UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS

CONNECTEDNESS DIMENSION OF THE SPECTRUM OF A RING AND LOCAL COHOMOLOGY

POR

PEDRO ANGEL RAMÍREZ MORENO

EN OPCIÓN AL GRADO DE MAESTRÍA EN CIENCIAS CON ORIENTACIÓN EN MATEMÁTICAS

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Abstract

The objective of this work is study the properties of connectedness dimension of the spectrum of a ring through the use of commutative algebra tools, mainly local cohomology. We also study how connectedness dimension is related to a special family of graphs whose vertices are minimal prime ideals.

Dedication

To my family.

Acknowledgements

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Contents

Chapter 1

Introduction

Algebraic Geometry gives us a way to connect Geometry and Algebra. We can study geometric properties through algebraic ones and vice versa.

For example, if K is an algebraic closed field, we know there is a dictionary between ideals of $K[x_1, \ldots, x_n]$ and algebraic varieties of the affine space A^n_K; radical ideals correspond to algebraic varieties, prime ideals correspond to irreducible varieties and maximal ideals correspond to singletons.

One of such properties is connectedness. For instance, The spectrum of a local ring is a connected space. This happens since the maximal ideal belongs to all the non empty closed sets of the space. But what about the subspaces of such spectrum?. The connectedness dimension of a ring is an invariant that lets us know a way to measure how connected a space by studying the connectedness of it subspaces.

One of the tools of commutative algebra used to study the connectedness of such spaces is local cohomology. In this thesis we study how local cohomology and the connectednes dimension of a ring are related.

The mains results of chapter 4 are:

Theorem 1.0.1. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 3$. Suppose there exists an $x \in \mathfrak{m}$ such that x is a non zero divisor of A and that (x) is a radical ideal. Let t be an integer such that $t \in [1, d-2]$. Then*

$$\Gamma_t(A/(x)) \text{ is connected} \Rightarrow \Gamma_t(A) \text{ is connected.}$$

As a consequence

$$\mathrm{c}(A) = \mathrm{c}(A/(x)) + 1.$$

Theorem 1.0.2. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring containing a field, of $\dim(A) = d \geq 3$, with separably closed residue field. Suppose there exists $x \in \mathfrak{m}$ such that x is a non zero divisor of A and that (x) is a radical ideal. Let t be an integer such that $t \in [1, d-2]$. Then*

$$\#\Gamma_t(A) = \#\Gamma_t(A/(x)).$$

The main results of chapter 5 are:

Theorem 1.0.3 ([HL90])**.** *Let $S = K[[x_1, \ldots, x_n]]$ be a power series ring over a separably closed field K. Let I be an ideal of S such that $d = \dim(S/I) \geq 2$. Then*

$$H_I^{n-1}(S) = 0 \Leftrightarrow \operatorname{Spec}^0(S/I) \text{ is connected.}$$

Corollary 1.0.4. *Let $S = K[[x_1, \ldots, x_n]]$ be a power series ring over a separably closed field K. Let I be an ideal of S such that $\dim(S/I) \geq 2$. Let $t = \# \operatorname{Spec}^0(S/I)$. Then $H_I^{n-1}(S) \cong E_S(K)^{t-1}$.*

In chapter 2 we review some of the background needed to study both things. We talk about some graphs that we can associate to a ring, the completion of a ring and injective modules.

In chapter 3 we review local cohomology and state some of its properties. We also give equivalent definitions of local cohomology.

In chapter 4 we talk about connectedness dimension of rings and we also define some graphs associated to a ring. Through these graphs we can study some aspects related to the connectedness dimension of a ring. In this chapter we also present new results regarding connectedness dimension of rings modulo certain elements of the ring.

Finally, in chapter 5 we talk about the relation of local cohomology and the graphs previously discussed in chapter 4.

Chapter 2

General Background

In this section we present some of the background knowledge we use during the next chapters. We present definitions for key concepts and also present some propositions. We provide proof for some of the propositions in this section, while others are simply announced.

2.1 Graphs Γ

In a Noetherian ring the amount of minimal primes is finite and non zero. We can construct a graph whose vertices are these minimal primes in the following way.

Definition 2.1.1 ([NnBSW19])**.** Let A be a Noetherian local ring of dimension d and fix an integer t such that $t \in [0, d]$. We define the graph $\Gamma_t(A)$ as a simple graph whose vertices are the minimal primes of A and there is and edge between $\mathfrak{p}$ and $\mathfrak{q}$ distinct minimal primes if and only if $\mathrm{ht}(\mathfrak{p} + \mathfrak{q}) \leq t$.

Notice that if $\Gamma_s(A)$ is a subgraph of $\Gamma_t(A)$ for every s and t such that $0 \leq s \leq t \leq d$. Then given such s and t, if $\Gamma_s(A)$ is connected , then $\Gamma_t(A)$ is connected too. $\Gamma_0(A)$ is connected if and only if A has only one minimal prime. Notice that if A has only one minimal prime, not only $\Gamma_0(A)$ is connected, but also $\Gamma_t(A)$ for every t. $\Gamma_d(A)$ is always connected not matter the amount of minimal primes of A.

2.2 Completion of a Ring

Several results of Chapters 4 and 5 regard complete local rings. To talk about them first we define the I-adic completion of a ring.

Definition 2.2.1. Let A be a ring and let I be an ideal of A. The I-adic completion of A is the ring

$$\varprojlim_{t} A/I^t$$

We denote it by $\hat{A}_I$ or simply $\hat{A}$ if it is clear we are talking about the I-adic completion of A.

There is a morphism ψ going from A to $\hat{A}_I$ which maps elements of A to the constant sequence of residue classes of that element.

We say that a ring A is complete with respect to its I-adic completion if this ψ is an isomorphism.

In the case of a local ring $(A, \mathfrak{m})$ we say A is complete if it is complete with respect to its $\mathfrak{m}$-adic completion.

Consider the natural maps

$$\phi_n : \hat{A}_I \to A/I^n$$

and define the ideals

$$\hat{I}_n := \ker \phi_n$$

The following proposition allows us to go to the quotient ring and keep applying some results of chapters 4 and 5 that were valid in the original ring.

Proposition 2.2.2. *Let A be a Noetherian complete local ring. Let A be an ideal of A. Then A/I is also a Noetheriean complete local ring.*

2.3 Connectedness Dimension

Connectedness dimension is a ring invariant and one of our main objects of study. We define it in the following way:

Definition 2.3.1. Let A be a ring. We define the connectedness dimension of A, and denote it as $\mathrm{c}(A)$, as

$$\mathrm{c}(A) = \min \left\{ \dim \left(\frac{A}{I} \right) \,\middle|\, \mathrm{Spec}(A) - V(I) \text{ is disconnected } \right\}$$

We take the convention that the empty set is disconnected.

We have the following proposition that allows us to check whenever an open subspace of $\operatorname{Spec}(A)$ is disconnected in terms of some ideals of A.

Proposition 2.3.2. *Let A be a Noetherian ring. Let $\mathfrak{a}$ be an ideal of A. Suppose $\operatorname{Spec}(A) - V(\mathfrak{a})$ is not empty. Then $\operatorname{Spec}(A) - V(\mathfrak{a})$ is disconnected if and only if there is $I, J \subseteq \mathfrak{a}$ ideals of A such that:*

1. $\sqrt{I}, \sqrt{J} \subsetneq \sqrt{\mathfrak{a}}$
2. $I \cap J = \sqrt{0}$
3. $\sqrt{I + J} = \sqrt{a}$

Proof. $(\Rightarrow)$ Since $\operatorname{Spec}(A) - V(\mathfrak{a})$ is disconnected and non empty, there are non empty open sets U, V such that they form a partition of $\operatorname{Spec}(A) - V(\mathfrak{a})$. Since $\operatorname{Spec}(A) - V(\mathfrak{a})$ is an open subspace of $\operatorname{Spec}(A)$, it follows that U is an open set of $\operatorname{Spec}(A)$. So $\operatorname{Spec}(A) - U = V(J')$ for some ideal J' of A. This means that $V = V(J') - V(\mathfrak{a})$. Similarly there is an ideal I' of A such that $\operatorname{Spec}(A) - V = V(I')$ and $U = V(I') - V(\mathfrak{a})$. Let $I = I' \cap \mathfrak{a}$ and $J = J' \cap \mathfrak{a}$. Notice that $U = V(I) - V(\mathfrak{a})$ and $V = V(J) - V(\mathfrak{a})$. By construction I and J are contained in $\mathfrak{a}$. Now we proceed to prove the three desired properties.

(1) Since $I \subseteq \mathfrak{a}$, we know that $\sqrt{I} \subseteq \sqrt{\mathfrak{a}}$. Suppose equality holds, then $U = V(I) - V(\mathfrak{a}) = \emptyset$, a contradiction. So $\sqrt{I} \subsetneq \sqrt{\mathfrak{a}}$. Similarly for J.

(2) We have the following chain of equalites

$$\begin{aligned} V(I \cap J) &= V(I) \cup V(J) \\ &= (\operatorname{Spec}(A) - U) \cup (\operatorname{Spec}(A) - V) \\ &= \operatorname{Spec}(A) - (U \cap V) \\ &= \operatorname{Spec}(A) \\ &= V(0) \end{aligned}$$

This means that $\sqrt{I \cap J} = \sqrt{0}$.

(3) We have the following chain of equalities

$$\begin{aligned} V(I+J) &= V(I \cup J) \\ &= V(I) \cap V(J) \\ &= (\mathrm{Spec}(A) - U) \cap (\mathrm{Spec}(A) - V) \\ &= \mathrm{Spec}(A) - (U \cup V) \\ &= V(\mathfrak{a}) \end{aligned}$$

This means that $\sqrt{I+J} = \sqrt{\mathfrak{a}}$.

($\Leftarrow$) Let $U = V(I) - V(\mathfrak{a})$ and $V = V(J) - V(\mathfrak{a})$. We prove that U, V form a partition of $\mathrm{Spec}(A) - V(\mathfrak{a})$. Since $I \subseteq \mathfrak{a}$, we know that $V(\mathfrak{a}) \subseteq V(I)$. This is an strict containment since $\sqrt{I} \subsetneq \sqrt{\mathfrak{a}}$. So $U \neq \emptyset$. Similarly for V.

We have the following chain of equalities

$$\begin{aligned} U \cup V &= (V(I) - V(\mathfrak{a})) \cup (V(J) - V(\mathfrak{a})) \\ &= (V(I) \cup V(J)) - V(\mathfrak{a}) \\ &= V(I \cap J) - V(\mathfrak{a}) \\ &= V(0) - V(\mathfrak{a}) \\ &= \mathrm{Spec}(A) - V(\mathfrak{a}) \end{aligned}$$

and also the following chain of equalities

$$\begin{aligned} U \cap V &= (V(I) - V(\mathfrak{a})) \cap (V(J) - V(\mathfrak{a})) \\ &= (V(I) \cap V(J)) - V(\mathfrak{a}) \\ &= V(I \cup J) - V(\mathfrak{a}) \\ &= \emptyset \end{aligned}$$

This means that U, V form a partition of $\mathrm{Spec}(A) - V(\mathfrak{a})$ by non empty sets. Thus U, V are open sets of $\mathrm{Spec}(A) - V(\mathfrak{a})$, so they form a disconnection of this space.

□

The following theorem gives us a lower bound for the connectedness dimension of a quotient ring. We use it in the proof of Theorem 4.0.16

Theorem 2.3.3 (Grothendieck's connectedness theorem)**.** *Let A be a Noetherian equidimensional complete local ring. Let I be a proper ideal of A. Then*

$$\mathrm{c}(A/I) \geq \min \{ \, \mathrm{c}(A), \dim(A) - 1 \, \} - \mathrm{ara}(I)$$

Related to the definition of connectedness dimension is another ring invariant regarding minimal primes of a ring.

Definition 2.3.4. Let A be a ring. We define the number $\mathrm{m}(A)$ as

$$\mathrm{m}(A) = \min \left\{ \dim \left(\frac{A}{\bigcap_{P \in S} P + \bigcap_{Q \in T} Q} \right) \;\middle|\; (S,T) \text{ is a partition of } \operatorname{Min}(A) \right\}$$

It turns out that the relation is that they are equal. So we can compute connectedness dimension just by focusing on the minimal primes of the ring.

Proposition 2.3.5. *Let A be a Noetherian local ring. Then $\mathrm{c}(A) = \mathrm{m}(A)$.*

Proof. $(\mathrm{c}(A) \geq \mathrm{m}(A))$ Take an ideal $\mathfrak{a}$ of A such that $\operatorname{Spec}(A) - V(\mathfrak{a})$ is disconnected and $\mathrm{c}(A) = \dim(A/\mathfrak{a})$.

Suppose $V(\mathfrak{a}) = \operatorname{Spec}(A)$. Then $\sqrt{\mathfrak{a}} = \sqrt{0}$, so $\dim(A/\mathfrak{a}) = \dim(A)$. This means that $\mathrm{c}(A) = \dim(A)$. But $\dim(A)$ is an upper bound for $\mathrm{m}(A)$, since this is the minimum of the dimension of quotient rings of A. So $\mathrm{c}(A) \geq \mathrm{m}(A)$.

Suppose $V(\mathfrak{a}) \subsetneq \operatorname{Spec}(A)$.

Suppose $\mathfrak{a} \subseteq P$ for some minimal prime P of A. Let $S = \{ P \}$ and $T = \operatorname{Min}(A) - \{ P \}$. This is a partition of the minimal primes of A. Let $I = P$ and $J = \bigcap_{Q \in T} Q$. Notice that $\operatorname{Spec}(A) - V(I + J)$ is disconnected by proposition 2.3.2. Since $\mathfrak{a} \subseteq I + J$, then $\dim(A/\mathfrak{a}) \geq \dim\left(\frac{A}{I+J}\right)$. But by definition of $\mathrm{m}(A)$, we know that $\dim\left(\frac{A}{I+J}\right) \geq \mathrm{m}(A)$. So $\mathrm{c}(A) \geq \mathrm{m}(A)$.

Suppose $\mathfrak{a} \nsubseteq P$ for every minimal prime P of A. By proposition 2.3.2, we know there are I, J ideals of A such that $U = V(I) - V(\mathfrak{a})$ and $V = V(J) - V(\mathfrak{a})$ form a disconnection of $\operatorname{Spec}(A) - V(\mathfrak{a})$. Since the minimal primes of A do not lie in $V(\mathfrak{a})$, each of them must lie either in U or in V, but not in both. Let $X = \{ P \in \operatorname{Min}(A) \mid P \in V(I) \}$ and $Y = \{ Q \in \operatorname{Min}(A) \mid Q \in V(J) \}$. Notice that X, Y is a partition of $\operatorname{Min}(A)$. By our choice of I and J we know that $\sqrt{I + J} = \sqrt{\mathfrak{a}}$. Since $I \subseteq \bigcap_{P \in X} P$ and $J \subseteq \bigcap_{Q \in Y} Q$, then $\sqrt{\mathfrak{a}} = \sqrt{I + J} \subseteq \sqrt{\bigcap_{P \in X} P + \bigcap_{Q \in Y} Q}$. So $\dim(A/\mathfrak{a}) \geq \dim\left(\frac{A}{\bigcap_{P \in X} P + \bigcap_{Q \in Y} Q}\right)$. Again by the definition of $\mathrm{m}(A)$, we have that $\dim\left(\frac{A}{\bigcap_{P \in X} P + \bigcap_{Q \in Y} Q}\right) \geq \mathrm{m}(A)$. So $\mathrm{c}(A) \geq \mathrm{m}(A)$.

($\mathrm{c}(A) \leq \mathrm{m}(A)$) Take a partition (S, T) of the minimal primes of A such that $\mathrm{m}(A) = \dim\left(\frac{A}{I+J}\right)$, where $I = \bigcap_{P \in S} P$ and $J = \bigcap_{Q \in T} Q$.

Now suppose $S = \emptyset$ or $T = \emptyset$. Say $S = \emptyset$. Then $\mathrm{m}(A) = \dim(A/J) = \dim(A/\sqrt{0}) = \dim(A)$. Since $\mathrm{c}(A)$ is the minimum of the dimension of quotient rings of A, it follows that $\mathrm{m}(A) \geq \mathrm{c}(A)$.

Now suppose $S \neq \emptyset$ and $T \neq \emptyset$ and let $\mathfrak{a} = I + J$. By Proposition 2.3.2, $V(I) - V(\mathfrak{a})$ and $V(J) - V(\mathfrak{a})$ form a disconnection of $\mathrm{Spec}(A) - V(\mathfrak{a})$. By definition of $\mathrm{c}(A)$, we have that $\dim(A/\mathfrak{a}) \geq \mathrm{c}(A)$, so $\mathrm{m}(A) \geq \mathrm{c}(A)$.

□

2.4 Injective Modules

Local cohomology is a right derived functor, so many of its properties can be deduced from the properties of injective modules. In this section we state and provide proofs for some of them. We start defining what an injective module is.

Definition 2.4.1. Let A be a ring and let E be an A-module. We say E is an injective module if one of the following equivalent conditions holds:

1. $\mathrm{Hom}_A(\_, E)$ is an exact contravariant functor.

2. The induced homomorphism $\mathrm{Hom}_A(N, E) \to \mathrm{Hom}_A(M, E)$ is surjective for every injective A-linear map $M \to N$.

3. Every A-linear map from a submodule M of N to E can be extended to a map from N to E.

Notice that injective modules do exist. The 0 module is an example of injective module. The definition 3 of injective module can be stated in terms of fixing $N = A$. In that case the A-submodules of A coincide with the ideals of A. In other words:

Proposition 2.4.2. *Let A be a ring and let E be an A-Module. E is injective if and only if for every ideal I of A and for every A-linear map $\phi : I \to E$, ϕ extends from A to E.*

Proof. Suppose E is injective. Let I be an ideal of A and let ϕ be an A-linear map from I to E. Since I is a submodule of A, it follows from the definition of injective modules that we can extend ϕ from A to E.

Now, let N be an A-module and M a submodule of N. Let $f : M \to E$ be an A-linear map. We show that we can extend f to a map from N to E.

Define the set

$$X = \bigcup_{M' \subseteq N} \{ \, g : M' \to E \mid g \text{ is } A\text{-linear} \, \} .$$

We define a partial order in X as follows, if $g_1, g_2 \in X$ we say that $g_1 \leq g_2$ whenever the domain of g_1 is contained in the domain of g_2 and g_2 is an extension of g_1.

Let $Y = \{ \, g \in X \mid f \leq g \, \}$. Notice that every ascending chain in Y has upper bound in Y. By Zorn's lemma Y has a maximal element. Let $h : L \to E$ be a maximal element of Y. Since $h \in Y$, h is an extension of f.

To complete the proof we must show that $L = N$, we do this by contradiction.

Suppose $L \subsetneq N$. This means there is an $x \in N$ such that $x \notin L$. Consider the ideal $I = (L : x)$ of A and let ϕ be the A-linear map $\phi : I \to E$ where $\phi(r) := h(rx)$. By hypothesis, we can extend ϕ to a map $\psi : A \to E$.

Consider the maps $h_1 : L \oplus A \to E$ and $h_2 : L \oplus A \to L + Ax$ defined respectively by $h_1(l, a) := h(l) + \psi(a)$ and $h_2(l, a) := l + ax$. Since h_2 is surjective, we have the isomorphism

$$L + Ax \cong \frac{L \oplus A}{\ker(h_2)}.$$

Let $(l, a) \in \ker(h_2)$. This means that $ax = -l \in L$, so $a \in I$. This implies that

$$\begin{aligned} h_1(l, a) &= h(l) + \psi(a) \\ &= h(-ax) + \psi(a) \\ &= -h(ax) + \phi(a) \\ &= -h(ax) + h(ax) \\ &= 0. \end{aligned}$$

Thus $\ker(h_2) \subseteq \ker(h_1)$, and so, we have a well defined A-linear map

$$H : L + Ax \cong \frac{L \oplus A}{\ker(h_2)} \to E,$$

such that $H(l + ax) = h(l) + \psi(a)$. Observe that H is an extension of h, and since $L \subsetneq L + Ax$, then $h < H$. This contradicts the maximality of h. □

Our first goal is to prove that every module embeds in an injective module. We need several results in order to do so. We begin with the following definition.

Definition 2.4.3. Let A be a domain and let M be an A-module. M is called divisible if one of the following equivalent conditions holds.

1. $aM = M$ for all $a \in A - \{0\}$.
2. For all $m \in M$ and for all $a \in A - \{0\}$, there is $m' \in M$ such that $m = am'$.

Proposition 2.4.4. *Let A be a domain and let M be an A-module. Then*

$$M \textit{ is injective } \Rightarrow M \textit{ is divisible.}$$

Furthermore, if A is a principal ideal domain, then

$$M \textit{ is injective } \Leftrightarrow M \textit{ is divisible.}$$

Proof. Suppose M is injective. Let $m \in M$ and let $a \in A - \{0\}$. Consider the ideal $I = (a)$. Let $\phi : I \to M$ such that $\phi(a) = m$, this map is well defined since A is a domain. Since M is injective, we know from Proposition 2.4.2 that there is an extension ψ of ϕ from A to M. Let $m' = \psi(1)$. Observe that

$$m = \phi(a) = \psi(a) = a\psi(1) = am',$$

and so, M is divisible.

Now let A be a principal ideal domain and suppose M is divisible. Let I be an ideal of $A$and let $\phi : I \to M$ be a A-linear map. We know $I = (a)$ for some $a \in A$ since A is a principal ideal domain. If $a = 0$, then ϕ is the zero function and we can extend it to the zero function from A to M. Suppose $a \neq 0$ and let $m = \phi(a)$. Since M is divisible, there is a $m' \in M$ such that $m = am'$. Let $\psi : A \to M$ such that $\psi(1) = m'$. Observe that

$$\psi(a) = a\psi(1) = am' = m,$$

and so, ψ is an extension of ϕ. Proposition 2.4.2 implies that M is injective. □

Proposition 2.4.5. *Let A be a ring and let $\phi : M \to N$ be an A-linear map. If M is divisible, then $\phi(M)$ is divisible too.*

Proof. Suppose M is divisible and let $a \in A - \{0\}$. Since M is divisible, we know that $aM = M$, so

$$a\phi(M) = \phi(aM) = \phi(M).$$

Thus $\phi(M)$ is injective too. □

Proposition 2.4.6. *$\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.*

Proof. Let $q = \frac{a}{b} \in \mathbb{Q}$ and let $n \in \mathbb{Z} - \{0\}$. Observe that $q' = \frac{a}{nb}$ is such that $q = nq'$. Thus $\mathbb{Q}$ is a divisible $\mathbb{Z}$ module, and since $\mathbb{Z}$ is a principal ideal domain, Proposition 2.4.4 that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Proposition 2.4.5 implies that $\mathbb{Q}/\mathbb{Z}$ is also divisible and from Proposition 2.4.4 we conclude that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. □

Observation 2.4.7. *Aqui va el lemma 0.3 de loc y tambien antes se debe agregar lo de la discusion previa del dual*

Theorem 2.4.8. *Let $B \to A$ be a ring homomorphism. Let M and N be A-modules and let L be a B-module. There is a natural isomorphism of A-modules*

$$\operatorname{Hom}_B(M \otimes_A N, L) \to \operatorname{Hom}_A(M, \operatorname{Hom}_B(N, L)).$$

Corollary 2.4.9. *Let A be a B-algebra. Let M be a flat A-module and let E be an injective B-module. Then $\operatorname{Hom}_B(M, E)$ is an injective A-module.*

Theorem 2.4.10. *Let A be a ring and let M be an A-module. There is an injective A-module E such that M embeds in E.*

2.5 Essential Extensions

Definition 2.5.1. Let A be a ring and let $h : M \to N$ be a monomorphism of A-modules. We say that h is an essential extension if any of the following equivalent conditions holds.

1. Every nonzero submodule of N has nonzero intersection with $h(M)$.
2. Every nonzero element of N has a nonzero multiple in $h(M)$.
3. If $\phi : N \to Q$ is a homomorphism of A-modules and ϕh is injective, then ϕ is injective.

Whenever we say that $M \subseteq N$ is an essential extension we understand h to be the inclusion map from M to N.

Essential extensions do exist. For any module M, any isomorphism of M is an essential extension. We call such essential extensions improper. Another example is the following, let A be a domain and let $K = \mathrm{Frac}(A)$, then K is an essential extension of A.

Proposition 2.5.2. *Let A be a ring. Let M, N and L be A-modules. Then*

1. *Let $f : M \to N$ and $g : N \to L$ be monomorphisms. Then*

$$gf \text{ is an essential extension } \Leftrightarrow f \text{ and } g \text{ are essential extensions.}$$

2. *If $M \subseteq L$ and $\{N_i\}_i$ is a family of modules such that $M \subseteq N_i \subseteq L$ for every i and $\bigcup_i N_i = L$, then*

$$M \subseteq L \text{ is an essential extension } \Leftrightarrow \forall i, M \subseteq N_i \text{ is an essential extension.}$$

3. *If $M \subseteq N$ then there exists a maximal submodule N' of N such that $M \subseteq N'$ is essential. (In this case we say that N' is a maximal essential extension of M within N).*

Proof. **(1)** Suppose f and g are essential extensions. Let $\phi : L \to Q$ be a homomorphism of A-modules such that $\phi g f$ is injective. This implies that ϕg is injective since f is an essential extension. Thus, ϕ is injective since g is an essential extension. We conclude that gf is an essential extension.

Conversely, suppose gf is an essential extension. Let $\phi : L \to Q$ be a homomorphism of A-modules such that ϕg is injective. Then $\phi g f$ is also injective since f is injective. Thus ϕ is injective since gf is an essential extension. We conclude that g is an essential extension. Now, let $n \in N - \{0\}$. Since g is injective, $g(n) \in L - \{0\}$. Then there is an $a \in A$ and a $m \in M$ such that $ag(n)$ is nonzero and $gf(m) = ag(n)$ since gf is an essential extension. So $g(f(m)) = g(an)$. Since g is injective we conclude that an is nonzero and $f(m) = an$. Thus, f is an essential extension.

(2) Suppose $M \subseteq L$ is an essential extension. (1) implies that $M \subseteq N_i$ is an essential extension for every i.

Conversely, suppose $M \subseteq N_i$ is an essential extension for every i. Let l be a nonzero element of L. There is an i such that $l \in N_i$, since $\bigcup_i N_i = L$. Since $M \subseteq N_i$ is an essential extension, then l has a nonzero multiple in M. This implies that $M \subseteq L$ is an essential extension.

(3) Let $X = \{ L \subseteq N \mid M \subseteq L \text{ is an essential extension } \}$. Notice that X is nonempty since $M \in X$. (2) implies that every ascending chain of elements of X has an upper bound in X. It follows from Zorn's lemma that X has a maximal element.

□

Definition 2.5.3. If $M \subseteq N$ is an essential extension and N has no proper essential extension we shall say that N is a maximal essential extension of M.

Proposition 2.5.4. *Let M_1, M_2, N_1, N_2 be A-modules such that $M_i \subseteq N_i$ is an essential extension for every i. Then:*

$$M_1 \oplus N_2 \subseteq N_1 \oplus N_2 \text{ is an essential extension.}$$

Proof. Let (n_1, n_2) be a nonzero element of $N_1 \oplus N_2$. Then $n_1 \neq 0$ or $n_2 \neq 0$. Say $n_1 \neq 0$. Since $M_1 \subseteq N_1$ is an essential extension, then there is an $a \in A$ such that $an_1 = m_1$ is a nonzero element of M_1. If $an_2 = 0$, then we are done since $a(n_1, n_2) = (m_1, 0)$ is a nonzero multiple of (n_1, n_2) in $M_1 \oplus M_2$. If $an_2 \neq 0$, then we can find a $b \in A$ such that $ban_2 = m_2$ is a nonzero element of M_2 since $M_2 \subseteq N_2$ is an essential extension. Then $ba(n_1, n_2) = (bm_1, m_2)$ is a nonzero multiple of (n_1, n_2) in $M_1 \oplus M_2$. We conclude that $M_1 \oplus M_2 \subseteq N_1 \oplus N_2$ is an essential extension. □

Notice that the previous proposition can be extended to arbitrarily large families of modules because of the direct sum.

Proposition 2.5.5. *Let A be a ring and let M, N be A-modules. Then*

$$M \oplus N \text{ is injective } \Leftrightarrow M, N \text{ are injective.}$$

Proof. Let π_1 and π_2 be the projection maps from $M \oplus N$ to M and N respectively.

Suppose $M \oplus N$ is injective. Let T be an A-module and let S be a submodule of T. Let $f : S \to M$ be an A-linear map and let $j : M \to M \oplus N$ be the inclusion map. Since $M \oplus N$ is injective there is a ψ that extends jf from T to $M \oplus N$. Notice that $\psi = (\pi_1\psi, \pi_2\psi)$. Let $s \in S$. Then

$$(f(s), 0) = jf(s) = \psi(s) = (\pi_1\psi(s), \pi_2\psi(s)),$$

so $f(s) = \pi_1\psi(s)$. This means that $\pi_1\psi$ is an extension of f from T to M. We conclude that M is injective. The proof for the injectivity of N is analogous.

Conversely, suppose M and N are injective. Let T be an A-module and let S be a submodule of T. Let $f : S \to M \oplus N$ be an A-linear map. Since M and N are injective, there is a ϕ that extends $\pi_1 f$ from T to M and a ψ that extends $\pi_2 f$ from T to N. Let $g : T \to M \oplus N$ be defined by $g = (\phi, \psi)$. Let $s \in S$. Then

$$f(s) = (\pi_1 f(s), \pi_2 f(s)) = (\phi(s), \psi(s)),$$

so $f(s) = g(s)$. This means that g is an extension of f from T to M. We conclude that $M \oplus N$ is injective. □

Proposition 2.5.6. *Let A be a ring. Let M be an A-module.*

1. *M is injective if and only if every essential extension of M is improper.*
2. *If M is an A-module and $M \subseteq E$ with E injective, then a maximal essential extension of M within E is an injective module.*
3. *If $M \subseteq E$ and $M \subseteq \tilde{E}$ are two maximal essential extensions of M, then there is an isomorphism of E with $\tilde{E}$ that is the identity map on M.*

Proof. **(1)** Let M be an injective A-module. Let $h : M \to N$ be an essential extension. Consider the indentity map $i : M \to M$. Since M is injective and h is a monomorphism, there is a $\phi : N \to M$ such that $i = \phi h$. Since h is an essential extension, then ϕ must be injective. Let $m \in M$. Then $m = i(m) = \phi(h(m))$, so ϕ is surjective. We conclude that $M \cong N$.

Conversely, suppose every essential extension of M is improper. From Theorem 2.4.10 we know there is an injective module E such that M embeds in E. This lets us regard M as a submodule of E.

Let $X = \{ L \subseteq E \mid L \cap M = 0 \}$. Notice that X is nonempty since $0 \in X$. Given an ascending chain of elements of X, the union of the elements of the chain is again a submodule L of E such that $L \cap M = 0$, so L is an upper bound of the chain and belongs to X. Zorn's lemma implies the existence of maximal elements of the set X.

Let N be a maximal element of X. Let π be the projection map from M to E/N. Notice that π is injective since $M \cap N = 0$. We proceed to prove that π is an essential extension. Let T be a non zero submodule of E/N. We can write $T = S/N$ for some submodule S of E such that $N \subsetneq S$. Suppose $T \cap \pi(M) = 0$, then

$$0 = T \cap \pi(M) = \frac{S}{N} \cap \frac{M+N}{N} = \frac{(M+N) \cap S}{S}.$$

This implies that $(M+N)\cap S \subseteq N$. Thus

$$M \cap S = M \cap (M \cap S) \subseteq M \cap ((M+N)\cap S) \subseteq M \cap N = 0.$$

We conclude that $S \in X$, which contradicts the maximality of N. Thus T has nonzero intersection with $\pi(M)$. This means that π is an essential extension.

Since every essential extension of M is improper, we conclude that π is an isomorphism between M and E/N. Thus $M+N=E$. Since $M+N=E$ and $M\cap N=0$, then $M\oplus N=E$. Proposition 2.5.5 implies that M is injective.

(2) Let S be a maximal essential extension of M within E. Let T be an A-module such that $S \subseteq T$ is an essential extension. Observe that $T \subseteq E$ since E is injective. Since $M \subseteq S$ and $S \subseteq T$ are essential extension, Proposition 2.5.2 implies that $M \subseteq T$ is an essential extension of M within E. The maximality of the essential extension $M \subseteq S$ implies that $S = T$. We conclude from (1) that S is injective.

(3) Suppose $f : M \to E$ and $g : M \to \tilde{E}$ are maximal essential extensions. We know from (1) that E is an injective module. This implies there is a $\phi : \tilde{E} \to E$ such that $f = \phi g$. Since g is an essential extension and f is injective, ϕ must also be injective. Proposition 2.5.2 implies that ϕ is injective. Since g is a maximal essential extension, ϕ must be bijective.

□

Definition 2.5.7. Let A be a ring. Let M and E be A-modules. If $M \subseteq E$ is a maximal essential extension of M over A we say that E as an injective hull for M and write $E = E_A(M)$, or $E = E(M)$ when the ring A is understood.

Note that every A-module has an injective hull.

Corollary 2.5.8. *Let A be a ring and let E be an injective A-module. The injective hull of E is E itself. That is, $E_A(E) = E$.*

Proof. This follows from (1) and (2) in Proposition 2.5.6. □

The following observation is key in the proof of Proposition 2.6.4.

Observation 2.5.9. *From the proof of (1) in Proposition 2.5.6 and from Proposition 2.5.5 we can deduce that given injective modules E', E such that $E' \subseteq E$, there is an injective submodule E'' of E such that $E = E' \oplus E''$.*

From the previous observation we can deduce the following theorem.

Proposition 2.5.10. *Let K be a field and let $E = E_K(K)$. Then $E = K$.*

Proof. Since E is a K-vector space, we know that $E = K \oplus E/K$. Observation 2.5.9 implies that K is an injective module. Corollary 2.5.8 implies that $E = K$. ☐

Proposition 2.5.11. *Let A be a ring and let M_1, M_2 be A-modules. Then*

$$E(M_1 \oplus M_2) \cong E(M_1) \oplus E(M_2).$$

Proof. We know that $M_1 \subseteq E(M_1)$ and $M_2 \subseteq E(M_2)$ are maximal essential extensions. Proposition 2.5.4 implies that $M_1 \oplus M_2 \subseteq E(M_1) \oplus E(M_2)$ is also an essential extension. Proposition 2.5.5 implies that $E(M_1) \oplus E(M_2)$ is also an injective module, so there is a $\phi : E(M_1 \oplus M_2) \to E(M_1) \oplus E(M_2)$ such that is an extension of the inclusion map $M_1 \oplus M_2 \subseteq E(M_1) \oplus E(M_2)$. Since $M_1 \oplus M_2 \subseteq E(M_1 \oplus M_2)$ is an essential extension, ϕ must be injective. The maximality of $M_1 \oplus M_2 \subseteq E(M_1 \oplus M_2)$ implies that ϕ is bijective. ☐

The previous proposition generalizes for any size of finite direct sums. If the direct sum is not finite, this might not be true. However, when the ring is Noetherian, the proposition holds for infinite direct sums too. We prove this later on.

We also state the following observations, which we use during the proof of Proposition 2.6.5.

Observation 2.5.12. *Let A be a ring and let $M \subseteq N$ be an essential extension of A-modules. Then $E_A(M) = E_A(N)$.*

Observation 2.5.13. *Let A be a ring and let M be an A-module. Suppose E is an injective A-module such that $M \subseteq E \subseteq E_A(M)$. Then $E = E_A(M)$.*

The concept of injective resolutions is key to the definitions of the local cohomology modules. We finish this section defining such resolutions.

Definition 2.5.14. Let A be a ring and let M be an A-module. We say that the complex

$$C : 0 \to M \to E_0 \to E_1 \to \dots$$

is an injective resolution of M if the E_i are injective modules and if C is an exact sequence. Moreover, we say that the resolution is minimal if $E_0 = E(M)$ and $E_i = E(\operatorname{Im}(E_{i-1} \to E_i))$ for every $i \geq 1$.

Proposition 2.5.15. *Let A be a ring and let M be an A-module. M has a minimal injective resolution.*

Notice also that any two minimal injective resolutions of the same module are isomorphic as complexes.

2.6 Injective Modules over Noetherian Rings

Proposition 2.6.1. *Let A be a ring and let M be an A-module. Let $\{N_i\}_i$ be a family of A-modules. Then there is a monomorphism*

$$\psi : \bigoplus_i \operatorname{Hom}_A(M, N_i) \to \operatorname{Hom}_A(M, \bigoplus_i N_i),$$

such that ψ is an isomorphism whenever M is finitely generated.

Proposition 2.6.2. *Let A be a Noetherian ring. Let $\{ E_i \}_i$ be a family A-modules and let $E = \bigoplus_i E_i$. If E_i is injective for every i, then E is injective.*

Proof. Suppose E_i is injective for every i and let I be an ideal of A. Since E_i is injective for every i, Proposition 2.4.2 and the definition of injective modules imply that $\operatorname{Hom}_A(A, E_i) \to \operatorname{Hom}_A(I, E_i)$ is surjective for every i. This implies that $\bigoplus_i \operatorname{Hom}_A(A, E_i) \to \bigoplus_i \operatorname{Hom}_A(I, E_i)$ is surjective. Since A is a Noetherian ring, A and I are finitely generated A-modules. Thus Proposition 2.6.1 implies that $\operatorname{Hom}_A(A, E) \to \operatorname{Hom}_A(I, E)$ is surjective. Proposition 2.4.2 and the definition of injective modules imply that E is injective. □

The following observation is used during the proof of 2.6.5.

Observation 2.6.3. *Given a field K and a K-vector space V, V is an injective K-vector space. This follows from Proposition 2.5.10 and Proposition 2.6.2.*

Proposition 2.6.4. *Let A be a Noetherian ring and let E be nonzero injective A-module. Let $X = \{ E_A(A/P) \mid P \in \operatorname{Spec}(A) \}$. Then E is a direct sum of elements of X.*

Proof. Let Y be the set of families of elements M of X such that M is embedded in E and the sum of the elements of the family is an internal direct sum in E. Since E is a nonzero module over a Noetherian ring, the set $\operatorname{Ass}_A(E)$ is not empty. Let $P \in \operatorname{Ass}_A(E)$. We know that A/P is embedded in E, so $E_A(A/P)$ is an element of X that is embedded in E. Since $E_A(A/P)$ is an internal direct sum in E, we conclude that the family $\{ E_A(A/P) \}$ belongs to Y. Thus Y is non empty. Consider any increasing chain of elements of Y. There is a family element of Y such that bounds above the chain. Zorn's lemma implies the existence of maximal elements of Y.

Let $\{ E_i \}_i$ be a maximal element of Y and let E' be the sum of the elements of such family. We know that $E' \subseteq E$. Observation 2.5.9 implies that $E = E' \oplus E''$, where E'' is an injective submodule of E.

Suppose $E'' \neq 0$, this implies that $\mathrm{Ass}_A(E'')$ is not empty. Let $Q \in \mathrm{Ass}_A(E'')$. We can think of A/Q as a submodule of E''. Thus $E_A(A/P)$ is also a submodule of E''. This means we can add $E_A(A/P)$ to our previous collection $\{ E_i \}_i$ to make a new collection which still belongs to Y. This contradicts the maximality of $\{ E_i \}_i$. Thus $E'' = 0$, which means that $E = E'$. □

Proposition 2.6.5. *Let A be a Noetherian ring and let P be a prime ideal. Let $E = E_A(A/P)$ and let $F = \mathrm{Frac}(A/P) = A_P/PA_P$.*

1. *$E_A(F) = E$.*

2. *For every $a \in A - P$, the A-module homomorphism $f_a : E \to E$ defined by $f_a(e) = ae$ is an automorphism.*

3. *E is an A_P-module. The scalar product is given by $\frac{b}{a}e := be'$, where e' is the unique element of E such that $e = ae'$.*

4. *$\mathrm{Ann}_E(P)$ is isomorphic to F as F-vector spaces.*

5. *$E_{A_P}(F) = E$.*

6. *$\mathrm{Ass}_A(E) = \{ P \}$. The annihilator of every nonzero element of E is P-primary. Every element of E is killed by a power of P.*

7. *Let $Q \in \mathrm{Spec}(A)$. $\mathrm{Hom}_{A_P}(F, E_A(A/Q)_P)$ is 0 when $Q \neq P$ and is isomorphic to F when $Q = P$.*

Proof. **(1)** We know that $A/P \subseteq \mathrm{Frac}(A/P) = F$ is an essential extension of A/P-modules. Now we consider A/P and F as A-modules via restriction of scalars. Let $x = \frac{\overline{a}}{\overline{w}}$ be a nonzero element of $\mathrm{Frac}(A/P)$. Observe that

$$w\frac{\overline{a}}{\overline{w}} = \frac{\overline{w}}{1}\frac{\overline{a}}{\overline{w}} = \frac{\overline{a}}{1} \neq 0,$$

since x is nonzero. Thus, x has a nonzero A-multiple in A/P. We conclude that $A/P \subseteq F$ is an essential extension of A-modules. Observation 2.5.12 implies that $E = E_A(F)$.

(2) Let $a \in A - P$. Let $g_a : F \to F$ be defined by $g_a\left(\frac{\overline{x}}{\overline{y}}\right) = \frac{\overline{ax}}{\overline{y}}$. Observe that this injective morphism. Extending the codomain of g_a gives us an injective morphism from F to E. Since $F \subseteq E$ is an essential extension of A-modules by (1), we conclude that $f_a : E \to E$ defined by $f(e) = ae$ is an injective morphism with image $aE \subseteq E$. Since E is injective, aE must also be injective. Thus, we have that $F \subseteq aE \subseteq E$. Observation 2.5.13 implies that $aE = E$. Hence, f_a is an automorphism.

(3) Notice that the automorphisms we defined in (2) imply that that the scalar product is well defined. And so, E is an A_P-module.

(4) First we prove that $\mathrm{Ann}_E(P) = \mathrm{Ann}_E(PA_P)$. Let $e \in \mathrm{Ann}_E(PA_P)$ and let $p \in P$. We know that $\frac{p}{1} \in PA_P$, thus $0 = \frac{p}{1}e = pe$. This means that $e \in \mathrm{Ann}_E(P)$. Now, let $e \in \mathrm{Ann}_E(P)$ and let $\frac{p}{w} \in PA_P$. We know that $p \in P$, thus $0 = \frac{0}{w} = \frac{pe}{w} = \frac{p}{w}e$. This means that $e \in \mathrm{Ann}_E(PA_P)$. We conclude that $\mathrm{Ann}_E(P) = \mathrm{Ann}_E(PA_P)$.

Since $\mathrm{Ann}_E(PA_P)$ is an A_P-submodule of E and PA_P is contained in $\mathrm{Ann}_{A_P}(\mathrm{Ann}_E(PA_P))$, we deduce that $\mathrm{Ann}_E(PA_P)$ is an $A_P/PA_P = F$-vector space. That is, $\mathrm{Ann}_E(P)$ is an F-vector space.

Observe that $F \subseteq \mathrm{Ann}_E(P) \subseteq E$. Since $F \subseteq E$ is an essential extension of A-modules, Proposition 2.5.2 implies that $F \subseteq \mathrm{Ann}_E(P)$ is also an essential extension of A-modules. Observe that $F \subseteq \mathrm{Ann}_E(P)$ is also an essential extension of F-vector spaces. Observation 2.5.12 implies that $E_K(K) = E_K(\mathrm{Ann}_E(P))$. Proposition 2.5.10 and Observation 2.6.3 imply that $K = \mathrm{Ann}_E(P)$.

(5) Since $F \subseteq E$ is an essential extension as A-modules, it is also an essential extension as A_P-modules. Let $E \subseteq M$ be an essential extension of A_P-modules. Let m be a nonzero element of M. Since $E \subseteq M$ is essential, there is nonzero A_P-multiple $e \in E$ of m, say $e = \frac{a}{m}x$, where $w \in A - P$. From (1) we can deduce that $we = am$ is a nonzero element of E. This means that m has a nonzero A-multiple in E. Thus, $E \subseteq M$ is an essential extension of A-modules. Since $F \subseteq E$ is a maximal essential extension of A-modules, we conclude that $E = M$. Hence, $E = E_{A_P}(F)$.

(6) Since E is an injective hull for A/P, we know that A/P is embedded in E. Thus $P \in \mathrm{Ass}_A(E)$. Now, let $Q \in \mathrm{Ass}_A(E)$. We know there exists a nonzero $e \in E$ such that $\mathrm{Ann}_A(e) = Q$. We know that $A/Q \cong Ae \subseteq E$. Since E is an essential extension of A/P and e is nonzero, there is an $a \in A$ such that ae is nonzero and $ae \in A/P$. We know that the annihilator of every nonzero element of A/P is P, hence $\mathrm{Ann}_A(ae) = P$. On the other hand, $ae \in Ae \cong A/Q$. We know that the annihilator of every nonzero element of A/Q is Q, hence $\mathrm{Ann}_A(ae) = Q$. Thus that $Q = P$. We conclude that $\mathrm{Ass}_A(E) = \{ P \}$.

Let e be a nonzero element of E. Let $I = \mathrm{Ann}_A(e)$. We know that $A/I \cong Ae \subseteq E$. Thus, $\mathrm{Ass}_A(A/I) \subseteq \mathrm{Ass}_A(E) = \{ P \}$. Since $e \neq 0$, then $I \neq A$. Thus A/I is a nonzero Noetherian ring. Hence, $\mathrm{Ass}_A(A/I) \neq \emptyset$. We conclude that $\mathrm{Ass}_A(A/I) = \{ P \}$. This implies that I is a P-primary ideal.

In a Noetherian ring every ideal contains a power of its radical. This means there is an n such that $P^n \subseteq I = \mathrm{Ann}_A(e)$, since I is P-primary. This means that $P^n e = 0$.

(7) Let $Q \in \mathrm{Spec}(A)$. Suppose $Q = P$. Consider E as an A_P-module as discussed in (2). Notice that the map $f : E \to E_P$ of A_P-modules is an isomorphism. We conclude that $\mathrm{Hom}_{A_P}(F, E_P) \cong \mathrm{Hom}_{A_P}(F, E)$.

Let $g \in \mathrm{Hom}_{A_P}(F, E)$. Let $\left[\frac{a}{w}\right] \in F$ and let $p \in P$. Observe that

$$pg\left(\left[\frac{a}{w}\right]\right) = \frac{p}{1} g\left(\left[\frac{a}{w}\right]\right) = g\left(\frac{p}{1}\left[\frac{a}{w}\right]\right) = g\left(\left[\frac{pa}{w}\right]\right) = g(0) = 0,$$

thus $\mathrm{Im}(g) \subseteq \mathrm{Ann}_E(P)$. Since $F \subseteq E$, we conclude that $\mathrm{Hom}_{A_P}(F, E) = \mathrm{Hom}_{A_P}(F, F)$. Since $\mathrm{Hom}_{A_P}(F, F) = F$, we have that $\mathrm{Hom}_{A_P}(F, E_P) \cong F$.

□

Proposition 2.6.6. *Let A be a ring and let M, N be A-modules such that $M \subseteq N$ is an essential extension. Then* $\mathrm{Ass}(M) = \mathrm{Ass}(N)$.

Proof. We proceed by double containment.

Suppose $P \in \mathrm{Ass}(M)$. Then there is an injection from A/P to M and since $M \subseteq N$, we have an inclusion from M to N. The composition of these maps gives us an injection from A/P to N. So $P \in \mathrm{Ass}(N)$.

Now suppose $P \in \text{Ass}(N)$. Then $P = \text{Ann}(n)$, for some $n \in N$ not zero. Since $M \subseteq N$ is essential then there is a non zero multiple of n in M, say rn. Now we proceed to prove that $\text{Ann}(n) = \text{Ann}(rn)$.

Suppose $x \in \text{Ann}(n)$. Then $xn = 0$, so $x(rn) = 0$ and hence $\text{Ann}(n) \subseteq \text{Ann}(rn)$. Now suppose $x \in \text{Ann}(rn)$. Then $xrn = 0$, so $xr \in \text{Ann}(n)$. Since $\text{Ann}(n) = P$ is prime, then $x \in \text{Ann}(n)$ or $r \in \text{Ann}(n)$, but since $rn \neq 0$, then $r \notin \text{Ann}(n)$ and hence $x \in \text{Ann}(n)$. Thus $\text{Ann}(rn) \subseteq \text{Ann}(n)$.

We conclude that $\text{Ann}(n) = \text{Ann}(rn)$ and so $\text{Ann}(rn) = P$. Then $P \in \text{Ass}(M)$. □

To finalize this section we present additional results regarding injective modules, this time their proof is omitted.

Proposition 2.6.7. *Let A be a Noetherian ring and let E be an injective A-module. Let $P \in \text{Spec}(A)$ and let $F = \text{Frac}(A/P) = A_P/PA_P$. Then the number of copies of $E(A/P)$ occurring in a representation of E as direct sum of modules of this form is $\dim_F \text{Hom}_{A_P}(F, E_P)$. Furthermore, $E(A/P)$ appears in such representation if and only if $P \in \text{Ass}_A(E)$.*

Corollary 2.6.8. *Let A be a Noetherian ring and let E be a nonzero injective A-module. E is indecomposable if and only $E \cong E(A/P)$ for some prime P of A.*

Theorem 2.6.9. *Let A be a Noetherian ring and let S be a multiplicative subset of A.*

1. *The injective modules over $S^{-1}A$ coincide with the injective A-modules E with the property that for every $E(A/P)$ occurring as a summand P does not meet S.*

2. *If E is any injective A-module then $S^{-1}E$ is an injective $S^{-1}A$-module.*

3. *If $M \subseteq N$ is an essential extension then $S^{-1}M \subseteq S^{-1}N$ is an essential extension. If $M \subseteq E$ is a maximal essential extension then $S^{-1}M \subseteq S^{-1}E$ is a maximal essential extension.*

Proposition 2.6.10. *Let A be a Noetherian ring and let M be a finitely generated A-module. Let*

$$C : 0 \to M \to E_0 \to E_1 \to \dots$$

be a minimal injective resolution of M. Let $P \in \text{Spec}(A)$. Then the number of copies of $E_A(A/P)$ ocurring in E_i is finite, for every i. Furthermore, this number is equal to $\dim_F \text{Ext}^i_{A_P}(F, M_P)$, where $F = \text{Frac}(A/P)$.

Proposition 2.6.11. *Let $(A, \mathfrak{m}, K)$ be a Noetherian local ring. A maximal essential extension of K over A is also a maximal essential extension of K over A is also a maximal essential extension of K over $\hat{A}$. That is, $E_A(K) \cong E_{\hat{A}}(K)$.*

Theorem 2.6.12. *Let $(A, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ be a local homomorphism of Noetherian local rings and suppose that S is module-finite over the image of A. Let E be an injective hull of K over A. Then $\operatorname{Hom}_A(S, E)$ is an injectuve hull L over S.*

Corollary 2.6.13. *If $S = A/I$, where $(A, \mathfrak{m}, K)$ is a Noetherian local ring, and $E = E_A(K)$, then the $\operatorname{Ann}_E(I) \cong \operatorname{Hom}_A(A/I, E)$ is an injective hull for K over S.*

Theorem 2.6.14. *Let $(A, \mathfrak{m}, K)$ be an Artin Local Aing. Then $E_A(K)$ is a module of finite length and its length is equal to the length of A.*

Lemma 2.6.15. *Let $(A, \mathfrak{m}, K)$ be any Noetherian local ring and let $^\vee$ denote $\operatorname{Hom}_A(\_, E)$, where $E = E_A(K)$. Then for every finite length module, $\ell(M^\vee) = \ell(M)$.*

Theorem 2.6.16. *Let $(A, \mathfrak{m}, K)$ be an Artin local ring and let $E = E_A(K)$. Then the map $A \to \operatorname{Hom}_A(E, E)$, which sends r to the map multiplication by r, is an isomorphism.*

Theorem 2.6.17. *A Noetherian local ring $(A, \mathfrak{m}, K)$ is injective as a module over itself if and only if the Krull dimension of A is zero and the socle of is one-dimensional as a K-vector space. Moreover $A \cong E_A(K)$ in this case.*

Theorem 2.6.18. *Let $(A, \mathfrak{m}, K)$ be a Noetherian local ring with $E = E_A(K)$ and let $^\vee$ denote the exact contravariant functor $\operatorname{Hom}_A(\_, E)$. There is a map $\tilde{A} \to \operatorname{Hom}_A(E, E)$ which is an isomorphism.*

Theorem 2.6.19. *Let $(A, \mathfrak{m}, K)$ be a local ring and let $E = E(K)$. E is an Artin A-module.*

Theorem 2.6.20. *Let $(A, \mathfrak{m}, k)$ be a local ring and let M be an A-module. The following conditions are equivalent:*

1. *Every element of M is killed by a power of m and the socle of M is a finite-dimensional vector space over K.*
2. *$\operatorname{Ass}(M) = \{m\}$ and the socle of M is a finite-dimensional vector space over K.*

3. M *is an essential extension of a finite dimensional* K*-vector space.*

4. *The injective hull of* M *is a finite direct sum of copies of* $E = E_A(K)$

5. M *can be embedded in a finite direct sum of copies of* E

6. M *is an Artinian* A*-module.*

Theorem 2.6.21. *Let* $(A, \mathfrak{m}, K)$ *be a complete Noetherian local ring, let* $E = E_A(K)$ *and let* $\_^\vee$ *denote the functor* $\mathrm{Hom}_A(\_, E)$*.*

1. *If* M *is an Artinian module then* $M^\vee$ *is a Noetherian module.*

2. *If* M *is a Noetherian module then* $M^\vee$ *is an Artinian module.*

3. *If* M *is an Artinian or Noetherian module then the induced map* $M \to M^{\vee\vee}$ *is an isomorphism.*

Chapter 3

Local Cohomology

Local cohomology is a powerful tool of homologic algebra that we use in order to study some objects in commutative algebra. There are several equivalent definitions of local cohomology. We begin with the next one and then we present alternative ones. We also state some of the properties of local cohomology that we use in the following chapters.

Let I and J be ideals of a ring A such that $I \subseteq J$. We know that there is a surjective map from A/I to A/J which just sends x modulo I to x modulo J. Such map induces a map from $\mathrm{Ext}^i_A(A/I, M)$ to $\mathrm{Ext}^i_A(A/I, M)$, for every A-module M.

Definition 3.0.1. Let A be a Noetherian ring and let M be an A-module. Let I be an ideal of A and let $i \in \mathbb{N}$. Consider the decreasing sequence of ideals given by the positive powers of I. Consider the following direct system induced by said powers

$$\mathrm{Ext}^i_A(A/I, M) \to \mathrm{Ext}^i_A(A/I^2, M) \to \mathrm{Ext}^i_A(A/I^3, M) \to \dots.$$

We define the ith local cohomology module of M with support in I as

$$H^i_I(M) = \varinjlim_t \mathrm{Ext}^i_A(A/I^t, M).$$

Observe that if we compute the direct limit with a subsequence $\{ I_t \}_t$ of $\{ I^t \}_t$ then

$$H^i_I(M) = \varinjlim_t \mathrm{Ext}^i_A(A/I_t, M).$$

Similarly, if $\{ J_t \}_t$ is a decreasing sequence of ideals of A such that is cofinal with $\{ I^t \}_t$, then

$$H_I^i(M) = \varinjlim_t \operatorname{Ext}_A^i(A/J_t, M).$$

In particular, if $I = (x_1, \dots, x_n)$, then the sequence $\{ I_t \}_t$ where $I_t = (x_1^t, \dots, x_n^t)$ is cofinal with $\{ I^t \}_t$. So we may use it to compute local cohomology.

The following proposition is useful at the moment of computing local cohomology modules since it lets us change the ideal of support for the modules without changing the modules themselves.

Proposition 3.0.2. *Let A be a Noetherian ring and let M be an A-module. Let I and J be ideals of A with the same radical. Then for every i*

$$H_I^i(M) \cong H_J^i(M).$$

Proof. This follows from the fact that the powers of I and the powers of J are cofinal. □

Definition 3.0.3. Let A be a ring and let I be an ideal of A. Let M be an A-module. We define

$$\Gamma_I(M) = \bigcup_{n \in \mathbb{N}} \operatorname{Ann}_M(I^n).$$

Notice that $\Gamma_I(\_)$ is a covariant additive functor of R-mods. We call this functor the I-torsion functor.

It turns out that if we compute the ith derived functor of Γ_I, this coincides with the ith local cohomology functor $H_I^i(\_)$. This follows from the fact that calculating cohomology commutes with direct limits, so our first definition of local cohomology is the same as taking the derived functors of the I-torsion functor.

So given a short exact sequence of R-modules

$$0 \to L \to M \to N \to 0$$

there is a long exact sequence

$$0 \to \cdots \to H_I^{i-1}(N) \to H_I^i(L) \to H_I^i(M) \to H_I^i(N) \to H_I^{i+1}(L) \to \dots$$

Since Ext is a right derived functor and local cohomology is defined in terms of Ext, then for every $i > 0$ and for every injective A-module M we have that $H_I^i(M) = 0$.

Proposition 3.0.4. *Let A be a Noetherian ring and let I be an ideal of A. Let M be an A-module. Then every element of $H_I^i(M)$ is killed by a power of I.*

Local cohomology is also related to the depth of certain modules.

Theorem 3.0.5. *Let A be a Noetherian ring and let I be an ideal of A. Let M be a finitely generated A-module. Then*

$$H_I^i(M) = 0 \textit{ for all } i \ \Leftrightarrow IM = M.$$

Furthermore, if $IM \neq M$, then

$$\operatorname{depth}_I(M) = \min \left\{ i \;\middle|\; H_I^i(M) \neq 0 \right\}.$$

Now we state the last one of our equivalent definitions of local cohomology. This definition relies in Kozsul cohomology.

Theorem 3.0.6. *Let A be a Noetherian ring and let $I = (x_1, \ldots, x_n)$ be an ideal of A. Then $H_I^i(\_) \cong H^\bullet(\underline{x}^\infty; \_)$ as functors.*

Corollary 3.0.7. *Let A be a Noetherian ring and let I be an ideal of A. Let $m = \min \left\{ \mu(J) \;\middle|\; \sqrt{I} = \sqrt{J} \right\}$. Then for $i > m$ we have that $H_I^i(M) = 0$.*

The following theorem gives us information about certain local cohomology modules in the local case. It is of vanishing nature.

Theorem 3.0.8. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated A-module with dimension d. Then $H_{\mathfrak{m}}^i(M) = 0$ for every $i > d$ and $H_{\mathfrak{m}}^d(M) \neq 0$.*

Chapter 4

Graphs and Connectedness Dimension

Our goal in this chapter is to relate two concepts we have previously discussed during Chapter 2: the graphs Γ_t of certain kinds of rings and their connectedness dimension.

We also present some results about the relation of the connectedness dimension of rings and the connectedness dimension of those rings modulo certain elements. Finally we present new results which can be seen as variations of that ones.

First we state some useful properties to have in mind during this chapter.

Proposition 4.0.1. *Let A be a Noetherian ring and let $I_1, \dots, I_n, J_1, \dots, J_m$ be ideals of A, then*

1. $\operatorname{ht}(\bigcap_{i=1}^n I_i) = \min \{ \operatorname{ht}(I_i) \mid 1 \le i \le n \}$.
2. $\sqrt{\bigcap_{i=1}^n I_i + \bigcap_{j=1}^m J_j} = \sqrt{\bigcap_{i=1}^n \bigcap_{j=1}^m (I_i + J_j)}$.
3. $\operatorname{ht}(\bigcap_{i=1}^n I_i + \bigcap_{j=1}^m J_j) = \min \{ \operatorname{ht}(I_i + J_j) \mid 1 \le i \le n, 1 \le j \le m \}$.

Proof. Let $I = \bigcap_{i=1}^n I_i$ and let $J = \bigcap_{j=1}^m J_j$.

(1) Since for every j we have that $I \subseteq I_j$, then $\operatorname{ht}(I) \le \operatorname{ht}(I_j)$ for every j, so $\operatorname{ht}(I) \le \min \{ \operatorname{ht}(I_i) \mid 1 \le i \le n \}$. Now take a minimal prime P of I such that $\operatorname{ht}(P) = \operatorname{ht}(I)$. By prime avoidance, there must be a j such that

$I_j \subseteq P$, which means that $\mathrm{ht}(I_j) \leq \mathrm{ht}(P)$. So we have the following chain of inequalities

$$\min \{ \mathrm{ht}(I_i) \mid 1 \leq i \leq n \} \leq \mathrm{ht}(I_j) \leq \mathrm{ht}(P) = \mathrm{ht}(I).$$

Thus $\mathrm{ht}(I) = \min \{ \mathrm{ht}(I_i) \mid 1 \leq i \leq n \}$.

(2) Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be ideals of A. Notice that

$$(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}+\mathfrak{c}) = \mathfrak{a}(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}) + \mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a} + \mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a} + \mathfrak{b} \cap \mathfrak{c} \subseteq (\mathfrak{a}+\mathfrak{b}) \cap (\mathfrak{a}+\mathfrak{c})$$

By taking radicals on this chain of subsets, we see that both ends have the same radical, so all the ideals in the chain have the same radical. In particular $\sqrt{\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c})} = \sqrt{(\mathfrak{a}+\mathfrak{b}) \cap (\mathfrak{a}+\mathfrak{c})}$. Now, since I and J are the intersection of finitely many ideals, we use the previous equality several times in order to conclude that $\sqrt{I+J} = \sqrt{\bigcap_{i=1}^{n} \bigcap_{j=1}^{m} (I_i + J_j)}$.

(3) We know from (2) that $\sqrt{I+J} = \sqrt{\bigcap_{i=1}^{n} \bigcap_{j=1}^{m} (I_i + J_j)}$. This means that $\mathrm{ht}(I+J) = \mathrm{ht}\left(\bigcap_{i=1}^{n} \bigcap_{j=1}^{m} (I_i + J_j)\right)$. By (1), we conclude that

$$\mathrm{ht}(I+J) = \min \{ \mathrm{ht}(I_i + J_j) \mid 1 \leq i \leq n, 1 \leq j \leq m \}.$$

□

The following proposition is important for the proof of Lemma 4.0.23.

Proposition 4.0.2. *Let A be a Noetherian ring and let I be an ideal of A. Then* $\mathrm{Min}(I) \subseteq \mathrm{Ass}_A(A/I)$.

Connectedness dimension and Γ graphs are defined in terms of dimension and height respectively. It is worth noting that in the setting we use during this chapter we have a way of relating these two concepts.

Proposition 4.0.3. *Let A be a Noetherian complete local equidimensional ring and let I be an ideal of A. Then*

$$\mathrm{ht}(I) + \dim(A/I) = \dim(A).$$

A graph is connected if and only if no matter how we partition its set of indices in two non empty sets, we can always find an edge between a vertex of one of these two disjoint sets and a vertex of the other one. This is the idea behind the next proposition, which we use in the proof of Proposition 4.0.8.

Proposition 4.0.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d with more than one minimal prime. Let t be an integer such that $t \in [1, d-1]$. Then $\Gamma_t(A)$ is connected if and only if $\mathrm{ht}(\bigcap_{P \in S} P + \bigcap_{Q \in T} Q) \le t$ for every (S, T) partition of $\mathrm{Min}(A)$ such that S and T are non empty.*

Proof. By Proposition 4.0.1, we know that given a (S, T) partition of $\mathrm{Min}(A)$, we have that

$$\mathrm{ht}\left(\bigcap_{P \in S} P + \bigcap_{Q \in T} Q\right) = \min\left\{\, \mathrm{ht}(P+Q) \mid P \in S, Q \in T \,\right\},$$

so there must be $\mathfrak{p} \in S$ and $\mathfrak{q} \in T$ such that $\mathrm{ht}\left(\bigcap_{P \in S} P + \bigcap_{Q \in T} Q\right) = \mathrm{ht}(\mathfrak{p}+\mathfrak{q})$. This means that for every (S, T) partition of $\mathrm{Min}(A)$ such that S and T are non empty , we have that

$$\mathrm{ht}\left(\bigcap_{P \in S} P + \bigcap_{Q \in T} Q\right) \le t \Leftrightarrow \exists \mathfrak{p} \in S, \mathfrak{q} \in T : \mathrm{ht}(\mathfrak{p}+\mathfrak{q}) \le t.$$

So, for any such partition (S, T), you can find an edge between S and T. This happens if and only if $\Gamma_t(A)$ is connected. □

In Chapter 2 we mentioned some results that would allow us to keep working with quotient rings. The following proposition is similar to those in Chapter 2.

Proposition 4.0.5. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \ge 1$. Let $x \in \mathfrak{m}$ such that x is not an element of any minimal prime of A. Then $\mathrm{ht}(Q) = 1$ for every minimal prime Q of (x), which means $\mathrm{ht}(x) = 1$, and $A/(x)$ is a Noetherian equidimensional complete local ring of dimension $d-1$.*

Proof. Let Q be a minimal prime of (x). By Krull's principal ideal theorem, we know that $\mathrm{ht}(Q) \le 1$. Since x is not in any minimal prime of A, we have that Q cannot be a minimal prime of A, since that would mean $x \in Q$. This means that $\mathrm{ht}(Q)$ is not zero. So $\mathrm{ht}(Q) = 1$.

We know that $A/(x)$ is a Noetherian complete local ring, we only need to show that it is also equidimensional and of dimension $d-1$.

The dimension of this ring is $d-1$ since we are going modulo a parameter.

We know that the minimal primes of $A/(x)$ are of the form $Q/(x)$, with Q a minimal prime of (x). Let Q be a minimal prime of (x). Then $\dim\left(\frac{A}{(x)} / \frac{Q}{(x)}\right) =$

$\dim(A/Q)$. We know that in A the equality $\operatorname{ht}(Q) + \dim(A/Q) = \dim(A) = d$ holds from Proposition 4.0.3, and since we have already established that $\operatorname{ht}(Q) = 1$, we conclude that $\dim(A/Q) = d - 1$. So $\dim\left(\frac{A}{(x)}/\frac{Q}{(x)}\right) = d - 1$. This means $A/(x)$ is also equidimensional. □

Given a Γ graph we focus our attention in its subgraph corresponding to certain subset of minimal primes. One way to study such subgraph is by doing specific quotients of the ring.

Proposition 4.0.6. *Let* $(A, \mathfrak{m})$ *be a Noetherian equidimensional complete local ring and let* I *be a proper ideal of* A *such that* $\operatorname{Min}(I) \subseteq \operatorname{Min}(A)$. *Then* A/I *is also a Noetherian equidimensional complete local ring and* $\dim(A/I) = \dim(A)$. *Furthermore, if* J *is an ideal of* A *such that* $I \subseteq J$ *then* $\operatorname{ht}(J) = \operatorname{ht}(J/I)$. *In addition, if* Σ *is the subgraph of* $\Gamma_t(A)$ *whose vertices are* $\operatorname{Min}(I)$, *then*

$$\Sigma \cong \Gamma_t(A/I).$$

Proof. We know that A/I is a Noetherian complete local ring. We also know that the minimal primes of A/I are the ideals of the form P/I with P minimal prime of A.

Observe that $\dim\left(\frac{A}{I}/\frac{P}{I}\right) = \dim(A/P) = \dim(A)$ for every minimal prime P of A since A is equidimensional. This means A/I is also equidimensional and $\dim(A/I) = \dim(A)$.

Let J be an ideal of A such that $I \subseteq J$. By Proposition 4.0.3 we have that $\operatorname{ht}(J/I) + \dim\left(\frac{A}{I}/\frac{J}{I}\right) = \dim(A/I)$, so $\operatorname{ht}(J/I) = \dim(A) - \dim(A/J)$. Proposition 4.0.3 implies that $\operatorname{ht}(J) = \dim(A) - \dim(A/J)$. Thus $\operatorname{ht}(J) = \operatorname{ht}(J/I)$.

The correspondence between vertices of Σ and vertices of $\Gamma_t(A/I)$ is given by assigning each minimal prime P of I to the minimal prime P/I of A/I. Thus the vertices are preserved. Notice that edges are also preserved since if there is a edge between P and Q minimal primes of I, then $\operatorname{ht}(P + Q) \leq t$. This is the same as saying that $\operatorname{ht}(P/I + Q/I) = \operatorname{ht}\left(\frac{P+Q}{I}\right) \leq t$ since $\operatorname{ht}(P + Q) = \operatorname{ht}\left(\frac{P+Q}{I}\right)$ by the previous paragraph. □

In particular we choose I to be exactly the intersection of the minimal primes corresponding to the part of the graph we want to focus our attention on.

The next proposition gives us more information about how the graphs work when we study the quotient ring with different ideals but with the same radical.

Observation 4.0.7. *Let* $(A, \mathfrak{m})$ *be a Noetherian equidimensional complete local ring and let* I, J *be ideals of* A *such that* $\sqrt{I} = \sqrt{J}$. *Then both* A/I *and* A/J *are*

Noetherian complete local rings of the same dimension and if A/I is equidimensional, then A/J is also equidimensional and $\Gamma_t(A/I) \cong \Gamma_t(A/J)$.

This follows from the fact that $\mathrm{Spec}(A/I) \cong \mathrm{Spec}(A/J)$.

Now we are ready to begin exploring the relations between connectedness dimension and the Γ graphs.

Proposition 4.0.8 ([NnBSW19])**.** *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 2$. Let t be an integer such that $t \in [1, d-1]$. Then*

$$\Gamma_t(A) \text{ is connected } \Leftrightarrow \mathrm{c}(A) \geq d - t.$$

As a consequence, the connectedness dimension is given by

$$\mathrm{c}(A) = \max\{\, i \mid \Gamma_{d-i}(A) \text{ is connected } \}\,.$$

Proof. We use Proposition 2.3.5, Proposition 4.0.1 and Proposition 4.0.3 to obtain the following chain of equalities.

$$\begin{aligned}
\mathrm{c}(A) &= \mathrm{m}(A) \\
&= \min_{(S,T) \text{ is a partition of } \mathrm{Min}(A)} \left\{ \dim\left(\frac{A}{\bigcap_{P\in S} P + \bigcap_{Q\in T} Q} \right) \right\} \\
&= \min_{(S,T) \text{ is a partition of } \mathrm{Min}(A)} \left\{ \dim(A) - \mathrm{ht}\left(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q \right) \right\} \\
&= d + \min_{(S,T) \text{ is a partition of } \mathrm{Min}(A)} \left\{ -\,\mathrm{ht}\left(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q \right) \right\} \\
&= d - \max_{(S,T) \text{ is a partition of } \mathrm{Min}(A)} \left\{ \mathrm{ht}\left(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q \right) \right\} \\
&= d - \max_{(S,T) \text{ is a partition of } \mathrm{Min}(A)} \left\{ \mathrm{ht}\left(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q \right) \right\}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{c}(A) \geq d - t \quad &\Leftrightarrow \quad d - \max_{(S,T) \text{ is a partition of } \mathrm{Min}(A)} \left\{ \mathrm{ht}\left(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q \right) \right\} \geq d - t \\
&\Leftrightarrow \quad \max_{(S,T) \text{ is a partition of } \mathrm{Min}(A)} \left\{ \mathrm{ht}\left(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q \right) \right\} \leq t,
\end{aligned}$$

which is the same as saying that for every (S,T) partition of $\mathrm{Min}(A)$ such that S and T are non empty $\mathrm{ht}\left(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q\right) \leq t$. Proposition 4.0.4 implies that this happens if and only if $\Gamma_t(A)$ is connected. □

We can compute connectedness dimension by counting how many of the $\Gamma_t(A)$ graphs with $t \in [0, d-1]$ are connected.

Even if a graph $\Gamma_t(A)$ is not connected we can also obtain information regarding its connected components.

Definition 4.0.9. Let G be a graph and let X be a topological space. We denote $\#G$ to the amount of connected components of G and denote $\#X$ to the amount of connected components of the space X.

Corollary 4.0.10 ([NnBSW19]). *Let $(A,\mathfrak{m})$ be a Noetherian equidimensional complete local ring of dimension $d \geq 2$ and let t be an integer such that $t \in [1, d-1]$. Then:*

$$\#\Gamma_t(A) = \max\{\, \#\operatorname{Spec}(A) - V(I) \mid \dim(A/I) < d-t \,\}.$$

Proof. Suppose $\Gamma_t(A)$ has only one connected component. This means $\Gamma_t(A)$ is connected, so we know $\mathrm{c}(A) \leq d-t$ by Proposition 4.0.8. From the definition of connectedness dimension, this means that for any ideal $\mathfrak{a}$ such that $\dim(A/a) < d-t$ the space $\operatorname{Spec}(A) - V(\mathfrak{a})$ is connected. Take $\mathfrak{a} = \mathfrak{m}$ and notice that $\dim(A/\mathfrak{m}) = 0$ since $\mathfrak{m}$ is maximal. This means that the collection of all $\#\operatorname{Spec}(A) - V(I)$ such that $\dim(A/I) < d-t$ is not empty and it is equal to $\{1\}$, so its maximum is also 1.

Now suppose $\Gamma_t(A)$ has $n > 1$ connected components. We denote by $\Sigma_1, \ldots,$ Σ_n the n connected components of $\Gamma_t(A)$. Define the ideals $\mathfrak{b}_i = \bigcap_{P\in\Sigma_i} P$ and the ideal $\mathfrak{a} = \bigcap_{i<j} \mathfrak{b}_i + \mathfrak{b}_j$. From Proposition 4.0.1 we know that $\mathrm{ht}(\mathfrak{a}) > t$ which is the same as saying that $\dim(A/\mathfrak{a}) < d-t$ by Proposition 4.0.3.

Consider the sets $V(\mathfrak{b}_i) - V(\mathfrak{a})$. We prove that they form a disconnection of $\operatorname{Spec}(A) - V(\mathfrak{a})$.

Suppose $V(\mathfrak{b}_i) - V(\mathfrak{a})$ is empty. Then $V(\mathfrak{b}_i) \subseteq V(\mathfrak{a})$, so $\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}_i}$. Thus $\mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{b}_i)$. Since $\mathfrak{b}_i$ is an intersection of minimal primes, Proposition 4.0.1 implies that $\mathrm{ht}(\mathfrak{b}_i) = 0$. Then, $t < \mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{b}_i) = 0$, a contradiction. Thus the sets $V(\mathfrak{b}_i) - V(\mathfrak{a})$ are not empty.

We have the chain of equalities

$$\bigcup_i V(\mathfrak{b}_i) = V\left(\bigcap_i \mathfrak{b}_i\right) = V\left(\bigcap_{P\in\mathrm{Min}(A)} P\right) = V(\sqrt{0}) = \operatorname{Spec}(A).$$

This means that $\bigcup_i (V(\mathfrak{b}_i) - V(\mathfrak{a})) = (\bigcup_i V(\mathfrak{b}_i)) - V(\mathfrak{a}) = \operatorname{Spec}(A) - V(\mathfrak{a})$.

Given i, j such that $i \neq j$ we have that $\mathfrak{a} \subseteq \mathfrak{b}_i + \mathfrak{b}_j$. So $V(\mathfrak{b}_i + \mathfrak{b}_j) \subseteq V(\mathfrak{a})$. This means that $V(\mathfrak{b}_i + \mathfrak{b}_j) - V(\mathfrak{a})$ is empty. But $(V(\mathfrak{b}_i) - V(\mathfrak{a})) \cap (V(\mathfrak{b}_j) - V(\mathfrak{a})) = V(\mathfrak{b}_i + \mathfrak{b}_j) - V(\mathfrak{a})$, thus $(V(\mathfrak{b}_i) - V(\mathfrak{a})) \cap (V(\mathfrak{b}_j) - V(\mathfrak{a}))$ is empty.

So the sets $V(\mathfrak{b}_i) - V(\mathfrak{a})$ form a partition of $\operatorname{Spec}(A) - V(\mathfrak{a})$ by non empty sets. Notice that they are open since $\operatorname{Spec}(A) - V(\mathfrak{a})$ is open. This means that they disconnect $\operatorname{Spec}(A) - V(\mathfrak{a})$ and then $n \leq \# \operatorname{Spec}(A) - V(\mathfrak{a})$.

We conclude that $\#\Gamma_t(A) \leq \max \{ \# \operatorname{Spec}(A) - V(I) \mid \dim(A/I) < d - t \}$.

Now let $\mathfrak{c}$ be an ideal of A such that $\dim(A/\mathfrak{c}) < d - t$, equivalently $\operatorname{ht}(\mathfrak{c}) > t$. Let $m = \# \operatorname{Spec}(A) - V(\mathfrak{c})$ and let $V(\mathfrak{c}_i) - V(\mathfrak{c})$ be its connected components.

If $\operatorname{Spec}(A) - V(\mathfrak{c})$ is connected, that is $m = 1$, then $n \geq m$.

Now suppose $\operatorname{Spec}(A) - V(\mathfrak{c})$ is disconnected. Notice that $\operatorname{Spec}(A) - V(\mathfrak{c})$ is not empty since that would mean that $\operatorname{ht}(\mathfrak{c}) = 0$, a contradiction.

Notice that the minimal primes of A are not in $V(\mathfrak{c})$, since that would mean that $\operatorname{ht}(\mathfrak{c}) = 0$. Also observe that each minimal prime of A belong to one and only one of the $V(\mathfrak{c}_i) - V(\mathfrak{c})$.

Now we prove that there is no edge between the minimal primes of A that belong to $V(\mathfrak{c}_i) - V(\mathfrak{c})$ and those who belong to $V(\mathfrak{c}_j) - V(\mathfrak{c})$, whenever $i \neq j$. Let $i \neq j$. We know that $V(\mathfrak{c}_i + c_j) - V(\mathfrak{c}) = (V(\mathfrak{c}_i) - V(\mathfrak{c})) \cap (V(\mathfrak{c}_j) - V(\mathfrak{c})) = \emptyset$. Then, $V(\mathfrak{c}_i + \mathfrak{c}_j) \subseteq V(\mathfrak{c})$, and so, $\sqrt{\mathfrak{c}} \subseteq \sqrt{\mathfrak{c}_i + \mathfrak{c}_j}$. Therefore $t < \operatorname{ht}(\mathfrak{c}) \leq \operatorname{ht}(\mathfrak{c}_i + \mathfrak{c}_j)$. Observe that there is no pair of minimal primes $P \in V(\mathfrak{c}_i) - V(\mathfrak{c})$ and $Q \in V(\mathfrak{c}_j) - V(\mathfrak{c})$ such that $\operatorname{ht}(P + Q) \leq t$, since that would contradict the fact that $\operatorname{ht}(\mathfrak{c}_i + \mathfrak{c}_j) > t$. This means that $n \geq m$.

We conclude that $\#\Gamma_t(A) \geq \max \{ \# \operatorname{Spec}(A) - V(I) \mid \dim(A/I) < d - t \}$.

□

Now we proceed to study what happens when we go modulo a parameter. We need additional tools in order to do so. The following lemma gives us information about the behaviour between minimal primes of a ring and the minimal primes of an ideal generated by a parameter. We only prove the second statement.

Lemma 4.0.11. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 1$. Let $x \in \mathfrak{m}$ such that x is not an element of any minimal prime of A. Then*

1. *For every minimal prime Q of (x), there is a minimal prime P of A such that $P \subseteq Q$.*

2. *For every minimal prime P of A, there is a minimal prime Q of (x) such that $P \subseteq Q$.*

Proof. Let P be a minimal prime of A. Notice that $\frac{A}{P}$ is also a Noetherian equidimensional complete local ring of dimension d, and that $\overline{x}$ is not contained in the unique minimal prime of $\frac{A}{P}$, so by Proposition 4.0.5 we know that $\frac{A}{P}/(\overline{x}) \cong \frac{A}{P+(x)}$ is also a Noetherian equidimensional complete local ring of dimension $d-1$ and that $\operatorname{ht}(\overline{x}) = 1$.

Since

$$\operatorname{ht}(\overline{x}) + \dim\left(\frac{\frac{A}{P}}{(\overline{x})}\right) = \dim\left(\frac{A}{P}\right),$$

we conclude that

$$\dim\left(\frac{A}{P+(x)}\right) = d-1.$$

We also know that

$$\operatorname{ht}(P+(x)) + \dim\left(\frac{A}{P+(x)}\right) = \dim(A),$$

and so

$$\operatorname{ht}(P+(x)) = 1.$$

Now take $Q \in \operatorname{Min}(P+(x))$ such that $\operatorname{ht}(Q) = \operatorname{ht}(P+(x))$. Since $(x) \subseteq P+(x)$ and $\operatorname{ht}(x) = \operatorname{ht}(P+(x))$ by Proposition 4.0.5, then $Q \in \operatorname{Min}(x)$. Finally $P \subseteq Q$ because $Q \in \operatorname{Min}(P+(x))$. □

From the last part of the proof of Lemma 4.0.11 we deduce that every $Q \in \operatorname{Min}(P+(x))$ such that $\operatorname{ht}(Q) = 1$ must be a minimal prime of (x). It turns out that every minimal prime of $P+(x)$ is of height 1. Then $\operatorname{Min}(P+(x)) \subseteq \operatorname{Min}(x)$. Furthermore, $\operatorname{Min}(P+(x))$ is the set of all $Q \in \operatorname{Min}(x)$ such that $P \subseteq Q$.

Corollary 4.0.12. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 1$. Let $x \in \mathfrak{m}$ such that x is not an element of any minimal prime of A. For every minimal prime P of A, we have that*

$$\operatorname{Min}(P+(x)) = \{Q \in \operatorname{Min}(x) \mid P \subseteq Q\}.$$

Proof. We proceed by double containment. Let $X = \{Q \in \operatorname{Min}(x) \mid P \subseteq Q\}$

Take a minimal prime Q of (x) that contains P. Since $P+(x) \subseteq Q$ and $\operatorname{ht}(P+(x)) = \operatorname{ht}(Q)$, then Q must be a minimal prime of $P+(x)$. So $X \subseteq \operatorname{Min}(P+(x))$.

Now, let $P \in \operatorname{Min}(A)$ and let $Q \in \operatorname{Min}(P + (x))$. We show that $\operatorname{ht}(Q) = 1$. From the proof of Lemma 4.0.11, we know that $\frac{A}{P+(x)}$ is a Noetherian equidimensional complete local ring of dimension $d - 1$. We know from Proposition 4.0.3 that

$$\operatorname{ht}\left(\frac{Q}{P + (x)}\right) + \dim\left(\frac{\frac{A}{P+(x)}}{\frac{Q}{P+(x)}}\right) = \dim\left(\frac{A}{P + (x)}\right).$$

Since $\frac{Q}{P+(x)} \in \operatorname{Min}\left(\frac{A}{P+(x)}\right)$, we get that $\operatorname{ht}\left(\frac{Q}{P+(x)}\right) = 0$. We conclude that

$$\dim\left(\frac{\frac{A}{P+(x)}}{\frac{Q}{P+(x)}}\right) = \dim\left(\frac{A}{Q}\right) = d - 1.$$

We also know that $\operatorname{ht}(Q) + \dim(A/Q) = \dim(A)$, and so $\operatorname{ht}(Q) = 1$. This means that Q is a minimal prime of (x) and we knew from the beginning that it contained P. So $\operatorname{Min}(P + (x)) \subseteq X$. □

The following definition plays a key role during the proof of Theorem 4.0.25.

Definition 4.0.13. Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 1$. Let $x \in \mathfrak{m}$ be such that x is not an element of any minimal prime of A. Given a minimal prime P of A, we define the dust of P to be the set

$$D(P) = \{Q \in \operatorname{Min}(x) \mid P \subseteq Q\}.$$

Furthermore if Σ is a subgraph of $\Gamma_t(A)$, then

$$D(\Sigma) = \bigcup_{P \in \Sigma} D(P).$$

Definition 4.0.14. Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 1$. Let $x \in \mathfrak{m}$ be such that x is not an element of any minimal prime of A. Let Σ be a subgraph of $\Gamma_t(A)$. Let Σ' be the subgraph of $\Gamma_t(A/(x))$ such that its vertices are given by $Q/(x)$ such that $Q \in D(\Sigma)$. We call Σ' the associated graph to Σ.

Notice that from Corollary 4.0.12 we know $D(P) = \operatorname{Min}(P + (x))$. From Lemma 4.0.11, we can deduce that $\bigcup_{P \in \operatorname{Min}(A)} D(P) = \operatorname{Min}(x)$.

Lemma 4.0.15. *Let* $(A, \mathfrak{m})$ *be a Noetherian equidimensional complete local ring of* $\dim(A) = d \geq 1$*. Let* $x \in \mathfrak{m}$ *such that* x *is a not an element of any minimal prime of* A*. Let* S *be a non empty subset of* $\operatorname{Min}(A)$ *and let* $I = \bigcap_{P \in S} P$*. Then*

$$\operatorname{Min}(I + (x)) = \bigcup_{P \in S} \operatorname{Min}(P + (x)).$$

Proof. We proceed by double containment.

First we prove that $\operatorname{Min}(I+(x)) \subseteq \bigcup_{P \in S} \operatorname{Min}(P+(x))$. Take a minimal prime Q of $I + (x)$. By prime avoidance Q contains a prime $P \in S$. But $I + (x) \subseteq P + (x) \subseteq Q$. This implies that Q is also a minimal prime of $P + (x)$. Notice this also means that $\operatorname{ht}(I + (x)) = 1$ since all the minimal primes of $P + (x)$ are of height 1 by Corollary 4.0.12.

Now, we prove that $\bigcup_{P \in S} \operatorname{Min}(P+(x)) \subseteq \operatorname{Min}(I+(x))$. Let $P \in S$ and let Q be a minimal prime of $P+(x)$. Since $I+(x) \subseteq P+(x)$ and $\operatorname{ht}(I+(x)) = \operatorname{ht}(P+(x)) = \operatorname{ht}(Q)$, we deduce that Q must also be a minimal prime of $I + (x)$. $\square$

In the previous setting let Σ be the subgraph of $\Gamma_t(A)$ whose vertices are the elements of S. Observe that $D(\Sigma) = \operatorname{Min}(I+(x))$, since $D(P) = \operatorname{Min}(P+(x))$.

Now we are ready for our study of connectedness dimension modulo a parameter. It turns out that if $\Gamma_t(A)$ is connected, then $\Gamma_t(A/(x))$ is also connected. The only moment when this is not necessarily true is when $t = 0$, as the following example shows.

Let K be a field and consider the power series ring $A = K[[s, t, u]]$. $\Gamma_t(A)$ is connected for every t since A is a domain, but $\Gamma_0(A/(stu))$ is not connected since it has more than one minimal prime.

Additionally, observe that we restrict t to be less or equal than $d - 2$. We do so because $\Gamma_{d-1}(A/(x))$ is connected regardless the connectedness of $\Gamma_{d-1}(A)$.

Theorem 4.0.16 ([NnBSW19])**.** *Let* $(A, \mathfrak{m})$ *be a Noetherian equidimensional complete local ring containing a field, of* $\dim(A) = d \geq 3$*, with separably closed residue field. Let* $x \in \mathfrak{m}$ *such that* x *is a not an element of any minimal prime of* A*. Let* t *be an integer such that* $t \in [1, d-2]$*. Then,*

$$\Gamma_t(A) \textit{ is connected} \Rightarrow \Gamma_t(A/(x)) \textit{ is connected}.$$

Proof. Suppose A has more than one minimal prime. Recall that $\operatorname{c}(A) = \dim(A)$ if and only if A has only one minimal prime, so $\operatorname{c}(A) < \dim(A)$ in this case. We know from Theorem 2.3.3 that

$$\operatorname{c}(A/(x)) \geq \min\{\, \operatorname{c}(A), \dim(A) - 1 \,\} - \operatorname{ara}(x).$$

Since $\text{ara}(x) = 1$ because (x) is a principal ideal, we conclude that

$$\text{c}(A/(x)) \geq \text{c}(A) - 1.$$

Since $\Gamma_t(A)$ is connected, Proposition 4.0.8 implies that $\text{c}(A) \geq d - t$. Thus,

$$\text{c}(A/(x)) \geq (d - t) - 1 = (d - 1) - t.$$

Once again Proposition 4.0.8 implies that $\Gamma_t(A/(x))$ is connected.

Now suppose A has only one minimal prime P. From Proposition 4.0.6 we know that $\Gamma_t(A) \cong \Gamma_t(A/P)$. Notice that A/P is a domain and that $\overline{x} \in A/P$ does not belong to any minimal prime.

From Zhang's work (proposition 2.2, reference 8) we know that since $\Gamma_1(A/P)$ is connected, we have that

$$\Gamma_1\left(\frac{\frac{A}{P}}{(\overline{x})}\right) \cong \Gamma_1\left(\frac{A}{P + (x)}\right)$$

is connected too.

This means that $\Gamma_t\left(\frac{A}{P+(x)}\right)$ is connected for every $t \in [1, d-2]$.

From Corollary 4.0.12 we know that $\text{Min}(P+(x)) = \{\, Q \in \text{Min}(x) \mid P \subseteq Q \,\}$, and since P is the only minimal prime of A, we deduce that every prime ideal contains P. This means that $\text{Min}(P + (x)) = \text{Min}(x)$. So $\sqrt{P + (x)} = \sqrt{(x)}$.

It follows from Proposition 4.0.5 and Observation 4.0.7 that

$$\Gamma_t\left(\frac{A}{P + (x)}\right) \cong \Gamma_t(A/(x)).$$

Thus, $\Gamma_t(A/(x))$ must be connected too. □

Corollary 4.0.17. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring containing a field, of $\dim(A) = d \geq 3$, with separably closed residue field. Let $x \in \mathfrak{m}$ such that x is a not an element of any minimal prime of A. Let t be an integer such that $t \in [1, d-2]$. Let Σ be a subgraph of $\Gamma_t(A)$ and let Σ' be the subgraph of $\Gamma_t(A/(x))$ associated to $D(\Sigma)$. Then,*

$$\Sigma \textit{ is connected} \Rightarrow \Sigma' \textit{ is connected.}$$

Proof. Suppose Σ is connected. Let I be the intersection of all the vertices of Σ. From Proposition 4.0.6 we know that $\Sigma \cong \Gamma_t(A/I)$, so $\Gamma_t(A/I)$ is also connected and Theorem 4.0.16 implies that $\Gamma_t\left(\frac{A}{I+(x)}\right)$ is also connected.

From Lemma 4.0.15 we know that $\sqrt{I+(x)} = \bigcap_{Q\in \operatorname{Min}(I+(x))} Q = \bigcap_{Q\in D(\Sigma)} Q$. Observation 4.0.7 implies that $\Gamma_t\left(\frac{A}{I+(x)}\right) \cong \Gamma_t\left(\frac{A}{\bigcap_{Q\in D(\Sigma)} Q}\right)$.

Notice that Proposition 4.0.6 implies that

$$\Gamma_t\left(\frac{A}{\bigcap_{Q\in D(\Sigma)} Q}\right) \cong \Gamma_t\left(\frac{\frac{A}{(x)}}{\bigcap_{Q\in D(\Sigma)} \frac{Q}{(x)}}\right) \cong \Sigma'.$$

We conclude that Σ' is also connected. □

The converse is also true if we add more restrictions to our paramater. In order to do so we need the following definition:

Definition 4.0.18 ([NnBSW19])**.** Let $(A,\mathfrak{m})$ be a Noetherian local ring. We define the following set of ideals:

$$\xi(A) = \left\{ P+Q \;\middle|\; P,Q \in \operatorname{Min}(A) \text{ such that } \sqrt{P+Q} \subsetneq \mathfrak{m} \right\}.$$

Notice that if the dimension of the ring is positive, all the minimal primes of A belong to $\xi(A)$.

The condition we add in order to obtain the converse of Theorem 4.0.16 is that the parameter is not in any minimal prime of any ideal of the set $\xi(A)$. This can be done using prime avoidance since there is only a finite amount of ideals in $\xi(A)$.

Theorem 4.0.19 ([NnBSW19])**.** *Let $(A,\mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 3$. Let $x \in \mathfrak{m}$ such that x is not in any minimal prime of any ideal of the set $\xi(A)$. Let t be an integer such that $t \in [1, d-2]$. Then,*

$$\Gamma_t(A/(x)) \textit{ is connected } \Rightarrow \Gamma_t(A) \textit{ is connected.}$$

Proof. Observe that if $\Gamma_t(A/(x))$ is connected, then $\Gamma_{t+1}(A)$ is connected too. This follows from the fact that for any ideal I such that $(x) \subseteq I$ we have that $\operatorname{ht}(I/(x)) + 1 = \operatorname{ht}(I)$, which is true by Proposition 4.0.3. Proposition 4.0.8 implies that $\operatorname{c}(A) \geq d-(t+1)$. Since $t \in [1, d-2]$, we get that $\operatorname{c}(A) \geq 1$.

Let (S,T) be a partition of $\operatorname{Min}(A)$ such that $\operatorname{c}(A) = \dim\left(\frac{A}{\bigcap_{P\in S} P + \bigcap_{Q\in T} Q}\right)$. If S or T is empty, then $\operatorname{c}(A) = \dim(A)$. This implies that A has only one minimal prime, so $\Gamma_t(A)$ is connected. Thus we can assume that neither of them is empty.

Suppose $\sqrt{P+Q} = \mathfrak{m}$ for every $P \in S, Q \in T$. Then, $\operatorname{ht}(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q) = d$ and $\dim\left(\frac{A}{\bigcap_{P\in S} P+\bigcap_{Q\in T} Q}\right) = 0$ by Proposition 4.0.3. This means

$\mathrm{c}(A) = 0$, a contradiction. We conclude that there must be at least some $P \in S$ and $Q \in T$ such that $\sqrt{P+Q} \subsetneq \mathfrak{m}$.

Observe that for any such P and Q, we have that x is not in any minimal prime of $P+Q$. Thus, $\overline{x}$ is a parameter of $\frac{A}{P+Q}$. This means that $\dim\left(\frac{A}{P+Q}\right) = \dim\left(\frac{A}{P+Q+(x)}\right) + 1$. Proposition 4.0.3 implies that $\mathrm{ht}(P+Q) = \mathrm{ht}(P+Q+(x)) - 1$.

With this last observation in mind we have that

$$\begin{aligned}
\mathrm{c}(A) &= \dim\left(\frac{A}{\bigcap_{P\in S} P + \bigcap_{Q\in T} Q}\right) \\
&= d - \mathrm{ht}\left(\bigcap_{P\in S} P + \bigcap_{Q\in T} Q\right) \\
&= d - \min\{\, \mathrm{ht}(P+Q) \mid P \in S, Q \in T \,\} \\
&= d - \min\{\, \mathrm{ht}(P+Q+(x)) - 1 \mid P \in S, Q \in T \,\} \\
&= d - \min\{\, \mathrm{ht}(P+Q+(x)) \mid P \in S, Q \in T \,\} + 1 \\
&= d - \min\{\, \mathrm{ht}((P+(x)) + (Q+(x))) \mid P \in S, Q \in T \,\} + 1 \\
&= d - \mathrm{ht}\left(\bigcap_{P\in S}(P+(x)) + \bigcap_{Q\in T}(Q+(x))\right) + 1 \\
&= \dim\left(\frac{A}{\bigcap_{P\in S}(P+(x)) + \bigcap_{Q\in T}(Q+(x))}\right) + 1 \\
&\geq \dim\left(\frac{A}{\bigcap_{\mathfrak{p}\in D(S)} \mathfrak{p} + \bigcap_{\mathfrak{q}\in D(T)} \mathfrak{q}}\right) + 1 \\
&= \dim\left(\frac{\frac{A}{(x)}}{\bigcap_{\mathfrak{p}\in D(S)} \frac{\mathfrak{p}}{(x)} + \bigcap_{\mathfrak{q}\in D(T)} \frac{\mathfrak{q}}{(x)}}\right) + 1 \\
&\geq \mathrm{c}(A/(x)) + 1.
\end{aligned}$$

Since $\Gamma_t(A/(x))$ is connected, $\mathrm{c}(A/(x)) \geq (d-1) - t$ by Proposition 4.0.8. So $\mathrm{c}(A) - 1 \geq (d-1) - t$. That is, $\mathrm{c}(A) \geq d - t$. We conclude, again by Proposition 4.0.4, that $\Gamma_t(A)$ is connected. □

Corollary 4.0.20. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 3$. Let $x \in \mathfrak{m}$ such that x is not in any minimal prime of* *any ideal of the set $\xi(A)$. Let t be an integer such that $t \in [1, d-2]$. Then*

$$\mathrm{c}(A) = \mathrm{c}(A/(x)) + 1.$$

Lemma 4.0.21. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring of $\dim(A) = d \geq 1$. Let $x \in \mathfrak{m}$ such that x is not in any minimal prime of A. Let t be an integer such that $t \in [1, d-2]$. Let Σ_i and Σ_j be subgraphs of $\Gamma_t(A)$. If the graphs Σ_i and Σ_j do not share any vertices and there are no edges between them, then $D(\Sigma_i)$ and $D(\Sigma_j)$ are disjoint. In particular the subgraphs $\Sigma_i^{'}$ and $\Sigma_j^{'}$ of $\Gamma_t(A/(x))$ associated to Σ_i and Σ_j respectively do not share vertices.*

Proof. Let $Q \in D(\Sigma_i) \cap D(\Sigma_j)$, then there are $P_i \in \Sigma_i$ and $P_j \in \Sigma_j$ such that $Q \in D(P_i) \cap D(P_j)$. So $P_i + P_j \subseteq Q$, this means that $\mathrm{ht}(P_i + P_j) \leq \mathrm{ht}(Q) = 1 \leq t$. So there is an edge between Σ_i and Σ_j, a contradiction. □

Theorem 4.0.22 ([NnBSW19])**.** *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring containing a field of $\dim(A) = d \geq 3$ with separably closed residue field. Let $x \in \mathfrak{m}$ such that x is not in any minimal prime of any ideal of the set $\xi(A)$. Let t be an integer such that $t \in [1, d-2]$. Then*

$$\#\Gamma_t(A) = \#\Gamma_t(A/(x)).$$

Proof. Suppose $\#\Gamma_t(A) = s$. Let $\Sigma_1, \ldots, \Sigma_s$ be the s connected components of $\Gamma_t(A)$. Let $\Sigma_1^{'}, \ldots, \Sigma_s^{'}$ be the subgraphs of $\Gamma_t(A/(x)$ associated to the sets $D(\Sigma_1), \ldots, D(\Sigma_s)$ respectively. We show that the associated graphs are the connected components of $\Gamma_t(A/(x)$.

Define ideals $\mathfrak{a}_i = \bigcap_{P \in \Sigma_i} P$. From Corollary 4.0.17 and its proof we know that $\Sigma_i^{'} \cong \Gamma_t\left(\frac{A}{\mathfrak{a}_i + (x)}\right)$ is also connected for each i.

From Lemma 4.0.21 we know that for distinct i and j, the graphs $\Sigma_i^{'}$ and $\Sigma_j^{'}$ do not share vertices. Then they are distinct connected subgraphs of $\Gamma_t(A/(x))$.

It remains to show that for every pair of distinct $\Sigma_i^{'}$ and $\Sigma_j^{'}$ there are no edges between them, so they are the connected components of $\Gamma_t(A/(x))$.

Suppose there is and edge between a pair of distinct $\Sigma_i^{'}$ and $\Sigma_j^{'}$. Then, the graph $\Gamma_t\left(\frac{A}{(\mathfrak{a}_i \cap \mathfrak{a}_j) + (x)}\right)$ is connected by Proposition 4.0.6 and Lemma 4.0.15. Since x is not in any minimal prime of any ideal of the set $\xi(A)$, we deduce that $\overline{x}$ is not in any minimal prime of any ideal of the set $\xi\left(\frac{A}{\mathfrak{a}_i + \mathfrak{a}_j}\right)$. This means that $\Gamma_t\left(\frac{A}{\mathfrak{a}_i \cap \mathfrak{a}_j}\right)$ is also connected by Theorem 4.0.24. This implies there is some edge between Σ_i and Σ_j, a contradiction. □

By changing a little bit the hypothesis of the previous four results, we show in the following results that we obtain the same results. First a lemma that helps us prove Theorem 4.0.24.

Lemma 4.0.23. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring. Suppose there is an $x \in \mathfrak{m}$ such that x is a non zero divisor of A and that (x) is a radical ideal. Let (S, T) be a partition of $\operatorname{Min}(A)$ such that S and T are non empty, and $I = \bigcap_{P \in S} P$ and $J = \bigcap_{Q \in T} Q$, then x is a non zero divisor of $\frac{A}{I+J}$. In particular x is not in any minimal prime of $I + J$.*

Proof. Let (S, T), I and J be as above and consider the exact sequence

$$0 \to \frac{A}{I \cap J} \to \frac{A}{I} \oplus \frac{A}{J} \to \frac{A}{I + J} \to 0.$$

This sequence induces a long exact sequence of Tor of the form

$$\ldots \to \operatorname{Tor}_1\left(\frac{A}{I} \oplus \frac{A}{J}, \frac{A}{(x)}\right) \to \operatorname{Tor}_1\left(\frac{A}{I+J}, \frac{A}{(x)}\right)$$

$$\to \frac{A}{I \cap J} \otimes \frac{A}{(x)} \to \left(\frac{A}{I} \oplus \frac{A}{J}\right) \otimes \frac{A}{(x)} \to \frac{A}{I + J} \otimes \frac{A}{(x)} \to 0.$$

Since x is a non zero divisor of A, then $\operatorname{Tor}_1(A/I, A/(x)) = \operatorname{Ann}_{A/I}(x)$. Let $\overline{a} \in \operatorname{Ann}_{A/I}(x)$. This means that $ax \in I = \bigcap_{P \in S} P$. Since x is not in any minimal prime of A, we conclude that a must belong to $\bigcap_{P \in S} P$. This means that $\overline{a} = 0$ and so, $\operatorname{Ann}_{A/I}(x) = 0$. Similarly $\operatorname{Tor}_1(A/J, A/(x)) = \operatorname{Ann}_{A/J}(x) = 0$. Then,

$$\operatorname{Tor}_1\left(\frac{A}{I} \oplus \frac{A}{J}, \frac{A}{(x)}\right) = \operatorname{Tor}_1\left(\frac{A}{I}, \frac{A}{(x)}\right) \oplus \operatorname{Tor}_1\left(\frac{A}{J}, \frac{A}{(x)}\right) = 0.$$

We also know that $\operatorname{Tor}_1\left(\frac{A}{I+J}, \frac{A}{(x)}\right) = \operatorname{Ann}_{\frac{A}{I+J}}(x)$ and by simplifying tensor products, we get

$$0 \to \operatorname{Ann}_{\frac{A}{I+J}}(x) \to \frac{A}{I \cap J + (x)} \to \frac{A}{I + (x)} \oplus \frac{A}{J + (x)} \to \frac{A}{I + J + (x)} \to 0.$$

Observe that

$$\begin{aligned}(x) &\subseteq \sqrt{0}+(x)\\ &= I\cap J+(x)\\ &\subseteq (I+(x))\cap(J+(x))\\ &\subseteq \sqrt{(I+(x))\cap(J+(x))}\\ &= \sqrt{I\cap J+(x)}\\ &= \sqrt{\sqrt{0}+\sqrt{(x)}}\\ &= \sqrt{0+(x)}\\ &= \sqrt{(x)}\\ &= (x),\end{aligned}$$

which means we have $I\cap J+(x) = (I+(x))\cap(J+(x))$. Since the sequence

$$0 \to \frac{A}{(I+(x))\cap(J+(x))} \to \frac{A}{I+(x)} \oplus \frac{A}{J+(x)} \to \frac{A}{(I+(x))+(J+(x))} \to 0$$

is exact, we conclude that $\operatorname{Ann}_{\frac{A}{I+J}}(x) = 0$. This means that x is a non zero divisor of $\frac{A}{I+J}$, so x is not in any minimal prime of $I+J$. $\square$

Theorem 4.0.24. *Let $(A,\mathfrak{m})$ be a Noetherian equidimensional complete local ring with $\dim(A) = d \geq 3$. Suppose there exists an $x \in \mathfrak{m}$ such that x is a non zero divisor of A and that (x) is a radical ideal. Let t be an integer such that $t \in [1, d-2]$. Then*

$$\Gamma_t(A/(x)) \text{ is connected} \Rightarrow \Gamma_t(A) \text{ is connected.}$$

As a consequence

$$\mathrm{c}(A) = \mathrm{c}(A/(x)) + 1.$$

Proof. We know $c(A) = \dim\left(\frac{A}{I+J}\right)$ where I and J are the intersection of all the elements of S and T respectively, for some partition (S,T) of $\operatorname{Min}(A)$.

From Lemma 4.0.23, we know that x is not an element of any minimal prime of $I+J$, so

$$\begin{aligned}
\mathrm{c}(A) &= \dim\left(\frac{A}{I+J}\right) \\
&= \dim\left(\frac{A}{I+J+(x)}\right)+1 \\
&\geq \dim\left(\frac{A}{\bigcap_{P\in S}\bigcap_{Q\in D(P)} Q+\bigcap_{P\in T}\bigcap_{Q\in D(P)} Q}\right)+1 \\
&= \dim\left(\frac{\frac{A}{(x)}}{\bigcap_{P\in S}\bigcap_{Q\in D(P)} \frac{Q}{(x)}+\bigcap_{P\in T}\bigcap_{Q\in D(P)} \frac{Q}{(x)}}\right)+1 \\
&\geq \mathrm{c}(A/(x))+1.
\end{aligned}$$

Suppose $\Gamma_t(A/(x))$ is connected. From Proposition 4.0.8 we have the inequality $\mathrm{c}(A/(x)) \geq (d-1)-t$, so $\mathrm{c}(A/x)+1 \geq d-t$. From our previous chain of inequalities, we get that $\mathrm{c}(A) \geq d-t$. We conclude that $\Gamma_t(A)$ is connected. $\square$

Theorem 4.0.25. *Let $(A, \mathfrak{m})$ be a Noetherian equidimensional complete local ring containing a field, of $\dim(A) = d \geq 3$, with separably closed residue field. Suppose there exists $x \in \mathfrak{m}$ such that x is a non zero divisor of A and that (x) is a radical ideal. Let t be an integer such that $t \in [1, d-2]$. Then*

$$\#\Gamma_t(A) = \#\Gamma_t(A/(x)).$$

Proof. Suppose $\#\Gamma_t(A) = s$. Let $\Sigma_1, \ldots, \Sigma_s$ be the s connected components of $\Gamma_t(A)$. Let $\Sigma_1^{'}, \ldots, \Sigma_s^{'}$ be the subgraphs of $\Gamma_t(A/(x)$ associated to the sets $D(\Sigma_1), \ldots, D(\Sigma_s)$ respectively. We show that the associated graphs are the connected components of $\Gamma_t(A/(x))$.

Define ideals $\mathfrak{a}_i = \bigcap_{P\in\Sigma_i} P$. From Corollary 4.0.17 and its proof we know that $\Sigma_i^{'} \cong \Gamma_t\left(\frac{A}{\mathfrak{a}_i+(x)}\right)$ is also connected for each i.

From Lemma 4.0.21 we know that for distinct i and j, the graphs $\Sigma_i^{'}$ and $\Sigma_j^{'}$ do not share vertices. Thus they are distinct connected subgraphs of $\Gamma_t(A/(x))$.

It remains to show that for every pair of distinct $\Sigma_i^{'}$ and $\Sigma_j^{'}$ there are no edges between them, so they are indeed the connected components of $\Gamma_t(A/(x))$.

For $i \neq j$, suppose there is an edge between $\mathfrak{q}_1/(x) \in \Sigma_i^{'}$ and $\mathfrak{q}_2/(x) \in \Sigma_j^{'}$. Let S be the set of vertices of $\Gamma_t(A)$ in Σ_i and let T be the set of vertices of $\Gamma_t(A)$ which are not in Σ_i. Notice that (S, T) is a partition of $\mathrm{Min}(A)$. Let I and J be the intersection of all the elements of S and T respectively. Take $\mathfrak{p}_1$ and $\mathfrak{p}_2$

such that $\mathfrak{q}_1 \in D(\mathfrak{p}_1)$ and $\mathfrak{q}_2 \in D(\mathfrak{p}_2)$. Since $I + J \subseteq \mathfrak{p}_1 + \mathfrak{p}_2 + (x)$, we have that $\mathrm{ht}(I + J) \leq \mathrm{ht}(\mathfrak{p}_1 + \mathfrak{p}_2 + (x))$. Suppose equality holds and take a minimal prime Q of $\mathfrak{p}_1 + \mathfrak{p}_2 + (x)$ such that $\mathrm{ht}(Q) = \mathrm{ht}(\mathfrak{p}_1 + \mathfrak{p}_2 + (x))$. Since $I + J$ and $\mathfrak{p}_1 + \mathfrak{p}_2 + (x)$ have the same height, Q must also be a minimal prime of $I + J$. But Lemma 4.0.23 prevents this from happening since $x \in Q$. We have that

$$\begin{aligned} \mathrm{ht}(I + J) + 1 &\leq \mathrm{ht}(\mathfrak{p}_1 + \mathfrak{p}_2 + (x)) \\ &\leq \mathrm{ht}(\mathfrak{q}_1 + \mathfrak{q}_2) \\ &= \mathrm{ht}(\mathfrak{q}_1/(x) + \mathfrak{q}_2/(x)) + 1. \end{aligned}$$

Thus $\mathrm{ht}(I + J) \leq \mathrm{ht}(\mathfrak{q}_1/(x) + \mathfrak{q}_2/(x)) \leq t$. From the proof of Proposition 4.0.4 we know this means there is an edge between some prime in S and some prime in T. Then, there is an edge between a vertex of Σ_i and a vertex of another connected component of $\Gamma_t(A)$, a contradiction.

We conclude that $\Sigma_1^{'}, \Sigma_2^{'}, \ldots, \Sigma_s^{'}$ are the connected components of $\Gamma_t(A/(x))$. $\square$

Chapter 5

Local Cohomology and Graphs

In this final chapter we study the connection between local cohomology and the connectivity of the punctured spectrum of a ring, that is, the subspace of the spectrum of a local ring in which we only remove the singleton containing the maximal ideal.

First we state some technical results we need in order to prove the main result in this chapter.

Theorem 5.0.1 (Hartshorne-Lichtenbaum). *Let $(A, \mathfrak{m})$ be a Noetherian complete local domain of dimension d. If I is a proper ideal of A and I is not $\mathfrak{m}$-primary, then $H^d_I(A) = 0$.*

Theorem 5.0.2 (Mayer-Viétoris). *Let I, J be ideals of a Noetherian ring A. Then for every A-mod M, there is a long exact sequence:*

$$\ldots \to H^i_{I+J}(M) \to H^i_I(M) \oplus H^i_J(M) \to H^i_{I\cap J}(M) \to H^{i+1}_{I+J}(M) \to \ldots.$$

Proof. Note that $I^n \cap J^n$ is cofinal with $(I \cap J)^n$ and that $I^n + J^n$ is cofinal with $(I + J)^n$.

Consider the exact sequence:

$$0 \to \frac{A}{I^n \cap J^n} \to \frac{A}{I^n} \oplus \frac{A}{J^n} \to \frac{A}{I^n + J^n} \to 0.$$

Let E be an injective A-mod. E induces the following exact sequence:

$$0 \to \operatorname{Hom}\left(\frac{A}{I^n \cap J^n}, E\right) \to \operatorname{Hom}\left(\frac{A}{I^n} \oplus \frac{A}{J^n}, E\right) \to \operatorname{Hom}\left(\frac{A}{I^n + J^n}, E\right) \to 0.$$

By taking the direct limit with respect to n, and using the fact of the cofinality of the sequence of ideals previously discussed, we get the following exact sequence:

$$0 \to H^0_{I+J}(E) \to H^0_I(E) \oplus H^0_J(E) \to H^0_{I\cap J}(E) \to 0,$$

which is the sames as:

$$0 \to \Gamma_{I+J}(E) \to \Gamma_I(E) \oplus \Gamma_J(E) \to \Gamma_{I\cap J}(E) \to 0.$$

Now let M be any A-mod. And consider an injective resolution of M. The previous exact sequence applied to every injective module of the resolution induces a commutative diagram such that it gives us long exact sequence in the cohomology of the Γ. That is the long exact sequence we were looking for. $\square$

Definition 5.0.3. Let $(A, \mathfrak{m})$ be a Noetherian local ring. We define the punctured spectrum of A as the space $\mathrm{Spec}(A) - \{ \mathfrak{m} \}$. We denote it as $\mathrm{Spec}^0(A)$.

We also write $V^0(I)$ to denote the closed subset $V(I) - \{ \mathfrak{m} \}$ of $\mathrm{Spec}^0(A)$.

The following result relates the punctured spectrum of a ring with its Γ_{d-1} graph, it also gives us a way to relate local cohomology to the connectedness of Γ_{d-1} graphs.

Notice also that it is quite similar to Proposition 4.0.8 but the ring need not be equidimensional nor complete.

Theorem 5.0.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Then, the following are equivalent.*

1. $\mathrm{Spec}^0(A)$ *is connected.*
2. $\Gamma_{d-1}(A)$ *is connected.*
3. $\mathrm{c}(A) \geq 1$.

Proof. Suppose $\Gamma_{d-1}(A)$ is disconnected. This happens if and only if there is a partition (S, T) of the minimal primes of A in non empty sets such that $\mathrm{ht}(P + Q) = d$ for every $P \in S, Q \in T$. From Proposition 4.0.1 we know this is equivalent to $\mathrm{ht}(I+J) = d$, where $I = \bigcap_{P\in S} P$ and $J = \bigcap_{Q\in T} Q$. So $\sqrt{I+J} = \mathfrak{m}$, $I \cap J = \sqrt{0}$ and $\sqrt{I}$ and $\sqrt{J}$ are proper subsets of $\mathfrak{m}$; otherwise, $\dim(A) = 0$. This happens if and only if $\mathrm{Spec}(A) - V(\mathfrak{m}) = \mathrm{Spec}^0(A)$ is disconnected by Proposition 2.3.2

Suppose $\operatorname{Spec}^0(A)$ is connected. If $\mathrm{c}(A) = 0$, then there is an ideal I such that $\dim(A/I) = 0$ and $\operatorname{Spec}(A) - V(I)$ is disconnected. But this means that I is $\mathfrak{m}$-primary, so $\operatorname{Spec}(A) - V(I) = \operatorname{Spec}(A) - V(\mathfrak{m}) = \operatorname{Spec}^0(A)$, a contradiction since $\operatorname{Spec}^0(A)$ is connected. Then $\mathrm{c}(A) \geq 1$. Conversely if $\mathrm{c}(A) \geq 1$, then the definition of connectedness dimension implies that $\operatorname{Spec}^0(A)$ must be connected since $\dim(A/\mathfrak{m}) = 0$. □

We also have an analogous to Corollary 4.0.10. First we state a result similar to Proposition 4.0.6 that is useful during the proof of this corollary.

Proposition 5.0.5. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and let I be a proper ideal of A such that $\operatorname{Min}(I) \subseteq \operatorname{Min}(A)$. Let Σ be the subgraph of $\Gamma_{d-1}(A)$ whose vertices are $\operatorname{Min}(I)$. Then*

$$\Sigma \cong \Gamma_{d'-1}(A/I),$$

where d' is the dimension of A/I.

Proof. The correspondence between vertices of Σ and vertices of $\Gamma_{d'-1}(A/I)$ is given by asigning the vertex P of Σ to the vertex P/I of $\Gamma_{d'-1}(A/I)$.

Observe there is an edge between P_1 and P_2 in Σ if and only if $\operatorname{ht}(P_1 + P_2) \leq d - 1$, which is equivalent to the ideal $P_1 + P_2$ being non $\mathfrak{m}$-primary. This is the same as $P_1/I + P_2/I$ being non $\mathfrak{m}/I$-primary, which happens if and only if $\operatorname{ht}(P_1/I + P_2/I) \leq d' - 1$, where $d' - 1$ is the dimension of A/I. We conclude that there is an edge between P_1 and P_2 in Σ if and only if there is an edge between P_1/I and P_2/I in $\Gamma_{d'-1}(A/I)$. □

Corollary 5.0.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Then*

$$\#\Gamma_{d-1}(A) = \# \operatorname{Spec}^0(A).$$

Proof. If $\Gamma_{d-1}(A)$ is connected the result follows from Theorem 5.0.4. Suppose $\Gamma_{d-1}(A)$ is disconnected and let $\Sigma_1, \ldots, \Sigma_s$ be the s connected components of $\Gamma_{d-1}(A)$. Define ideals $\mathfrak{a}_i = \bigcap_{P \in \Sigma_i} P$. Observe that the $V^0(\mathfrak{a}_i)$'s are a partition of $\operatorname{Spec}^0(A)$ by open sets. So $s \leq \# \operatorname{Spec}^0(A)$.

Let d_i be the dimension of $A/\mathfrak{a}_i$. From Proposition 5.0.5 we know that $\Sigma_i \cong \Gamma_{d_i-1}(A/\mathfrak{a}_i)$. Since Σ_i is connected, then $\Gamma_{d_i-1}(A/\mathfrak{a}_i)$ is also connected and by Theorem 5.0.4 $\operatorname{Spec}^0(A/\mathfrak{a}_i)$ is also connected. Since $\operatorname{Spec}^0(A/\mathfrak{a}_i) \cong V^0(a_i)$, it follows that $s = \# \operatorname{Spec}^0(A)$. □

Theorem 5.0.7 (Hochster-Huneke)**.** *Let* $(A, \mathfrak{m})$ *be a Noetherian complete local domain of dimension* d*. Let* I *be a proper ideal of* A *such that* $\dim(R/I) \geq 2$*. Then*

$$H_I^{d-1}(A) = 0 \Rightarrow \operatorname{Spec}^0(A/I) \text{ is connected.}$$

Proof. We prove the contrapositive. Suppose $\operatorname{Spec}^0(A/I)$ is disconnected. From Proposition 2.3.2 we know there are non $\mathfrak{m}/I$-primary ideals $J_1/I, J_2/I \subseteq \mathfrak{m}/I$ such that $\sqrt{J_1/I + J_2/I} = \mathfrak{m}/I$ and $J_1/I \cap J_2/I = \sqrt{0}$. So $\sqrt{J_1 + J_2} = \mathfrak{m}$ and $\sqrt{J_1 \cap J_2} = \sqrt{I}$.

From the Mayer-Viétoris sequence for J_1 and J_2 we get the exact sequence

$$H_{J_1 \cap J_2}^{d-1}(A) \to H_{J_1+J_2}^{d}(A) \to H_{J_1}^{d}(A) \oplus H_{J_2}^{d}(A).$$

Since J_i/I is not $\mathfrak{m}/I$-primary, then J_i is not $\mathfrak{m}$-primary. So Theorem 5.0.1 implies that $H_{J_i}^d(A) = 0$. Notice also that since $J_1 \cap J_2$ and I have the same radical, then $H_{J_1 \cap J_2}^{d-1}(A) = H_I^{d-1}(A)$ by Proposition 3.0.2. $J_1 + J_2$ and $\mathfrak{m}$ also have the same radical, so $H_{J_1+J_2}^d(A) = H_{\mathfrak{m}}^d(A)$ by Proposition 3.0.2. Then our sequence becomes

$$H_I^{d-1}(A) \to H_{\mathfrak{m}}^d(A) \to 0.$$

We know from Theorem 3.0.8 that $H_{\mathfrak{m}}^d(A) \neq 0$. The exactness of our sequence implies that $H_I^{d-1}(A) \neq 0$. □

Proposition 5.0.8. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 2$*. Let* P *be a minimal prime of* A*. Then*

$$\operatorname{Spec}^0(A) \text{ is connected } \Rightarrow \dim(A/P) \geq 2.$$

Proof. Suppose $\operatorname{Spec}^0(A)$ is connected. We proceed by contradiction. Suppose there is a minimal prime P of A such that $\dim(A/P) < 2$. Note that if P is the only minimal prime of A then $\dim(A) = \dim(A/P) < 2$, a contradiction. Then, we can assume that A has more than one minimal prime.

If $\dim(A/P) = 0$, then P is $\mathfrak{m}$-primary. Hence, there is no edge between P and any other minimal prime of A in $\Gamma_{d-1}(A)$, that is, $\Gamma_{d-1}(A)$ is disconnected. This contradicts the fact that $\Gamma_{d-1}(A)$ is connected from Theorem 5.0.4.

If $\dim(A/P) = 1$, then there is a prime P' of A such that $P \subsetneq P' \subsetneq \mathfrak{m}$. Let $I = P$ and J be the intersection of all the minimal primes of A except P. Note that I and J are non $\mathfrak{m}$-primary subsets of $\mathfrak{m}$ such that $I \cap J = \sqrt{0}$ and $\sqrt{I + J} = \mathfrak{m}$. Thus, Proposition 2.3.2 implies that $\operatorname{Spec}(A) - V(\mathfrak{m}) = \operatorname{Spec}^0(A)$ is disconnected, a contradiction. □

We state the following lemma without providing proof.

Lemma 5.0.9. *Let $S = K[[x_1, \dots, x_n]]$ be a power series ring over a separably closed field K. Let $P \in \operatorname{Spec}(S)$ such that $\dim(S/P) \geq 2$. Then $H_P^{n-1}(S) = 0$.*

Theorem 5.0.10 ([HL90])**.** *Let $S = K[[x_1, \dots, x_n]]$ be a power series ring over a separably closed field K. Let I be an ideal of S such that $d = \dim(S/I) \geq 2$. Then*

$$H_I^{n-1}(S) = 0 \Leftrightarrow \operatorname{Spec}^0(S/I) \text{ is connected.}$$

Proof. Since S is a Noetherian complete local ring of dimension n such that $\dim(S/I) \geq 2$, it follows from Theorem 5.0.7 that if $H_I^{n-1}(S) = 0$, then $\operatorname{Spec}^0(S/I)$ is connected.

Now suppose $\operatorname{Spec}^0(S/I)$ is connected, then $\Gamma_{d-1}(S/I)$ is connected by Theorem 5.0.4. Let t be the number of minimal primes of S/I. We proceed by induction on t.

Suppose $t = 1$. Let P/I be the minimal prime of S/I, with P prime of S that contains I. From Proposition 5.0.8 we know that

$$\dim\left(\frac{\frac{S}{I}}{\frac{P}{I}}\right) = \dim\left(\frac{S}{P}\right) \geq 2.$$

Lemma 5.0.9 implies that $H_P^{n-1}(S) = 0$. Since P/I is the only minimal prime of S/I, we deduce that P is the only minimal prime of I, that it, $\sqrt{I} = P$. From Proposition 3.0.2 we know that $H_I^{n-1}(S) = 0$.

Suppose the result holds for $t-1$ minimal primes and suppose S/I has t minimal primes. Let $P_1/I, \dots, P_{t-1}/I, P_t/I$ be the minimal primes of S/I, where $P_1, \dots, P_{t-1}, P_t$ are the minimal primes of I. Observe that there is a vertex of $\Gamma_{d-1}(S/I)$ such that if we remove that vertex. The graph $\Gamma_{d-1}(S/I)$ would still be connected. Suppose P_t/I is such vertex. Let Σ be the subgraph of $\Gamma_{d-1}(S/I)$ whose vertices are $P_1/I, \dots, P_{t-1}/I$ and set $J = \bigcap_{i=1}^{t-1} P_i$. By Proposition 5.0.5 we know that

$$\Sigma \cong \Gamma_{d'-1}\left(\frac{\frac{A}{I}}{\bigcap_{i=1}^{t-1} \frac{P_i}{I}}\right) \cong \Gamma_{d'-1}(S/J),$$

where d' is the dimension of S/J. We conclude that $\Gamma_{d'-1}(S/J)$ is connected too.

From the Mayer-Viétoris sequence for J and P_t we get the exact sequence

$$H_J^{n-1}(S) \oplus H_{P_t}^{n-1}(S) \to H_{J\cap P_t}^{n-1}(S) \to H_{J+P_t}^{n}(S) \to H_J^n(S) \oplus H_{P_t}^n(S).$$

Notice $\dim(S/J) \geq 2$ by Proposition 5.0.8. Since $\Gamma_{d'-1}(S/J)$ is connected and it has $t-1$ minimal primes, then by our induction hypothesis we know that $H^{n-1}_J(S) = 0$.

Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal ideal of S. Since $\Gamma_{d-1}(S/I)$ is connected, there is a $j \in [1, t-1]$ such that $\operatorname{ht}(P_j/I + P_t/I) \leq d-1$, so $P_j + P_t$ is not $\mathfrak{m}$-primary. By construction of J we know that $J + P_t \subseteq P_j + P_t$, so $J + P_t$ is not $\mathfrak{m}$-primary. Observe that neither J nor P_t are $\mathfrak{m}$-primary since they are contained in $J + P_t$. Theorem 5.0.1 implies that

$$H^n_{J+P_t}(S) = H^n_J(S) = H^n_{P_t}(S) = 0.$$

Then our sequence becomes

$$0 \to H^{n-1}_{J\cap P_t}(S) \to 0 \to 0$$

so $H^{n-1}_{J\cap P_t}(S) = 0$ and since $J \cap P_t = \sqrt{I}$, we conclude by Proposition 3.0.2 that $H^{n-1}_I(S) = 0$. □

Corollary 5.0.11. *Let $S = K[[x_1, \ldots, x_n]]$ be a power series ring over a separably closed field K. Let I be an ideal of S such that $\dim(S/I) \geq 2$. Let $t = \#\operatorname{Spec}^0(S/I)$. Then $H^{n-1}_I(S) \cong E_S(K)^{t-1}$.*

Proof. We proceed by induction on t.

Suppose $t = 1$. This means that $\operatorname{Spec}^0(S/I)$ is connected. It follows from Theorem 5.0.10 that $H^{n-1}_I(S) = 0$. Observe that $E_K(S)^{t-1} = 0$ since $t = 1$. So in this case $H^{n-1}_I(S) = E_S(K)^{t-1}$.

Suppose the result holds for $t-1$ connected components. Suppose $\operatorname{Spec}^0(S/I)$ has t connected components, namely $V^0(J_1/I), \ldots, V^0(J_t/I)$. Let $\mathfrak{a} = \bigcap_{i=1}^{t-1} J_i$.

From the Mayer-Viétoris sequence for $\mathfrak{a}$ and J_t we get the short exact sequence

$$H^{n-1}_{\mathfrak{a}+J_t}(S) \to H^{n-1}_{\mathfrak{a}}(S) \oplus H^{n-1}_{J_t}(S) \to H^{n-1}_{\mathfrak{a}\cap J_t}(S) \to H^n_{\mathfrak{a}+J_t}(S).$$

Observe that

$$V^0\left(\frac{\mathfrak{a}}{I} + \frac{J_t}{I}\right) = V^0\left(\frac{\mathfrak{a}}{I}\right) \cap V^0\left(\frac{J_t}{I}\right) = \bigcup_{i\neq t}\left(V^0\left(\frac{J_i}{I}\right) \cap V^0\left(\frac{J_t}{I}\right)\right) = \emptyset.$$

This implies that $\mathfrak{a}+J_t$ is a $\mathfrak{m}$-primary ideal, so from Proposition 3.0.2 we conlude that $H^{n-1}_{\mathfrak{a}+J_t}(S) = H^{n-1}_{\mathfrak{m}}(S)$ and $H^n_{\mathfrak{a}+J_t}(S) = H^n_{\mathfrak{m}}(S)$. Since S is a Gorenstein ring, $H^{n-1}_{\mathfrak{m}}(S) = 0$ and $H^n_{\mathfrak{m}}(S) = E_S(K)$.

We know from the proof of Corollary 5.0.6 that $\operatorname{Spec}^0(S/J_t)$ is connected. Theorem 5.0.10 implies that $H^{n-1}_{J_t}(S) = 0$. From the proof of Corollary 5.0.6 we know that $\operatorname{Spec}^0(S/\mathfrak{a})$ has $t-1$ connected components, so $H^{n-1}_{\mathfrak{a}}(S) \cong E_S(K)^{t-2}$ by our induction hypothesis.

Since $\mathfrak{a} \cap J_t = \sqrt{I}$, Proposition 3.0.2 implies that $H^{n-1}_{\mathfrak{a}\cap J_t}(S) = H^{n-1}_I(S)$.

Then our sequence becomes

$$0 \to E_S(K)^{t-2} \to H^{n-1}_I(S) \to E_S(K)$$

. Since $E_S(K)^{t-2}$ is injective, the sequence splits. Hence,

$$H^{n-1}_I(S) = E_S(K)^{t-2} \oplus E_S(K) = E_S(K)^{t-1}.$$

□

This last proposition relates the depth of a ring with the connectivity of the punctured spectrum of a ring.

Proposition 5.0.12. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring such that* $\operatorname{depth}(A) \geq 2$*. Then* $\operatorname{Spec}^0(A)$ *is connected.*

Proof. We proceed by contradiction. Suppose $\operatorname{Spec}^0(A)$ is not connected. From Proposition 2.3.2 we know there are ideals $I, J \subseteq \mathfrak{m}$ such that both of them are not $\mathfrak{m}$-primary, $I \cap J = \sqrt{0}$ and $\sqrt{I+J} = \mathfrak{m}$.

From the Mayer-Viétoris sequence for I and J we get the exact sequence

$$H^0_{I+J}(A) \to H^0_I(A) \oplus H^0_J(A) \to H^0_{I\cap J}(A) \to H^1_{I+J}(A).$$

From Theorem 3.0.5 we know that $H^0_{\mathfrak{m}}(A) = H^1_{\mathfrak{m}}(A) = 0$ since $\operatorname{depth}(A) \geq 2$.

Since $I \cap J = \sqrt{0}$, we know from Proposition 3.0.2 that $H^0_{I\cap J}(A) = H^0_0(A)$. From the definition of local cohomology it follows that $H^0_0(A) = A$.

From the definition of local cohomology we know that $H^0_I(A) = \Gamma_I(A)$ and $H^0_J(A) = \Gamma_J(A)$.

Then our sequence becomes

$$0 \to \Gamma_I(A) \oplus \Gamma_J(A) \to A \to 0.$$

Since A is indecomposable, we get $\Gamma_I A = A$ or $\Gamma_J(A) = A$. Say $\Gamma_I(A) = A$. This implies that $\sqrt{I} = \sqrt{0}$, so $V^0(I) = \operatorname{Spec}^0(A)$, contradicting the fact that $V^0(I)$ and $V^0(J)$ form a disconnection of $\operatorname{Spec}^0(A)$. □

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