{S}_{ij} = \sqrt{S_{ij}} \qquad \Rightarrow \qquad \hat{S}_{ij} \geq \frac{S_{ij}}{\sqrt{S}}$$

$$\text{As} \qquad S_{ij} \leq S, \tag{6.8}$$

$$\text{applying the square root in both sides} \qquad \sqrt{S_{ij}} \leq \sqrt{S}, \tag{6.9}$$

$$\text{when } S_{ij} = 0 \qquad \sqrt{S_{ij}} = 0, \tag{6.10}$$

$$\text{when } S_{ij} > 0$$

$$\text{we can divide by } S_{ij} \qquad \frac{\sqrt{S_{ij}}}{S_{ij}} \leq \frac{\sqrt{S}}{S_{ij}}, \tag{6.11}$$

$$\text{and simplifying and rearranging terms} \qquad \sqrt{S_{ij}} \geq \frac{S_{ij}}{\sqrt{S}}, \tag{6.12}$$

$$\therefore\ x \in Fo \Rightarrow x \in Fr \qquad \wedge \qquad Fo \subseteq Fr.$$

□

As a relaxation, MINLP-R provides a lower bound for MINLP2. Moreover, if its optimal solution is contained in the feasible region of MINLP2, this solution provides an upper bound. To be feasible to MINLP2, the solution must satisfy the constraint expressed by Equation 4.40, but evaluating the original function of mean and variance of the demand, Equations (4.41) and (4.42), in the next way:

$$\tau\sqrt{\sum_{i\in I}\rho_{ij}\ddot{D}_{ij}} + z_e\sqrt{\sum_{i\in I}\ell_{ij}\ddot{S}_{ij}} \leq q_j X_j \quad \forall j \in J, \tag{6.13}$$

where $\ddot{D}_{ij}$ and $\ddot{S}_{ij}$ are the mean and variance, respectively, of the served demand by the distribution center $j$ and supplied by the plant $i$, according the optimal solution in MINLP-R model.

## 6.2 PIECEWISE APPROXIMATION

In the second approximation developed , instead of using secants in the square root functions, we now use a piecewise linear approximation for solving the model (4.30–4.44) by adding proper variables and constraints. The approximation is made over the same interval than the original functions and consists of a sequence of linear segments. It is known that the larger the number of linear pieces is used, the better approximation is obtained, but also it is increased the computational performance impact, since greater CPU time is needed to solve the transformed problem.

The basic idea is to subdivide the interval where we want to approximate the nonlinear function by introducing vertices, that we call break points and to determine the value in the original function at each of these vertices, then to connect them by lines to obtain a piecewise linear function.

Several formulations for the piecewise linearization have been proposed in the literature. The most common are the incremental cost, the convex combination by SOS1 and by SOS2 (Tomlin, 1988). SOS1, or a special ordered set of type 1, is a set of variables where no more than one set member may be non-zero, and positive in the feasible solution. In SOS2, or special ordered sets of type 2, at most two can be nonzero, and if two are nonzero, they must be consecutive in their ordering. Here, the used formulation follows the model proposed by Bazaraa et al. (1993).

### 6.2.1 PROPOSED MODEL

We denote the break points of the function $\hat{D}_{ij}$ by $\rho_b^D$ and the break points of the function $\hat{S}_{ij}$ by $\rho_b^S$ with $b = 0, 1, \ldots, \kappa$. Then, $\forall i \in I, j \in J$, the function $\hat{D}_{ij}$ can be approximately linearized over the interval $\left[\rho_0^D, \rho_\kappa^D\right]$ as:

$$L\left(\hat{D}_{ij}\right) = \sum_{b=1}^{\kappa} \sqrt{\rho_b^D} \lambda_{bij}^D \tag{6.14}$$

$$D_{ij} = \sum_{b=1}^{\kappa} \rho_b^D \lambda_{bij}^D \tag{6.15}$$

$$\lambda_{0ij}^D \leq F_{0ij}^D \tag{6.16}$$

$$\lambda_{bij}^D \leq F_{b-1,ij}^D + F_{bij}^D \qquad (b = 1, \cdots, \kappa - 1) \tag{6.17}$$

$$\lambda_{\kappa ij}^D \leq F_{\kappa-1,ij}^D \tag{6.18}$$

$$\sum_{b=0}^{\kappa-1} F_{bij}^D = 1 \tag{6.19}$$

$$\sum_{b=0}^{\kappa} \lambda_{bij}^D = 1 \tag{6.20}$$

$$F_{bij}^D \in \{0,1\}, \lambda_{bij}^D \geq 0 \tag{6.21}$$

Likewise, $\forall i \in I, j \in J$, the approximation for $\hat{S}_{ij}$ in the interval $\left[\rho_0^S, \rho_\kappa^S\right]$ is established as:

$$L\left(\hat{S}_{ij}\right) = \sum_{b=1}^{\kappa} \sqrt{\rho_b^S} \lambda_{bij}^S \tag{6.22}$$

$$S_{ij} = \sum_{b=1}^{\kappa} \rho_b^S \lambda_{bij}^S \tag{6.23}$$

$$\lambda_{0ij}^S \leq F_{0ij}^S \tag{6.24}$$

$$\lambda_{bij}^S \leq F_{b-1,ij}^S + F_{bij}^S \qquad (b = 1, \ldots, \kappa - 1) \tag{6.25}$$

$$\lambda_{\kappa ij}^S \leq F_{\kappa-1,ij}^S \tag{6.26}$$

$$\sum_{b=0}^{\kappa-1} F_{bij}^S = 1 \tag{6.27}$$

$$\sum_{b=0}^{\kappa} \lambda_{bij}^S = 1 \tag{6.28}$$

$$F_{bij}^S \in \{0,1\}, \lambda_{bij}^S \geq 0 \tag{6.29}$$

$$\sum_{j\in J} u_j X_j + \sum_{j\in J}\sum_{k\in K} \mu_k \acute{c}_{jk} Y_{jk} + \sum_{j\in J}\sum_{i\in I} \acute{a}_{ij} D_{ij} + \sum_{i\in I}\sum_{j\in J}\sum_{b\in B} \eta h_{ij} \sqrt{\rho_b^D} \lambda_{bij}^D + \cdots$$
$$\cdots + \sum_{i\in I}\sum_{j\in J}\sum_{b\in B} \phi t_{ij} \sqrt{\rho_b^S} \lambda_{bij}^S \qquad (6.30)$$

$$\tau \sum_{i\in I}\sum_{b\in B} h_{ij} \sqrt{\rho_b^D} \lambda_{bij}^D + z_e \sum_{i\in I}\sum_{b\in B} t_{ij} \sqrt{\rho_b^S} \lambda_{bij}^S \leq q_j X_j \forall j \in J \qquad (6.31)$$

This model introduces $\kappa$ extra binary variables, $\kappa+1$ continuous variables and $\kappa+5$ constraints in each approximation, i.e $D_{ij}$ and $S_{ij}$. Lin et al. (2013) show the comparatives of others proposals for piecewise approximation, with fewer variables and constraints. However, for mixed integer models, fewer constraints do not imply necessarily better solutions, in much cases, is the opposite.

CHAPTER 7

# COLUMN GENERATION

Preliminary computational experiments showed that solving directly formulations MINLP and MILP by using a commercial optimizer is not viable, for the incapability of getting the optimum value and solving large instances.

Other approach for solving MILP may be to enumerate all possible combinations of feasible groups of retailers and assign them to a plant and to a distribution center. This gives a finite but very large number of groups. Just considering the retailers, the largest number of combinations is equal to the power set of retailers, namely $2^m$. Even if these assignments are limited by physical constraints such as production capacity or storage capacity, the number of possibilities is impractical.

However, the performed experiments allowed us to know about the structure of the problem and its behavior, so we decided to keep formulation MILP and apply decomposition techniques, in order to solve larger instances.

Since most of the variables will be non-basic and take a value of zero in the optimal solution, we decided to use a column generation approach, where the appealing idea is to work only with a sufficiently meaningful subset of variables (Desrosiers and Lübbecke, 2005). Each variable is called *column*, and it is used if it has potential to improve the objective function. In the next section we expose how the column generation is applied to our problem.

## 7.1 MODELS

### 7.1.1 MASTER PROBLEM

The master problem selects sets of retailers and assigns them to a plant and to a distribution center to minimize the system cost, taking into account the production capacity, the storage capacity and the service to all the retailers. The model is the same as the one described through Equations (4.47–4.51), the only difference lies in the fact that here not all possible variables are used, reason for calling it *Restricted Master Problem*.

### 7.1.2 PRICING PROBLEM

The pricing problem consists in finding a column (variable) with a negative reduced cost or to prove that no such column exists. In our problem, a column is a variable that represents a group formed by retailers, a plant and a distribution center with negative reduce cost. Therefore, if a column with negative reduced cost exists the pricing problem will always identify it (Barnhart et al., 1998).

We will denote by $\pi_k$, $\sigma_i$ and $\gamma_{ij}$ the dual values corresponding to constraints (4.48–4.50) respectively. Variables used for defining the assignments in formulation MINLP are used again, they are variables $Z_{ij}$, which take the value of one if distribution center located at $j$ is served by plant $i$, and variables $Y_{jk}$ that take the value of one if retailer $k$ is served by distribution center $j$.

We will also need the auxiliary variables:

$D_{ij}$ : Covered demand at distribution center $j$ and supplied from plant $i$.

$W_{ij}$ : Cost for ordering, holding inventory and transporting product regarding plant $i$, distribution center $j$ and subset of retailers $b$.

$S_{ij}$ : Used capacity at distribution center $j$ when plant $i$ supplies it.

Notice that, variables $W_{ij}$ and $S_{ij}$ have the same concept of $\omega_{bij}$ (Equation 4.45) and $s_{bij}$ (Equation 4.46) but now they are variables and their values are not explicitly calculated for all possible combination of elements, they are obtained after optimizing the model. Relating to parameters, there is the upper bound ($d^u_{ij}$) on the total amount of product shipped from plant $i$ to distribution center $j$ (See Equation 4.1) and an upper bound ($w^u_{ij}$) on the total weighted cost of product shipped from plant $i$ to distribution center $j$ equal to:

$$w^u_{ij} = \max_k\{\acute{c}_{jk}\}D + \acute{a}_{ij}D + \phi\sqrt{\sum_k \upsilon_k l_{ij}} + \eta\sqrt{\rho_{ij}D} \tag{7.1}$$

The pricing problem is the next:

$$\min \quad \sum_i\sum_j W_{ij} - \sum_k\sum_j \pi_k Y_{kj} - \sum_i\sum_j \sigma_i D_{ij} - \sum_i\sum_j \gamma_j S_{ij} \tag{7.2}$$

s.t:

$$W_{ij} = \sum_k \mu_k \acute{c}_{jk} Y_{kj} + \acute{a}_{ij} D_{ij} + \phi\sqrt{\sum_k \upsilon_k l_{ij} Y_{kj}} + \eta\sqrt{\rho_{ij} D_{ij}} \qquad \forall i \in I,\ j \in J \tag{7.3}$$

$$S_{ij} = \tau\sqrt{\rho_{ij} D_{ij}} + z_e\sqrt{\sum_k \upsilon_k l_{ij} Y_{kj}} \qquad \forall i \in I,\ j \in J \tag{7.4}$$

$$W_{ij} \le w^u_{ij} Z_{ij} \qquad \forall i \in I,\ j \in J \tag{7.5}$$

$$S_{ij} \le d^u_{ij} Z_{ij} \qquad \forall i \in I,\ j \in J \tag{7.6}$$

$$\sum_i\sum_j Z_{ij} = 1 \tag{7.7}$$

$$\sum_i D_{ij} \ge \sum_k \mu_k Y_{kj} \qquad \forall j \in J \tag{7.8}$$

$$D_{ij} \le D Z_{ij} \qquad \forall i \in I,\ j \in J \tag{7.9}$$

$$\sum_j Y_{kj} = 1 \qquad \forall k \in K \tag{7.10}$$

$$W_{ij}, S_{ij}, D_{ij} \ge 0 \qquad \forall i \in I,\ j \in J \tag{7.11}$$

$$Z_{ij}, Y_{kj} \in \{0,1\} \qquad \forall i \in I,\ j \in J \tag{7.12}$$

Equation (7.7) is the convexity constraint that asserts only one assignment, the remaining constraints define or limit the variables.

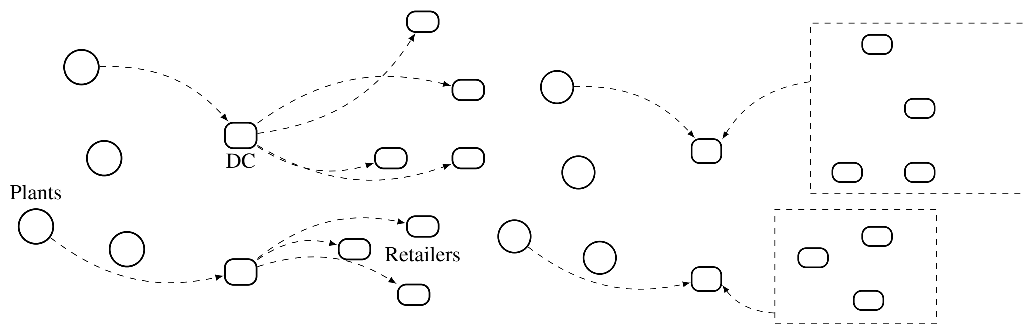
There is a drawback in this formulation. We again have non-linearities in the functions, so the optimal solution is not guaranteed. For this reason, we decided to approximate the nonlinear functions obtaining the model expressed in Section 7.2.2.

## 7.2 SOLUTION PROCESS

In this section, we explain the four basic steps of the proposed column generation method. In the first step, initial columns are required for solving the master problem, which is the second step. Then, we solve the pricing problem and we get new columns to add to the master problem and so on, until we prove that the optimal solution has been reached. Finally, the master problem is again optimized, but with the integer constraint, since usually the last master problem does not satisfy the integrality conditions. These steps are explained next in more detail.

### 7.2.1 INITIAL COLUMNS

For creating the initial columns, we propose to change the idea of allocating retailers by the idea of assigning groups of retailers to a distribution center. Figure 7.1(a) and Figure 7.1(b) illustrates both ways for allocating.




(a) ***Allocation of retailers*** (b) ***Allocation of group of retailers***

Figure 7.1: Different approaches to allocation

The generation of initial columns basically consists of creating groups of retailers of different sizes, assess the amount of product demanded by each group and assign them to a plant, respecting the capacity constraints.

### 7.2.2 ADDING COLUMNS

#### 7.2.2.0 APPROXIMATE METHOD

Since the pricing model is non linear, we decided to approximate the square root terms:

$$V_{ij} = \sqrt{\sum_k \upsilon_k l_{ij} Y_{kj}} \tag{7.13}$$

$$P_{ij} = \sqrt{\rho_{ij} D_{ij}} \tag{7.14}$$

$$\min \quad \sum_i \sum_j W_{ij} - \sum_k \sum_j \pi_k Y_{kj} - \sum_i \sum_j \sigma_i D_{ij} - \sum_i \sum_j \gamma_j S_{ij} \tag{OF}$$

s.t:

$$W_{ij} = \sum_k \mu_k \acute{c}_{jk} Y_{kj} + \acute{a}_{ij} D_{ij} + \phi V_{ij} + \eta P_{ij} \qquad \forall i \in I,\ j \in J \tag{C1}$$

$$S_{ij} = \tau P_{ij} + z_e V_{ij} \qquad \forall i \in I,\ j \in J \tag{C2}$$

$$W_{ij} \leq w^u_{ij} Z_{ij} \qquad \forall i \in I,\ j \in J \tag{C3}$$

$$S_{ij} \leq d^u_{ij} Z_{ij} \qquad \forall i \in I,\ j \in J \tag{C4}$$

$$\sum_i D_{ij} \geq \sum_k \mu_k Y_{kj} \qquad \forall j \in J \tag{C5}$$

$$V_{ij} - m^v \hat{\ell}_{ij} \sum_k \upsilon_k Y_{kj} = \hat{\ell}_{ij} \left( \sqrt{\sum_k \upsilon_k} - m^v \sum_k \upsilon_k \right) \qquad \forall i \in I,\ j \in J \tag{C6}$$

$$P_{ij} - m^P \hat{\rho}_{ij} D_{ij} = \hat{\rho}_{ij} \left( \sqrt{D} - m^P D \right) \qquad \forall i \in I,\ j \in J \tag{C7}$$

$$D_{ij} \leq D Z_{ij} \qquad \forall i \in I,\ j \in J \tag{C8}$$

$$\sum_i \sum_j Z_{ij} = 1 \tag{C9}$$

$$\sum_j Y_{kj} = 1 \qquad \forall k \in K \tag{C10}$$

$$W_{ij}, S_{ij}, D_{ij}, V_{ij}, P_{ij} \geq 0 \qquad \forall i \in I,\ j \in J \tag{C11}$$

$$Z_{ij}, Y_{kj} \in \{0, 1\} \qquad \forall i \in I,\ j \in J \tag{C12}$$

#### 7.2.2.0 EXACT METHOD

It is not necessary to select the column with the most negative reduced cost. Any column with negative reduced cost will improve the solution quality (Barnhart et al., 1998). Knowing that, we can improve the efficiency of our implementation when the pricing problem must be a intensively computed.

### 7.2.3 INTEGER MASTER PROGRAM

Sometimes, when no column prices out for entering to the base in the master program, the actual solution does not satisfy the integrality conditions. Branch and Price, which is a generation of branch-and-bound with LP relaxation, allows column generation.

CHAPTER 8

# COMPUTATIONAL EXPERIMENTS

In this chapter we will describe the computational experiments which have been designed for evaluation the performance of our proposed solutions.

## 8.1 ASSERTING CAPACITY CONSTRAINT

As we say before, the demand is attended with a certain service level as consequence of uncertainty parameters. Once the demand is present the actual service level and the fulfill of capacity may differ from the estimates, in order to verify this, a simulation was executed. Instances with different values of parameters, that affect the computational complexity, were generated. Once the instances were solved, a demand ($\mu_s$) was simulated for each one, then the optimal configuration is tested of be feasible, that means two things:

- verify if the capacity limitation is fulfill in the distribution center open. The used capacity ($C_j$) in the distribution center is calculated by:

$$C_j = Q + R - D_j^s, \tag{8.1}$$

where $D_j^s$ := Demand simulated during lead time in distribution center $j$.

$$D_j^s = \sum_k \sum_i \mu_s \ell_{ij} Y_{jk} Z_{ij}, \tag{8.2}$$

- compare a simulated demand during lead time versus the inventory in distribution center $j$, which is supplied by plant $i$. This inventory has to be large enough to keep the service level, i.e. the instance solution must meet the condition:

$$D_j^s \leq z_\alpha \sum_k \sum_i \sigma_k \hat{\ell}_{ij} Y_{kj} Z_{ij} + \acute{\mu}_j, \quad \forall i, j \tag{8.3}$$

Notice that Equation (8.1) and Equation (4.8) are the same, but the value of demand is changed for the simulated one. This value is generated randomly with a uniform distribution in the same range in which the mean of dairy demand in the instances was originally generated and in a larger range. The parameters of the instances are:

Table 8.1: Factors considered in the instances evaluated.

| Factor | Levels |
|---|---|
| Service level ($\alpha$) | 75%, 98% |
| Probability of fulfill capacity ($\gamma$) | 80%, 95% |
| Number of plants ($p$) | 4, 6 |
| Number of distribution centers ($n$) | 6, 8 |
| Number of retailers ($m$) | 10, 13 |
| Lead time [days] | [1–8], [1–24] |

We get 64 groups of combinations with 10 different instances of each group. The demand was simulated, in the two ranges of demand, 1000 times for each instance in a program implemented in PYTHON[1] v.3.3.3. The instances are classified in four types, as in Table 8.2.

Table 8.2: Classification of the instances in the simulations.

| Instances | LT [days] | $\mu_s$ [units] |
|---|---|---|
| A | [1–8] | [5,60] |
| B | [1–24] | [5,60] |
| C | [1–8] | [10,155] |
| D | [1–24] | [10,155] |

The instances were modeled and optimized through GAMS[2] v.24.2.3 and CPLEX[3] v.12.6. The results are shown in the Figure 8.1. The bar chart shows the percentage

[1] https://www.python.org/
[2] http://gams.com//
[3] http://www-01.ibm.com/software/commerce/optimization/cplex-optimizer/

of feasible distribution center, i.e. the total ration of distribution centers to which the inventory assigned has not exceed the capacity. A feasible instance is the one with any exceed distribution center. The opposite case, an infeasible instance is, for example, when certain instance got a optimal solution and let say, four distribution centers were selected, this solution is feasible and optimal theoretically, but when the simulation is running, one distribution center is insufficient for keep the inventory, in that case, even if the other distribution center are capable to manage the inventory all the time (2000 scenarios), then the instance is consider infeasible. Of course, the percentage is less than the percentage of feasible distribution center. For last, the ratio of meet demand in all cases is of 100 %. We can see that the safety stock is enough to meet the demand, even with a low service level (75%). That indicates, that it is possible to change the probability of service level ($\alpha$) and the probability of fulfill the capacity ($\gamma$) and decrease the inventory, which also means to decrease the cost.

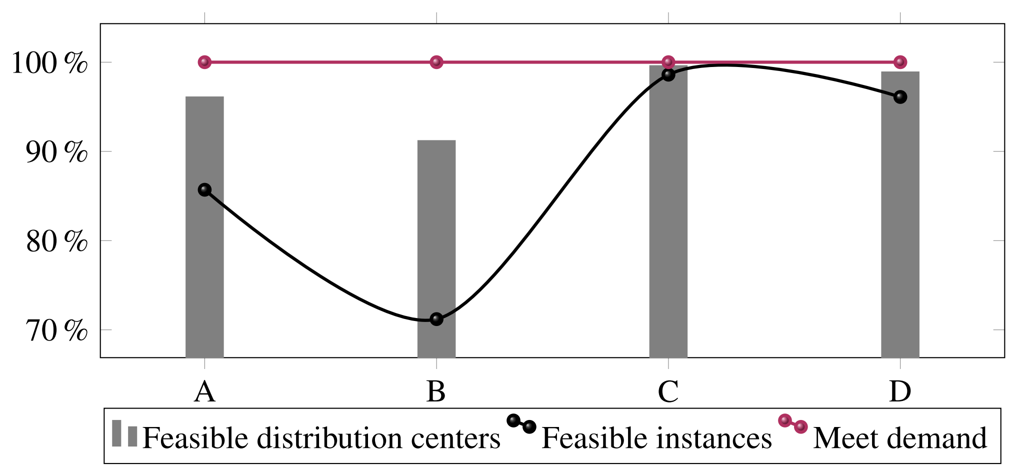



Figure 8.1: Simulation results

In the Table 8.3, the ratio of infeasible scenarios is shown according the values of the estimate probability of fulfill the capacity and the desirable service level. The results oscillate just a little, although, a bigger value of service level implies a bigger ratio of infeasibility, but it is not significant.

Table 8.3: Service level vs fulfill capacity

| Probability of fulfill capacity ($\gamma$) | Service level ($\alpha$) | Infeasible scenarios |
|---|---|---|
| 80% | 75% | 14.40% |
| 95% | 75% | 13.48% |
| 80% | 98% | 15.46% |
| 95% | 98% | 13.97% |

Summarizing,the capacity constraint and the inventory has been modeled correctly and the inventory-management goal is reached.

## 8.2 COLUMN GENERATION - INITIAL COLUMNS

This was programmed in PYTHON [4] 2.7, the program uses classes for defining the plants, the distribution centers and the retailers. Lists and dictionaries are also used. The complete source code, the data processing and also, some examples of program execution can be found in `https://github.com/NellyMonserrat/heuristico.git`.

The performance of the program is evaluated according to the columns generation time, this was achieved through an experiment developed in a laptop computer with Intel Core i5 CPU @ 2.3 GHz processor, 6 GB of RAM following the listed parameters in Table 8.4. It shows the lower level, upper level and the size variation of each parameter.

Table 8.4: Variation of parameters

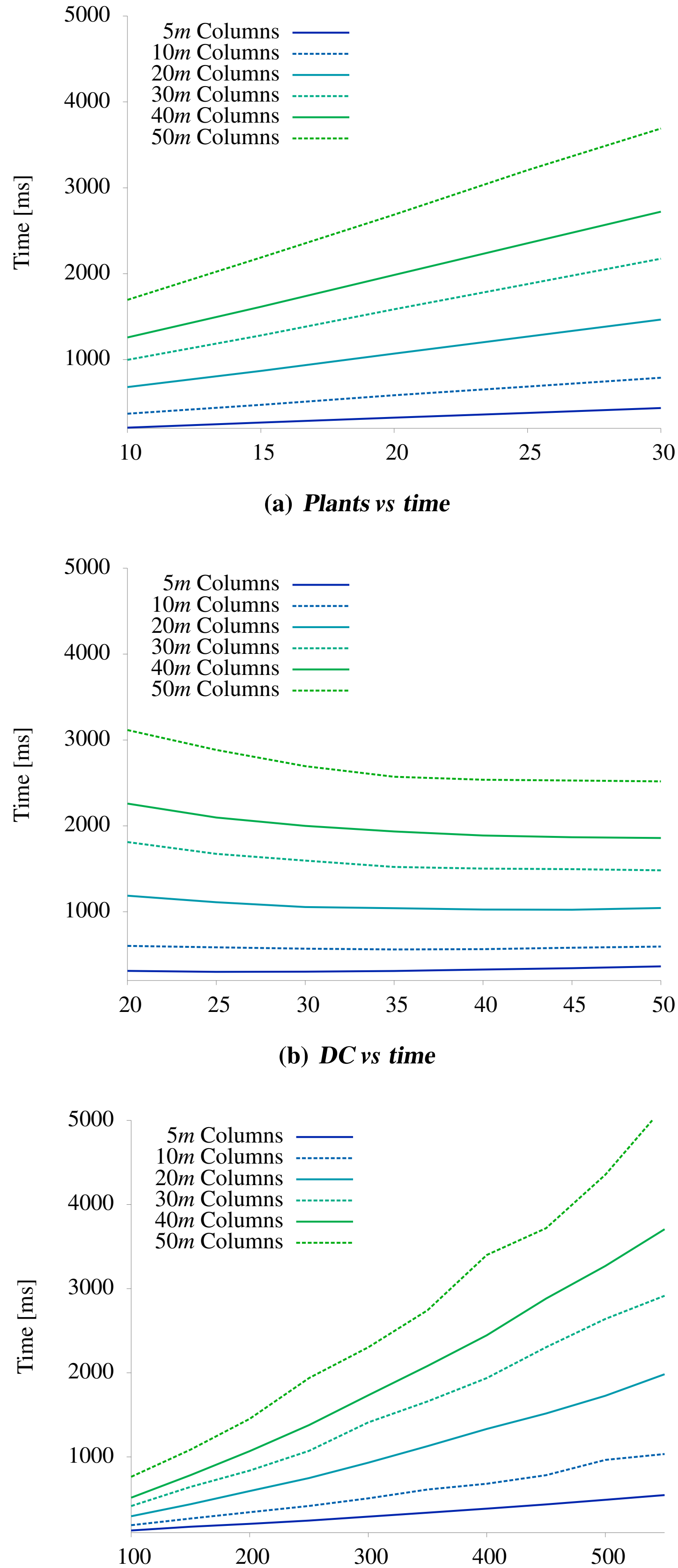
| Parameters | Levels | Size variation |
|---|---|---|
| Number of plants $(p)$ | 10 – 30 | 10 |
| Number of distribution centers $(n)$ | 20 – 50 | 5 |
| Number of retailers $(m)$ | 100 – 550 | 50 |
| Desirables columns | $10m$ – $50m$ | 10 |

Each possible combination was repeated 10 times. The experimental results can be observed in the Figure 8.2. Horizontal axis shows the number of elements in the instances, plants (Figure 8.2(a)), distribution centers (Figure 8.2(b)) and retailers (Figure

[4] `https://www.python.org/`

8.2(c)). The vertical axis represents the number of milliseconds taken to create six different number of columns, that increased according the number of total retailers ($m$).

Both in Figure 8.2(a) and in Figure 8.2(c), for all series, the mean of the required time increases progressively according to the number of the plants and the distribution centers, a different performance is shown in the Figure 8.2(b). When the number of centers increases, the mean time to generate the columns is lower, the reason is once we have created the group of retailers, the assignment only consists in comparing the amount of product required for each grouping (a calculation already done) with the storage capacity, this operation has the complexity of $\mathscr{O}(1)$. Namely, as increased the number of potential distribution centers, it is easier to generate a column.




**(a)** ***Plants vs time***



**(b)** ***DC vs time***



**(c)** ***Retailers vs time***

Figure 8.2: Generation time of initial columns

CHAPTER 9

# CONCLUSIONS

The accuracy and the computational efficiency of the piecewise linear approximation depends on the selection of break points so, it is value the study of break point selection strategies to get good solution in reasonable time. In our specific case, for the square root function, some ideas to explore are: to select more break points inside the interval with the largest approximation error, increase iteratively the number of points, adding points at the midpoint of each interval of the existing points.

We have proven that the decision problem corresponding to the Location-Inventory Problem (LIP) is NP-complete with a reduction from the Bin Packing Problem, establishing that LIP is at least as difficult as the Bin Packing Problem, which is known to be NP-complete. Therefore, the optimization version of the Location-Inventory Problem is NP-hard. We represent inventory management in terms of the EOQ model, but the particularities of the model do not affect the proof, which indicates that the complexity of the LIP does not depend on the inventory model used.

Knowing that LIP is NP-hard suggests that no exact algorithm can be expected to efficiently solve large instances and the computation times in attempting such a solution may be infeasible long. However, due to the strategic nature of the cost-minimization problem that LIP represents, an optimal solution would be valuable, since the savings between an optimal and a feasible solution may be large.

We demonstrate that the reformulation and approximations are capable of providing optimal solutions without using specialized algorithms. This contribution implies that additional variants of the problem, could be solvable by optimization software.

# NOMENCLATURE

**Sets**

$B$ The collection of nonempty subsets of the retailers, index by $b = 1,2,...,2^m$.

$I$ Set of plants, indexed by $i = 1,2,...p$.

$J$ Set of candidate distribution centers sites, indexed by $j = 1,2,...n$.

$K$ Set of retailers, indexed by $k = 1,2,...m$.

**Costs**

$\beta$ Weight factor of the shipment cost.

$\theta$ Weight factor of the inventory cost.

$a_{ij}$ Fixed cost per shipment from plant $i$ to the distribution center $j$.

$c_{jk}$ Unit shipment cost from the distribution center $j$ to retailer $k$.

$f_j$ Fixed cost for placing an order from the distribution center $j$.

$g_{ij}$ Variable cost per shipment from plant $i$ to the distribution center $j$.

$h$ Annual holding cost per item.

$u_j$ Fixed annual cost for locating the distribution center $j$.

**Supply chain**

$\acute{a}_{ij}$ Weighted annual shipment cost from the plant $i$ to the distribution center $j$ per item.

$\acute{c}_{jk}$ Weighted annual shipment cost from the distribution center $j$ to the retailer $k$ per item.

$\alpha$ Probability of meeting the demand during lead time.

$\ell_{ij}$ Lead time from plant $i$ to distribution center $j$.

$\mu_k$ Mean of the daily demand for retailer $k$.

$\sigma_k$ Variance of the daily demand for retailer $k$.

$\rho_{ij}$ Cost parameter for sending product from the plant $i$ to the distribution center $j$.

$C_j$ Used capacity in the distribution center $j$.

$D$ Total mean dairy demand of all set of retailers.

$d_{ij}^u$ Upper bound on the total amount of product sending from the plant $i$ to the distribution center $j$.

$p_i$ Production capacity in plant $i$.

$q_j$ Storage capacity in the distribution center $j$.

$r$ Number of working days in a year.

$z_\alpha$ Value of the standard normal random variable corresponding to cumulative probability of $\alpha$.

**Inventory management**

$Q$ Order quantity, the amount of product to be asked to the plants.

$R$ Reorder point, which is when to place an order.

**Mathematical models**

MILP Mixed Integer Linear Programming

MINLP Mixed Integer Nonlinear Programming

**Modelling**

$\acute{\sigma}$ Variance of demand in distribution center $j$ during lead time.

$M_j$ Minimum probable demand in distribution center $j$ during lead time.

$z_e$ Standard normal distribution value, that accumulates the probability of not incurring in stock out and not exceeding the storage capacity.

$\acute{\mu}_j$ Mean of demand in distribution center $j$ during lead time.

$C_j$ Current demand during lead time.

$\gamma$ Probability of the expected minimum demand in distribution center $j$ during lead time.

$D_a$ Total annual demand in distribution center

$x$ Number of orders per year.

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