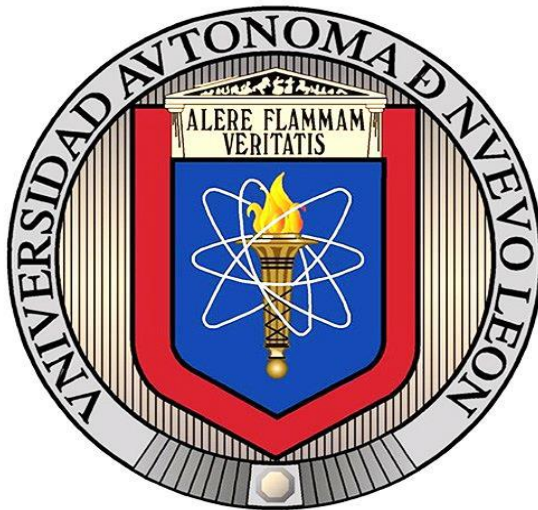


UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS



TESIS

**PROGRAMACIÓN BINIVEL Y EQUILIBRIOS CONJETURADOS:
RESULTADOS TEÓRICOS Y ALGORITMOS NUMÉRICOS**

POR
JOSÉ GUADALUPE FLORES MUÑOZ

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NOVIEMBRE, 2019

Universidad Autónoma de Nuevo León
Facultad de Ciencias Físico Matemáticas
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Los miembros del Comité de Tesis recomendamos que la Tesis “Programación Binivel y Equilibrios Conjeturados: Resultados Teóricos y Algoritmos Numéricos”, realizada por el alumno José Guadalupe Flores Muñiz, con número de matrícula 1483923, sea aceptada para su defensa como opción al grado de Doctorado en Ciencias con Orientación en Matemáticas.

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AGRADECIMIENTOS

Quiero agradecer principalmente a mi familia por darme los recursos y el apoyo necesario para realizar mis estudios desde preescolar hasta la licenciatura. Agradezco también a la Universidad Autónoma de Nuevo León (UANL), a la Facultad de Ciencias Físico Matemáticas (FCFM), y al Consejo Nacional de Ciencia y Tecnología (CONACyT), por apoyarme económicamente y darme la oportunidad de realizar mis estudios de Maestría y Doctorado. Además, agradezco en general a mis compañeros de posgrado que han estado conmigo durante el transcurso de 5 años. Por último, quiero agradecer especialmente, primero, al ex-director de la FCFM, el Dr. Rogelio Sepúlveda, por darme su mano durante mi último semestre de Maestría, lo cual me permitió terminar correctamente el grado de Maestría y proseguir con el Doctorado, segundo, al director del CICFM, el Dr. Romeo Selvas, por ..., y tercero, a la Dra. Nataliya Kalashnykova y al Dr. Viacheslav Kalashnikov por compartirme sus conocimientos, experiencias, y consejos, durante el transcurso de mi Maestría y Doctorado. Finally, I want to thank Dr. Vladik Kreinovich of the University of Texas at El Paso, for accepting me as an exchange student under his guidance, the knowledge he shared with me helped me through my Master's and PhD degrees.

Para abarcar a un público más general, de ahora en adelante, el contenido de esta tesis será presentado en inglés.

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ABSTRACT

This thesis presents the fruit of 3 years of research. During this time 3 works were developed, each one with its own mathematical formulations and results. These works are, of course, related to each other and will be further developed in the near future.

The first work of this thesis is presented in chapter 1 and addresses the problem of defining an optimality criterion for a semi-public company in a semi-mixed duopoly model. Here, we have two agents competing, the semi-public company and a private firm, both producing a homogeneous good to satisfy the demand in the market. The private firm, as usual, seeks to maximize its net profit, while the semi-public company has a commitment to watch over the economy of the population, but at the same time, does not neglect its own profit. The compromise between these two objectives for the semi-public company is described by a parameter $\beta \in (0, 1]$, where $\beta \rightarrow 0$ represents that the semi-public company thinks only for its own net profit, and $\beta = 1$ represents that the semi-public company cares solely for the economy of the population without seeking its own benefit.

This kind of semi-mixed oligopoly model has been considered before, but only dealing with a fixed parameter β . In our research, we studied the behavior of the consistent conjectural variations equilibrium (CCVE), the Cournot-Nash equilibrium and the perfect competitions equilibrium (the most important types of equilibrium for classic and mixed oligopoly models), as functions of β . After analyzing the results, we develop an optimality criterion for the socialization level that allows the semi-public company to take care of the population's economy, as well as its own profit.

These results are already published in Kalashnikov-Jr et al. (2017).

In the classic and mixed oligopoly model, in order to define the CCVE, two types of demand must be contemplated, the passive demand that depends upon the price of the product in the market, and an active demand that is independent of the price. This assumption is essential to define the consistency of the conjectural variations equilibrium (CVE). However, this is a disadvantage when defining the consistency of the CVE for

other kinds of market models (e.g. human migration and financial market models) where the notion of active demand doesn't exist.

This obstacle can be overcome thanks to the results obtained in the second work of this thesis presented in chapter 2. In this second work, a classic oligopoly market is considered (i.e. there are only private firms in the market which maximize their own net profit), presenting the results for the consistency of the CVE (similar to the results of the first work for the CCVE in the semi-mixed duopoly model). In addition, the classic oligopoly market is modeled in another way, as a bilevel multi-leader-follower game where the leaders and followers in both levels are the same private firms of the original oligopoly model. In this formulation for the classic oligopoly market as a bilevel game, the strategies for the agents in the lower-level are their output volumes of the product in the market while their strategies in the upper-level are their conjectural variations. It is shown that for any strategy profile for the upper-level, there exists a unique lower-level Nash equilibrium, which implies that the upper-level game is well defined. This upper-level game is then named the meta-game. Finally, it is shown that (under certain assumptions) a strategy profile of conjectural variations is a Nash equilibrium for the meta-game if, and only if, these conjectural variations are consistent for the original oligopoly model.

This result creates a bridge between two different concepts, the CCVE and the Nash Equilibrium, which also gives us the idea of how to define the CCVE in other types of market models (without the need of an active demand), as the solution of a bilevel multi-leader-follower game.

These results are already published in Kalashnikov et al. (2019), Kalashnykova et al. (2016), Kalashnikov, Kalashnykova and Flores-Muñiz (2018*b*) and presented in the following conferences:

- V Congreso Nacional de la Sociedad Mexicana de Investigación de Operaciones (SMIO 2016), Guadalajara, México, October 2016.
- Variational Inequalities, Nash Equilibrium Problems and Applications (VINEPA 2018), Reggio Calabria, Italy, March 2018.
- 2nd International Workshop on Bilevel Programming (IWOBIP 2018), Lille, France, June 2018.
- International Conference on Variational Analysis and Nonsmooth Optimization (IC-VANO 2018), Halle, Germany, 28 June-July 2018.

Although an equivalence is found between the CCVE as the solution of a bilevel game, the numerical results presented are computed with the theory corresponding to

the consistency of the conjectural variations, which gives us an efficient way to find the CCVE. However, an efficient algorithm to find the solution of the proposed meta-game is yet to be developed.

Thus, for the third work of this thesis, presented in chapter 3, we start to study the theory required for the construction of efficient algorithms to solve bilevel programming problems. The problem chosen for this goal was the Tolls Optimization Problem (TOP) which has one important similarity with the meta-game and is modeled as a linear bilevel optimization problem where its lower-level is a many-person game.

The linear bilevel formulation of the TOP arises from the assumption that the delay cost for a single unit to travel through an arc is a constant. However, this assumption is unrealistic in the presence of congestion. Thus, in this third work of the thesis, we extend the linear bilevel formulation of the TOP by letting the delay cost be a linear function depending upon the total amount of drivers traveling along the same arc.

This new assumption leads to a linear-quadratic bilevel formulation for the TOP where the lower-level is a convex quadratic problem, which is a special case of the more general convex optimization problem appearing in the lower-level of the meta-game. Thus, the theory and algorithms developed to solve the linear-quadratic bilevel formulation of the TOP will be useful in the future to construct an algorithm to efficiently solve the lower-level of the meta-game.

The efficient solution of the lower-level problem in bilevel optimization is especially important since its optimal solution is required in order to evaluate the upper-level functions.

These results are already published in Kalashnikov, Kreinovich, Flores-Muñiz and Kalashnykova (2016), Flores-Muñiz et al. (2017), Kalashnikov, Kalashnykova and Flores-Muñiz (2018a), Kalashnykova, Flores-Muñiz and Kalashnikov (2018), Kalashnykova, Kalashnikov and Flores-Muñiz (2018), Kalashnikov, Flores-Muñiz and Kalashnykova (2016), Kalashnikov, Flores-Muñiz, Kalashnykova and Kreinovich (2017), Kalashnikov, Flores-Muñiz and Kalashnykova (2017), Kalashnikov, Flores-Muñiz and Kalashnykova (2018), Kalashnikov, Kalashnykova and Flores-Muñiz (2018c) and presented in the following conferences:

- Congreso Internacional de Logística y Cadena de Suministro (CiLOG 2016), Yucatan, México, October 2016.
- 2nd International Conference on Gas, Oil and Petroleum Engineering (GOPE 2018), Texas, United States, February 2018.

- International Conference on Optimization (SIGOPT 2018), Kloster Irsee, Germany, March 2018.
- 2nd International Workshop on Bilevel Programming (IWOBIP 2018), Lille, France, June 2018.
- International Conference on Unconventional Modelling, Simulation and Optimization - Soft Computing and Meta Heuristics (UMSO 2018), Kitakyushu, Japan, December 2018.

Finally, although it is not included in this thesis, an application of this work has been exposed in the following conferences:

- 2019 INFORMS-ALIO International Conference, Cancún, México, June 2019.
- 30th European Conference on Operational Research, Glasgow, United Kingdom, June 2019.
- EUROPT Workshop 2019, Dublin, Ireland, June 2019.

Now, without further ado, we present below the first chapter of the thesis.

CONSISTENT CONJECTURAL VARIATIONS EQUILIBRIUM IN A SEMI-MIXED DUOPOLY

1.1 INTRODUCTION

During the last 15 years, researchers in the field of mathematical economics have extensively and intensively studied mixed oligopoly models. In contrast to the classical oligopoly, a mixed oligopoly, apart from standard producers who seek to maximize their net profit, usually includes (at least one) public company trying to optimize another objective function involving indicators of the firm's social responsibility. Many such models include an agent who maximizes the domestic social surplus (cf., Cornes and Sepahvand (2003); Fershtman (1990); Matsumura (2003); Matsushima and Matsumura (2004); Matsumura and Kanda (2005)). An income-per-worker function replaces the standard net profit objective function in some other publications (cf., Ireland and Law (1982); Bonin and Putterman (1987); Stephen (1982); Putterman (2008)). Other researchers (Saha and Sen-sarma (2013); Mumcu et al. (2001)) have studied a third kind of mixed duopoly, in which an exclusive participant aims to maximize a convex combination of its net profit and domestic social surplus. This work addresses such a company as *semi-public*.

In many of the aforementioned works, the authors investigated mixed oligopolies by making use of the classical Cournot-Nash, Hotelling, or Stackelberg models. The notion of *conjectural variations equilibrium* (CVE) first introduced by Bowley (1924) and Frisch (1933) opens another way of the agents' reaction to the market challenge, which attracts ever-growing interest on part of the related researchers. In CVE, competitors behave as follows: each producer selects its most favorable strategy having supposed that every opponent's action is a *conjectural variation function* of its own strategical variation. For example, as Laitner (1980) states, "Although the firms make their output decisions simultaneously, plan changes are always possible before production begins". In other words,

in contrast to the Cournot-Nash approach, here, every company assumes that its choice of the own output volume will affect her competitors' reaction. The consequently arising prediction (or, conjectural variation) function is the central point of conjectural variation decision-making, or the conjectural variations equilibrium (CVE).

As is mentioned in Figuières et al. (2004) and Giocoli (2005), the notion of CVE has been the topic of abundant theoretical discussions (cf., Lindh (1992)). Notwithstanding this, economists have extensively used various forms of CVE to predict the outcome of non-cooperative behavior in many areas of economics. The literature on conjectural variations has focused mainly on two-player games (cf., Figuières et al. (2004)) because a serious conceptual difficulty arises if the number of agents is greater than two (cf., Figuières et al. (2004); Kalashnikov, Bulavsky, Kalashnykova and Castillo-Pérez (2011)).

In order to overcome conceptual hurdles arising in many-player games, a new tool was developed in Bulavsky (1997); namely, instead of imposing very restrictive additional assumptions (like the identity of players in the oligopoly), it was supposed that every player makes conjectures only about the variations of the market-clearing (equilibrium) price as a consequence of (infinitesimal) variations of the same player's output volume. Knowing the opponents' conjectures (the *influence coefficients*), each firm applies a verification procedure in order to determine whether its influence coefficient is *consistent* with those of the remaining agents.

In papers Kalashnikov, Bulavsky, Kalashnykova and Castillo-Pérez (2011) and Kalashnykova et al. (2011), the authors extended the ideas of Bulavsky (1997) to the mixed duopoly and oligopoly cases, respectively. They defined *exterior equilibrium* as a CVE state with the conjectures fixed in an extrinsic manner. This sort of CVE was proved to exist uniquely, which helped introduce the notion of *interior equilibrium* as the exterior equilibrium with consistent conjectures (influence coefficients). All these instruments, the consistency criteria, consistency verification procedures, and existence theorems for the interior equilibrium were developed and demonstrated in Kalashnikov, Bulavsky, Kalashnykova and Castillo-Pérez (2011) and Kalashnykova et al. (2011).

In the next series of papers, Kalashnikov et al. (2013) and Kalashnikov et al. (2014), the aforesaid constructions were extended to the case of a semi-mixed duopoly, where similar to Saha and Sensarma (2013) and Mumcu et al. (2001), the semi-public company strives to maximize a convex combination of the net profit and domestic social surplus. The results of numerical experiments with a test model of a market of electricity resembling that of Liu et al. (2007), both with and without a semi-public producer, showed that the consumer gains more if the semi-public producer follows the CVE strategy as compared to the Nash-Cournot equilibrium. Furthermore, in Kalashnikov et al. (2014), the authors

declared a guess that there must exist such a value of the combination parameter (also interpreted as the public firm's socialization level) that brings up the “equivalence” (in a certain sense) of the consistent conjectural variations equilibrium (CCVE) and the classical Cournot-Nash one. This equivalence permits a socially responsible municipality to diminish (cancel) subsidies paid either to the private company (in order to compensate its losses when following the consistent conjectures) or to the consumers (to reimburse them the higher retail price of the good if both the competing semi-public firm and the private company both are stuck to the Cournot-Nash conjectures).

In this chapter, we present mathematically rigorous proofs of the above-mentioned guess. In other words, we establish the existence of the value of the combination coefficient (also known as the semi-public enterprise's socialization level) such that the private producer's profit is the same in the CCVE and Cournot-Nash equilibrium states, which makes the subsidies from the authorities either to the producer or to the consumers unnecessary.

1.2 MODEL SPECIFICATION

Consider a semi-mixed duopoly with two producers where $i = 0$ is a semi-public company and $i = 1$ is a private firm. The companies supply a homogeneous produce under the expenditure estimated by the *cost functions* $f_i(q_i)$, $i = 0, 1$, where $q_i \geq 0$ is the *output volume* by producer i . The market-clearing supply is specified by a demand (inverse price) function $G = G(p)$, whose argument p is the price suggested by the suppliers. An active demand D is nonnegative and independent of the price. The equilibrium between the demand and supply for a given price p is described by the following *balance equality*:

$$q_0 + q_1 = G(p) + D. \quad (1.1)$$

We assume the following properties of the model's data.

A 1.1. *The demand function $G = G(p) \geq 0$ has finite values for all $p \geq 0$, and is continuously differentiable with $G'(p) < 0$.*

A 1.2. *For each $i = 0, 1$, the cost function $f_i(q_i)$ is quadratic with zero overhead costs, i.e.,*

$$f_i(q_i) = \frac{1}{2}a_i q_i^2 + b_i q_i, \quad (1.2)$$

where

$$a_i > 0, \quad b_i > 0, \quad i = 0, 1. \quad (1.3)$$

Moreover, it is assumed that

$$b_0 \leq b_1. \quad (1.4)$$

The private firm $i = 1$ selects his output $q_1 \geq 0$ in order to maximize its net profit function

$$\pi_1(p, q_1) = pq_1 - f_1(q_1), \quad (1.5)$$

whereas the semi-public company $i = 0$ decides its output volume $q_0 \geq 0$ so as to maximize the convex combination of *domestic social surplus* and the net profit function

$$S(p, q_0, q_1) = \beta \left(\int_0^{q_0+q_1} p(x)dx - pq_1 - f_0(q_0) \right) + (1 - \beta)(pq_0 - f_0(q_0)), \quad (1.6)$$

where $0 < \beta \leq 1$.

Here, domestic social surplus involving the integral in (1.6) is usually interpreted as the money gained by the (domestic) consumer when he/she acquires the good at the lower price (established in the market) than that expected by him/her *before* the semi-public company entered the market (see the more detailed interpretation by the well-known Japanese mathematicians and economists: Matsumura (2003); Matsushima and Matsumura (2004); Matsumura and Kanda (2005)).

According to our concept of conjectural variations equilibrium (CVE), we assume that both producers (semi-public and private) *conjecture* about variations in the market-clearing price p as a function of the perturbations in their output quantities. In terms of the first derivative, the latter assumption might be described by a *conjectured* dependency of (infinitesimal) affine variations of the price p upon (infinitely small) perturbations of the supply quantities q_i . Within this framework, the first order optimum condition depicting equilibrium reduces to the form: for the semi-public company ($i = 0$)

$$\frac{\partial S}{\partial q_0} = p - [\beta q_1 - (1 - \beta)q_0] \frac{\partial p}{\partial q_0} - f'_0(q_0) \begin{cases} = 0, & \text{if } q_0 > 0, \\ \leq 0, & \text{if } q_0 = 0. \end{cases} \quad (1.7)$$

A similar first order optimality condition for the private producer ($i = 1$) yields

$$\frac{\partial \pi_1}{\partial q_1} = p + q_1 \frac{\partial p}{\partial q_1} - f'_1(q_1) \begin{cases} = 0, & \text{if } q_1 > 0, \\ \leq 0, & \text{if } q_1 = 0. \end{cases} \quad (1.8)$$

On that account, in order to predict the (instantaneous) behavior of supplier i , one need make use of the first order derivative

$$\frac{\partial p}{\partial q_i} \equiv -\nu_i \quad (1.9)$$

rather than the (exact) functional dependency of p upon q_i . Even more, the latter dependency is extremely hard to estimate in a many-person game with several decision-makers. Here, the negative sign is applied in order to have nonnegative values of ν_i . Surely, the conjectured (first-order) dependency of p upon q_i should guarantee the concavity of the i -th producer's conjectured profit as a function of its supply, which implies the maximum of the producer's revenue. Under the assumption that the cost functions $f_i(q_i)$ are quadratic and strictly convex (assumption **A1.1**), it suffices supposing the coefficient ν_i (referred to as the i -th producer's *influence coefficient*) to be nonnegative and constant. In this case, the conjectured dependency of the semi-public company's objective function variations upon the production output $q_0 + \eta_0$ has the form

$$\begin{aligned} \hat{S}(q_0 + \eta_0) = & \beta \left[\int_0^{q_0 + q_1 + \eta_0} p(x) dx - (p - \nu_0 \eta_0) q_1 - f_0(q_0 + \eta_0) \right] \\ & + (1 - \beta) [(p - \nu_0 \eta_0)(q_0 + \eta_0) - f_0(q_0 + \eta_0)], \end{aligned} \quad (1.10)$$

while the local maximum condition at $\eta_0 = 0$ is provided by the relation:

$$\begin{cases} p = -\nu_0 [\beta q_1 - (1 - \beta) q_0] + a_0 q_0 + b_0, & \text{if } q_0 > 0, \\ p \leq -\beta \nu_0 q_1 + b_0, & \text{if } q_0 = 0. \end{cases} \quad (1.11)$$

Similarly, the private firm's profit presumes a local dependency upon the production output $q_1 + \eta_1$ in the form

$$\hat{\pi}_1(q_1 + \eta_1) = (p - \nu_1 \eta_1)(q_1 + \eta_1) - f_1(q_1 + \eta_1), \quad (1.12)$$

which permits formulating the maximum condition at $\eta_1 = 0$ as follows:

$$\begin{cases} p = \nu_1 q_1 + a_1 q_1 + b_1, & \text{if } q_1 > 0, \\ p \leq b_1, & \text{if } q_1 = 0. \end{cases} \quad (1.13)$$

If the producers' conjectures concerning market-clearing price were assigned exogenously (like it was done in Bulavsky and Kalashnikov (1994) and Bulavsky and Kalashnikov (1995)), one might assume the values ν_i to be functions of q_i and p . Notwithstanding

that, we use the approach from Bulavsky (1997), where the conjectures in the equilibrium are determined simultaneously with the market-clearing price p and the output volumes q_i by a special verification procedure. In this circumstance, the influence coefficients are the solution of a nonlinear system of equations found for the equilibrium only. In section 1.4, such equilibrium state is referred to as *interior* and is described by the extended vector $(p, q_0, q_1, \nu_0, \nu_1)$.

1.3 EXTERIOR EQUILIBRIUM

In order to describe the verification procedure, we need first to define a more elementary concept of equilibrium referred to as *exterior* (cf., Bulavsky (1997) with the parameters ν_i assigned exogenously).

Definition 1.1. *A vector (p, q_0, q_1) is named exterior equilibrium for the fixed coefficients $\nu_i \geq 0, i = 0, 1$, if the market is balanced, i.e., equation (1.1) is satisfied, and the optimality conditions for the semi-public company and private firm, (1.11) and (1.13) respectively, hold.*

In what follows, we are going to consider only the case when the set of producers with strictly positive output volumes is fixed, i.e., it does not depend upon the values ν_i of the influence coefficients. In order to assure this feature, we make the following assumption:

A1.3. *For the price $p_0 = b_1$, the following inequality holds:*

$$\frac{p_0 - b_0}{a_0} < G(p_0). \quad (1.14)$$

The latter assumption, together with **A1.1** and **A1.2**, guarantees that, for all non-negative values of $\nu_i, i = 0, 1$, there always exists a unique solution of the optimality conditions (1.11) and (1.13) satisfying the balance equation (1.1), i.e, the exterior equilibrium. Moreover, the latter three conditions (1.1), (1.11) and (1.13) can hold simultaneously if and only if $p > p_0$, or equivalently, if and only if $q_i > 0$ for all $i = 0, 1$.

Lemma 1.1. *Let assumptions **A1.1-A1.3** be valid. Then, for all nonnegative values of $\nu_i, i = 0, 1$, supply values q_i are strictly positive (i.e., $q_i > 0, i = 0, 1$) at any exterior equilibrium if and only if $p > p_0$.*

Proof. See appendix A.

We are now in a position to formulate the main result of this section.

Theorem 1.1. *Under assumptions **A1.1-A1.3**, for any $\beta \in (0, 1]$, $D \geq 0$ and $\nu_i \geq 0$, $i = 0, 1$, there exists uniquely the exterior equilibrium (p, q_0, q_1) depending continuously upon the parameters (D, ν_0, ν_1) . The equilibrium price p as a function of these parameters is continuously differentiable with respect to D and ν_i , $i = 0, 1$. Moreover $p(D, \nu_0, \nu_1) > p_0$ and*

$$\frac{\partial p}{\partial D} = \frac{1}{\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) - G'(p)}. \quad (1.15)$$

Proof. See appendix A.

1.4 INTERIOR EQUILIBRIUM

Now we are in a position to define the concept of interior equilibrium. To do that, we first describe the procedure of verification of the influence coefficients ν_i exactly as it was introduced in Bulavsky (1997). Assume that the system is in the exterior equilibrium (p, q_0, q_1) that occurs for some given β , D and ν_i , $i = 0, 1$. One of the producers, say k , $k \in \{0, 1\}$, temporarily changes its behavior by abstaining from the maximization of its conjectured objective function, subtracts its produce q_k from the total demand and makes infinitesimal fluctuations around the latter. In mathematical terms, this is equivalent to restricting the list of producers to the subset $I_{-k} = \{1 - k\}$ with the output q_k subtracted from the active demand and the balance equation restated in the form:

$$q_{1-k} = G(p) + D - q_k. \quad (1.16)$$

Then, variations of the production output by producer k are equivalent to the corresponding active demand fluctuations in the form

$$dD_k = d(D - q_k) = -dq_k. \quad (1.17)$$

If we treat these variations as infinitesimal, we can assume that by observing the corresponding variations of the equilibrium price in the equilibrium attained among the remaining participants, producer k can evaluate the derivative of the equilibrium price with respect to the active demand, i.e, its influence coefficient.

Applying formula (1.15) from theorem 1.1 to calculate the derivatives, one has to remember that producer k is temporally absent from the equilibrium model, hence, one

has to exclude the terms corresponding to $i = k$ from the denominator. Having that in mind, we obtain the following criterion:

Definition 1.2 (Consistency criterion). *In the exterior equilibrium (p, q_0, q_1) , the influence coefficients ν_i , $i = 0, 1$, are named consistent if the following equalities hold:*

$$\nu_0 = \frac{1}{\frac{1}{\nu_1 + a_1} - G'(p)} \quad (1.18)$$

and

$$\nu_1 = \frac{1}{\frac{1}{(1 - \beta)\nu_0 + a_0} - G'(p)}. \quad (1.19)$$

We are now ready to define the concept of interior equilibrium.

Definition 1.3. *A vector $(p, q_0, q_1, \nu_0, \nu_1)$ is named interior equilibrium if, for the influence coefficients $\nu_i \geq 0$, $i = 0, 1$, the vector (p, q_0, q_1) is the exterior equilibrium, and the consistency criterion is valid for all ν_i , $i = 0, 1$.*

The conjectural variations equilibrium is called consistent if the corresponding influence coefficients meet the consistency criterion (definition 1.2).

Theorem 1.2. *Under assumptions **A1.1-A1.3**, there exists the interior equilibrium.*

Proof. See appendix A.

Now, express the demand function's derivative with $\tau = G'(p)$, and replace the consistency conditions (1.18) and (1.19) with the formulas below:

$$\nu_0 = \frac{1}{\frac{1}{\nu_1 + a_1} - \tau} \quad (1.20)$$

and

$$\nu_1 = \frac{1}{\frac{1}{(1 - \beta)\nu_0 + a_0} - \tau}, \quad (1.21)$$

where $\tau \in (-\infty, 0]$.

When $\tau \rightarrow -\infty$ the solutions of system (1.20) and (1.21) tend the (unique) limit solution $\nu_i = 0$, $i = 0, 1$. For any finite values of τ , we establish the following proposition.

Proposition 1.1. *For all $\tau \leq 0$, there exists a unique solution $\nu_i = \nu_i(\tau)$, $i = 0, 1$, of system (1.20) and (1.21), which continuously depends upon τ . In addition, $\nu_i(\tau) \rightarrow 0$ whenever $\tau \rightarrow -\infty$, and $\nu_i(\tau)$ strictly grows and tends to $\nu_i(0)$ as $\tau \rightarrow 0$, $i = 0, 1$.*

Proof. See appendix A.

1.5 A SPECIAL CASE: AN AFFINE DEMAND FUNCTION

Assume that the demand function $G(p)$ is affine, i.e.,

$$G(p) := -Kp + T, \quad (1.22)$$

where, $K > 0$, $T > 0$.

Under this extra assumption, theorem 1.2 entails the next corollary.

Corollary 1.1. *Under assumptions **A1.1-A1.3**, for all $\beta \in (0, 1]$, the demand function of type (1.22) implies the uniqueness of the interior equilibrium.*

Proof. See appendix A.

This section mainly targets at the study of the behavior (as a function of the parameter β) of the three most popular equilibrium kinds: 1) the consistent conjectural variations equilibrium (CCVE), 2) the Cournot-Nash equilibrium, and 3) the perfect competition equilibrium.

1.5.1 CONSISTENT CONJECTURAL VARIATIONS EQUILIBRIUM

The conjectural variation equilibrium (CVE) is called consistent if the influence coefficients at the interior CVE meet the consistency principle represented by systems (1.18) and (1.19).

For all $\beta \in (0, 1]$, corollary 1.1 provides for the existence of the unique interior equilibrium $(p^*(\beta), q_0^*(\beta), q_1^*(\beta), \nu_0^*(\beta), \nu_1^*(\beta))$. The behavior of the CCVE is described in the following result:

Theorem 1.3. *For the affine demand function $G(p)$ from (1.22), the price $p^*(\beta)$, the supply outputs $q_i^*(\beta)$, $i = 0, 1$, and the influence coefficients $\nu_i^*(\beta)$, $i = 0, 1$, characterizing the interior equilibrium, together with total market supply $G^*(\beta) = q_0^*(\beta) + q_1^*(\beta)$, are*

continuously differentiable by $\beta \in (0, 1]$. Furthermore, $q_0^*(\beta)$ and $G^*(\beta)$ strictly increase, whereas $p^*(\beta)$, $\nu_0^*(\beta)$, $\nu_1^*(\beta)$ and $q_1^*(\beta)$ strictly decrease.

Proof. See appendix A.

1.5.2 COURNOT-NASH EQUILIBRIUM

Below, we will examine the comportment of the (exterior) Cournot-Nash equilibrium as a function of the parameter β .

The well-known Cournot-Nash conjecture

$$\omega_i = \frac{\partial G}{\partial q_i} = 1, \quad i = 0, 1, \quad (1.23)$$

in the proposed framework is equivalent to the next conjecture:

$$\nu_i = -\frac{\partial p}{\partial q_i} = -\frac{1}{G'(p)} = -\frac{1}{K}, \quad i = 0, 1. \quad (1.24)$$

Theorem 1.1 implies that, for all $\beta \in (0, 1]$, there exists uniquely the Cournot-Nash equilibrium, denoted by $(p^c(\beta), q_0^c(\beta), q_1^c(\beta))$. The behavior of the Cournot-Nash equilibrium is described in the following result:

Theorem 1.4. *For the affine demand function $G(p)$ described in (1.22), the price $p^c(\beta)$ and the supply values $q_i^c(\beta)$, $i = 0, 1$, from the Cournot-Nash equilibrium, are continuously differentiable with respect to $\beta \in (0, 1]$. Moreover, $p^c(\beta)$ and $q_1^c(\beta)$ strictly decrease, whereas $q_0^c(\beta)$ strictly increase.*

Proof. See appendix A.

It is quite evident that the Cournot-Nash equilibrium in our framework does not satisfy the consistency criterion, i.e., it is not interior (consistent) equilibrium.

1.5.3 PERFECT COMPETITION EQUILIBRIUM

Now, the comportment of the (exterior) perfect competition equilibrium as a function of the parameter β will be evaluated.

The perfect competition conjecture

$$\omega_i = 0, \quad i = 0, 1, \quad (1.25)$$

in our framework is described with the subsequent conjecture:

$$\nu_i = \frac{\partial p}{\partial q_i} = 0, \quad i = 0, 1. \quad (1.26)$$

For every $\beta \in (0, 1]$, theorem 1.1 guarantees that there exist uniquely the exterior equilibrium implementing the perfect competition. The latter will be represented by $(p^t(\beta), q_0^t(\beta), q_1^t(\beta))$. Once more, in order to conduct a comparative study of all three types of equilibrium in our affine framework, we will establish the clear-cut formulas for the perfect equilibrium supplies and price in the next result.

Theorem 1.5. *For the affine demand function $G(p)$ described in (1.22), the price $p^t(\beta)$ and the output volumes $q_i^t(\beta)$, $i = 0, 1$, related to the perfect competition equilibrium, are invariant for all $\beta \in (0, 1]$ and are described by the clear-cut expressions:*

$$p^t = \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)}{a_0 + a_1 + a_0 a_1 K}, \quad (1.27)$$

$$q_0^t = \frac{a_1 (G(b_0) + D) + (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}, \quad (1.28)$$

$$q_1^t = \frac{a_0 (G(b_1) + D) - (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}. \quad (1.29)$$

Proof. See appendix A.

Like in the Cournot-Nash case, one can easily see that the perfect competition equilibrium within our framework does not meet the consistency criterion, thus, it is non-interior (inconsistent) equilibrium.

1.5.4 COMPARISON OF CONSISTENT CVE WITH COURNOT-NASH AND PERFECT COMPETITION EQUILIBRIUMS

In this subsection, taking advantage of the simple (affine) demand function, we will deduce some comparative statics results. These are always interesting for evaluating the strong and weak points of different concepts of more or less similar nature.

Theorem 1.6. *For the affine demand function $G(p)$ from (1.22), the price functions in the CCVE, $p^*(\beta)$, the Cournot-Nash equilibrium, $p^c(\beta)$, and the perfect competition equilibrium, p^t , satisfy the following inequalities:*

$$p^t < \lim_{\beta \rightarrow 0} p^*(\beta), \quad (1.30)$$

and

$$p^*(\beta) < p^c(\beta), \quad \forall \beta \in (0, 1]. \quad (1.31)$$

Proof. See appendix A.

Inequality (1.30), in general, does not hold when $\beta \rightarrow 1$. The latter is a very curious result because the perfect competition equilibrium price p^t is usually the lowest in the market while the Cournot-Nash equilibrium price is the highest (as seen in (1.31)). Moreover, in some cases, it may happen that $p^t > p^c(\beta)$ for the values of β near 1 (e.g., see table 1.4).

1.5.5 OPTIMALITY CRITERION FOR β

In order to find an optimal (in some sense) value for the “socialization” level β of the semi-public company, we study the behavior of the private firm’s profit function in the two equilibrium states: CCVE and Cournot-Nash equilibrium.

The function $\pi_1(p, q_1)$ given by (1.5) is continuously differentiable with respect to p and q_1 , while $p^*(\beta)$, $q_1^*(\beta)$, $p^c(\beta)$ and $q_1^c(\beta)$, are continuously differentiable with respect to β . Therefore, for the equilibrium states CCVE and Cournot-Nash, we have that the private firm’s net profit values,

$$\pi_1^*(\beta) = p^*(\beta)q_1^*(\beta) - \frac{1}{2}a_1q_1^*(\beta)^2 - b_1q_1^*(\beta), \quad (1.32)$$

in the interior equilibrium (CCVE), as well as the similar values

$$\pi_1^c(\beta) = p^c(\beta)q_1^c(\beta) - \frac{1}{2}a_1q_1^c(\beta)^2 - b_1q_1^c(\beta), \quad (1.33)$$

in the Cournot-Nash exterior equilibrium, are continuously differentiable by $\beta \in (0, 1]$.

Theorem 1.7. *The functions $\pi_1^*(\beta)$ and $\pi_1^c(\beta)$ are strictly decreasing with respect to $\beta \in (0, 1]$. Moreover, the following inequalities hold:*

$$\pi_1^*(1) > \pi_1^c(1) \quad (1.34)$$

and

$$\lim_{\beta \rightarrow 0} \pi_1^*(\beta) < \lim_{\beta \rightarrow 0} \pi_1^c(\beta). \quad (1.35)$$

Proof. See appendix A.

Directly from the proof of the last theorem, we conclude that there exists the value $\bar{\beta}$ such that $\pi_1^*(\bar{\beta}) = \pi_1^c(\bar{\beta})$. We now assume that the semi-public firm is socially responsible, and making use of the subsidy policy, it economically motivates the private firm to change its Cournot-Nash strategy to the consistent CVE comportment, or pays subsidies to the consumers to compensate the highest price in the Cournot-Nash equilibrium. The choice of this parameter $\bar{\beta}$ allows the semi-public company not to pay subsidies either to the private company or to the consumers.

With this idea in mind, we introduce the following definition:

Definition 1.4. *The value of the parameter $\bar{\beta} \in (0, 1)$ such that $\pi_1^*(\bar{\beta}) = \pi_1^c(\bar{\beta})$ is called optimal socialization level.*

From theorem 1.7, it follows immediately that, for the duopoly model considered in this paper, we can always find the optimal socialization level for the semi-public company. In other words, the following result has been established above:

Theorem 1.8. *Under assumptions **A1.1-A1.3**, there exists the value of $\bar{\beta} \in (0, 1)$ such that $\pi_1^*(\bar{\beta}) = \pi_1^c(\bar{\beta})$. In other words, the optimal socialization level exists.*

1.6 NUMERICAL RESULTS

In this section, we rely on the data of the numerical experiments exposed in the work of Liu et al. (2007). Here, we describe the experiments in more detail.

The inverse demand function is given by

$$p(G, D) = 50 - 0.02(G + D) = 50 - 0.02(q_0 + q_1). \quad (1.36)$$

Then, solving (1.36) for $G + D$ yields the demand function

$$G(p) + D = -50p + 2500. \quad (1.37)$$

The producers' cost functions are quadratic and are described by (1.2), where the values a_i and b_i are given in table 1.1.

We calculate and compare three types of equilibrium: the consistent conjectural variations equilibrium (CCVE), the Cournot-Nash equilibrium, and the perfect competition equilibrium. The influence coefficients for the CCVE are determined by equations (1.18)

i	b_i	a_i
0	2.0	0.02
1	1.75	0.0175
2	3.25	0.00834

Table 1.1: Experiments' input data

and (1.19). For the Cournot-Nash equilibrium, the influence coefficients are given by the equality (1.24), while for the perfect competition equilibrium, they have the value from (1.26).

Based on the data of table 1.1, we proceed to perform the numerical experiments for the following three instances:

Experiment 1.1. *Firm $i = 0$ is public and firm $i = 2$ is private.*

Experiment 1.2. *Firm $i = 0$ is public and firm $i = 1$ is private.*

Experiment 1.3. *Firm $i = 2$ is public and firm $i = 1$ is private.*

In each instance, we handle the following notation for each of the three kinds of equilibrium: **CNE** for Cournot-Nash Equilibrium, **CCVE** for Consistent Conjectural Variations Equilibrium, and **PCE** for Perfect Competition Equilibrium.

1.6.1 EXPERIMENT 1.1

For this instance, firm $i = 0$ is semi-public and firm $i = 2$ is private, so the semi-public firm is stronger than the private firm; that is, the inequality $b_0 \leq b_1$ holds (assumption **A1.2**). The numerical results of this experiment are shown in table 1.2.

From the results of table 1.2, we see that the behavior of variables is as described in the theorems of the previous sections. The numerical results show that, for socialization levels $0 < \beta \leq 0.50$, the private firm's profit is higher in the Cournot-Nash equilibrium than in the CCVE, but for $0.75 \leq \beta \leq 1$, its profit is higher in the CCVE equilibrium than in the Cournot-Nash. Then, the optimal socialization level lies within the interval $0.50 < \beta_{\text{optimal}} < 0.75$. Furthermore, as a result of the numerical experiment, the approximate optimal value $\beta_{\text{optimal}} = 0.55262$ is found, for which the private firm's net profit is almost the same both in the Cournot-Nash (CNE) and the Consistent Conjectural Variations Equilibrium (CCVE). The corresponding CCVE (interior equilibrium) is presented as follows: $(p^*, q_0^*, q_1^*, \nu_0^*, \nu_1^*) = (18.07, 829.5, 766.8, 0.009830, 0.01099)$. This means that, if the semi-public company $i = 0$ accepts its objective function as a mixture of 55 % of domestic social surplus and 45 % of its (would-be) net profit, then, the private (foreign)

		$\omega_i = -G'(p)\nu_i$		
		CNE	CCVE	PCE
$\beta = 0.25$	ω_0	1.0	0.49870	0.0
	ω_2	1.0	0.57870	0.0
	p	22.609318	19.486879	13.595420
	q_0	686.424683	710.318787	579.770996
	q_2	683.109314	815.337402	1240.458008
	G	1369.533936	1525.656250	1820.229004
	π_0	14124.014648	13194.797852	11644.427734
	π_2	11278.648438	10466.422852	6416.529297
$\beta = 0.50$	ω_0	1.0	0.49275	0.0
	ω_2	1.0	0.55480	0.0
	p	20.858637	18.324474	13.595420
	q_0	835.733032	808.190979	579.770996
	q_2	621.335083	775.585388	1240.458008
	G	1457.068115	1583.776367	1820.229004
	π_0	19391.527344	19203.304688	19927.513672
	π_2	9331.004883	9183.150391	6416.529297
$\beta = 0.75$	ω_0	1.0	0.48590	0.0
	ω_2	1.0	0.52850	0.0
	p	18.868885	17.108099	13.595420
	q_0	1005.430603	911.731628	579.770996
	q_2	551.125122	732.863464	1240.458008
	G	1556.555664	1644.595093	1820.229004
	π_0	25023.082031	25747.185547	28210.599609
	π_2	7341.368652	7916.434082	6416.529297
$\beta = 1.0$	ω_0	1.0	0.47835	0.0
	ω_2	1.0	0.50000	0.0
	p	16.587503	15.846337	13.595420
	q_0	1200.000122	1020.859924	579.770996
	q_2	470.624695	686.823181	1240.458008
	G	1670.624756	1707.683105	1820.229004
	π_0	31014.878906	32875.441406	36493.683594
	π_2	5353.354980	6684.358887	6416.529297

Table 1.2: Results of experiment 1.1

competitor is indifferent to the choice of the Cournot-Nash or CCVE model to generate its supply because its net profit is the same in both cases. This can be considered as a win-win outcome for the local authorities since they need not either subsidize the consumers (in order to reimburse the higher price of the commodity in the Cournot-Nash equilibrium) nor pay compensation to the private firm for having accepted the CCVE equilibrium model (which uses to decrease the private firm's net profit as compared to that in the Cournot-Nash equilibrium).

In the previous sections, we made use of assumption **A1.2**. In the following two experiments, we will consider the case when this assumption is not met to see how our model behaves.

1.6.2 EXPERIMENT 1.2

In this instance, firm $i = 0$ is semi-public and firm $i = 1$ is private, so the semi-public firm is weaker than the private firm. The numerical results of this experiment are shown in table 1.3.

The results shown in table 1.3 demonstrate that the variables still behave as described in the theorems of the previous sections, even though assumption **A1.2** is not met.

In this second experiment, we have that $\beta_{\text{optimal}} = 0.62905$ and its corresponding interior equilibrium is as follows: $(p^*, q_0^*, q_1^*, \nu_0^*, \nu_1^*) = (19.46, 905.4, 627.2, 0.01175, 0.01098)$.

1.6.3 EXPERIMENT 1.3

For this instance, firm $i = 2$ is semi-public and firm $i = 1$ is private, so the semi-public firm now is even weaker than the private firm in comparison to the previous experiment. Again, we observe that the variables behave according to our model. The numerical results of this experiment are presented in table 1.4.

For the third instance, we have that $\beta_{\text{optimal}} = 0.80324$ and its corresponding interior equilibrium is as follows: $(p^*, q_0^*, q_1^*, \nu_0^*, \nu_1^*) = (13.33, 1359, 474.8, 0.01099, 0.006886)$.

From the results of the numerical experiments, we can see that the weaker semi-public company (as compared to the private company), the closer to 1 its optimal socialization level.

		$\omega_i = -G'(p)\nu_i$		
		CNE	CCVE	PCE
$\beta = 0.25$	ω_0	1.0	0.59445	0.0
	ω_2	1.0	0.59110	0.0
	p	23.938864	21.564388	17.181818
	q_0	711.353699	746.033264	759.090942
	q_2	591.703064	675.747375	881.818176
	G	1303.056763	1421.780640	1640.909180
	π_0	14790.943359	14083.678711	12493.647461
	π_2	10065.734375	9393.968750	6804.028809
$\beta = 0.50$	ω_0	1.0	0.59000	0.0
	ω_2	1.0	0.56425	0.0
	p	22.112148	20.202539	17.181818
	q_0	851.401917	848.829346	759.090942
	q_2	542.990601	641.043823	881.818176
	G	1394.392578	1489.873169	1640.909180
	π_0	19596.324219	19344.347656	19225.105469
	π_2	8476.324219	8233.186523	6804.028809
$\beta = 0.75$	ω_0	1.0	0.58485	0.0
	ω_2	1.0	0.53400	0.0
	p	20.010050	18.733961	17.181818
	q_0	1012.562927	960.608276	759.090942
	q_2	486.934662	602.693542	881.818176
	G	1499.497559	1563.301758	1640.909180
	π_0	24847.169922	25176.449219	25956.564453
	π_2	6816.779297	7057.777832	6804.028809
$\beta = 1.0$	ω_0	1.0	0.57895	0.0
	ω_2	1.0	0.50000	0.0
	p	17.565216	17.155014	17.181818
	q_0	1200.000122	1082.066895	759.090942
	q_2	421.739105	560.182312	881.818176
	G	1621.739258	1642.249268	1640.909180
	π_0	30578.642578	31659.882813	32688.021484
	π_2	5113.585938	5883.829590	6804.028809

Table 1.3: Results of experiment 1.2

		$\omega_i = -G'(p)\nu_i$		
		CNE	CCVE	PCE
$\beta = 0.25$	ω_0	1	0.57135	0
	ω_2	1	0.4581	0
	p	21.52866	18.10005	13.16771
	q_0	896.135986	981.770264	1189.17383
	q_2	527.430908	613.227173	652.440613
	G	1423.5669	1594.99744	1841.6145
	π_0	18097.7637	16920.0293	14375.8008
	π_2	7997.77197	6735.87891	3724.68872
$\beta = 0.50$	ω_0	1	0.5625	0
	ω_2	1	0.41105	0
	p	18.86553	16.055681	13.16771
	q_0	1100.30933	1141.02332	1189.17383
	q_2	456.414124	556.192444	652.440613
	G	1556.72339	1697.21582	1841.6145
	π_0	24250.3164	23585.2266	22854.6621
	π_2	5989.02344	5249.89893	3724.68872
$\beta = 0.75$	ω_0	1	0.5519	0
	ω_2	1	0.3569	0
	p	15.652787	13.826693	13.16771
	q_0	1346.61963	1318.49219	1189.17383
	q_2	370.740997	490.173065	652.440613
	G	1717.3606	1808.66528	1841.6145
	π_0	31259.9844	31230.5996	31333.5254
	π_2	3951.65552	3817.31006	3724.68872
$\beta = 1.0$	ω_0	1	0.539	0
	ω_2	1	0.2943	0
	p	11.700713	11.425176	13.16771
	q_0	1649.61206	1515.0188	1189.17383
	q_2	265.352356	413.722351	652.440613
	G	1914.96436	1928.74121	1841.6145
	π_0	39263.8008	40014.6523	39812.3867
	π_2	2024.34131	2505.13232	3724.68872

Table 1.4: Results of experiment 1.3

1.7 CONCLUSIONS AND FUTURE RESEARCH

In this chapter, we presented mathematically rigorous proofs of the conjectures (cf., Kalashnikov et al. (2014)) concerning the behavior of the semi-public company and private firm of a semi-mixed duopoly of a homogeneous good. The main difference of this work from the classical duopoly models is in the presence of one producer who maximizes not its net profit, but the convex combination of the latter with the domestic social surplus. Moreover, we not only studied the classical Cournot-Nash and perfect competition equilibriums in the model, but also the consistent conjectural variations equilibrium (CCVE) introduced and examined previously by numerous authors.

We demonstrated the existence (and in the case of the affine demand function, the uniqueness) of the CCVE and provided elements of comparative static analysis by evaluating the relationships between the equilibrium price and equilibrium production outputs of both the semi-public and private producers in the aforementioned equilibrium types.

Finally, the role of the convex combination parameter $\beta \in (0, 1]$ involved in the definition of the objective function of the semi-public (socially responsible) company was discussed and investigated. Since this parameter can be considered as reflecting the semi-public company's socialization level, we introduced a criterion to estimate its optimal value, namely, we proposed to admit the value of this parameter as desirable (optimal) if, for this parameter value, the net profits of the private firm under the CCVE and the Cournot-Nash equilibrium conditions coincide. It can be reasonable when taking into account that, with such profit equality, the socially responsible authorities need not pay any subsidies either to the private producer (to compensate its financial losses if switching from the Cournot-Nash strategy to the consistent conjectures prevailing in CCVE), or to the consumers (in the case when the equilibrium price under Cournot-Nash equilibrium turns out to be much higher than it would be in the consistent conjectural variations equilibrium, CCVE). Under the additional assumption about the affine nature of the model's demand function, the existence of such an optimal parameter value $\bar{\beta} \in (0, 1)$ is proven.

The linearity of the demand function is a serious restriction. Hence, one of the aims for future research is to relax this condition and extend the obtained results to the semi-mixed duopoly with the demand function being not necessarily affine. The next step of our research plan is to investigate the role of the socialization level parameter in order to find its optimal value in the semi-mixed oligopoly, wherein more than one private firms compete.

CONSISTENT CONJECTURAL VARIATIONS COINCIDE WITH THE NASH SOLUTION IN THE META-MODEL

2.1 INTRODUCTION

As mentioned in chapter 1, the concept of the *conjectural variations equilibrium* (CVE) was first proposed by Bowley (1924) and Frisch (1933) to extend the concept of a solution to a static (Cournot) model (game).

The papers Bulavsky and Kalashnikov (1994, 1995) and the monograph Isac et al. (2002) introduce and examine a new form of the CVE, in which the conjectural variations (represented via the *influence coefficients* of each agent) were used to bring about a new equilibrium concept distinct from that of Cournot-Nash.

For instance, in Isac et al. (2002), the classical oligopoly model was extended to the conjectural oligopoly as follows. Instead of the usual Cournot-Nash assumptions, all producers $i = 1, \dots, n$, considered the conjectural variations described below:

$$G_i(q_i + \eta_i) = G + \eta_i \omega_i(G, q_i). \quad (2.1)$$

Here, G is the current total quantity of the product cleared in the market, q_i and $q_i + \eta_i$ are, respectively, the present and the expected supplies by the i -th agent, whereas $G_i(q_i + \eta_i)$ is the total cleared market volume *conjectured* by the i -th agent as a response to changing her own supply from q_i towards $q_i + \eta_i$. The conjecture function ω_i was referred to as the i -th agent's *influence quotient (coefficient)*. Recall that the usual Cournot-Nash model assumes $\omega_i \equiv 1$ for all $i = 1, \dots, n$. Under general enough assumptions concerning the properties of the influence coefficients $\omega_i = \omega_i(G, q_i)$, cost functions $f_i(q_i)$, and the inverse demand function (or, price function) $p = p(G)$, new existence and uniqueness

results for the conjectural variations equilibrium (CVE) were obtained. This approach was further developed in Kalashnikov et al. (2009); Kalashnikov, Cordero and Kalashnikov-Jr (2010).

An interesting comparison of the Cournot-Nash and Bertrand models is provided in Kreps and Scheinkman (1983). The Bertrand's model of oligopoly, which considers the perfect competition, assumes that: 1) there is a competition over prices; and 2) production follows the realization of the demand. The authors from Kreps and Scheinkman (1983) demonstrate that both of these assumptions are required. In more detail, they study a two-stage oligopoly game where, first, there is a simultaneous production, and, second, after the production levels are made public, there is a price competition. Under rather mild assumptions about the demand, the authors show in Kreps and Scheinkman (1983) that the unique equilibrium is the Cournot-Nash one. This illustrates that solutions to oligopoly games depend on both the strategic variables employed and the context (game form) in which those variables are employed.

A different example of a two-stage game can be found in Murphy and Smeers (2005). Similar studies were presented in the unpublished manuscript Kimbrough et al. (2014) dealing mainly with forward markets but still providing important insight into the consistent conjectural variations equilibrium regarding the many-stage oligopoly model. The authors examine the impact of the conjectures about the players' knowledge on the outcome of the game, where the outcomes are consistent with their conjectures. Then they deduce the similar result obtained by Allaz and Vila (1993) on forward markets but under different assumptions of knowledge from consistent conjectural variations. All the above-mentioned papers deal in various ways with games with conjectural variations equilibrium (CVE).

In chapter 1, we brought up the approach proposed in Bulavsky (1997) to define the consistent CVE, where each agent applies a verification procedure and checks if its influence coefficient is consistent (compatible) with those of the rest of the agents. Exactly the same verification formulas were obtained independently (10 years later) in Liu et al. (2007) establishing the existence and uniqueness of consistent conjectural variation equilibrium in an electricity market. However, to do that, the authors of Liu et al. (2007) made use of a much more complicated optimal control technique when searching the system's steady states (a similar approach was employed in Driskill and McCafferty (1989)). Moreover, in Liu et al. (2007), the inverse demand function is linear, and the agents' cost functions are quadratic, whereas Bulavsky (1997) allows nonlinear and even non-differentiable demand functions as well as arbitrary convex cost functions for the agents.

As shown in chapter 1 and Kalashnikov, Bulavsky, Kalashnykova and Castillo-Pérez (2011), in general, the consistent conjectures are *not* the Cournot-Nash conjectures. In other words, at a consistent CVE, each agent i does not use the Cournot-Nash equilibrium concept since it *does not* assume that all other agents are dead set on the (equilibrium) production volumes q_j , $j \neq i$. In Kalashnikov, Bulavsky, Kalashnykova and López-Ramos (2017), a meta-model is introduced, in which *not* the players' production volumes q_i but their *conjectures* ν_i serve as the players' strategies. The remarkable fact was demonstrated: the consistent (for the original oligopoly) conjectures ν_i^* , while in general not being the Cournot-Nash ones for this game, *are* the optimal Cournot-Nash strategies in the above-mentioned meta-model. In other words, if each player i assumes that the rest of the players stick to their consistent CVE conjectures ν_j^* , $j \neq i$, then, its consistent conjecture ν_i^* is optimal for player i , too. This means that the vector of (consistent in the original oligopoly) conjectures $(\nu_1^*, \dots, \nu_n^*)$, coincides with the classical Cournot-Nash equilibrium in the meta-game. Similar results were claimed (without proof) in Kalashnykova et al. (2012) but only for quadratic cost functions.

However, since the meta-model allows the agents to select their strategies from \mathbb{R}^n , and this isn't a compact set, the existence of the Cournot-Nash equilibrium in the meta-model has to be guaranteed by some extra assumptions (similar difficulties related to the price equilibrium were run into and overcome by Kress and Pesch (2016)). Under those assumptions, the complete equivalence of the consistent CVE in the original oligopoly and the meta-model has been established in this chapter.

2.2 MODEL SPECIFICATION

Consider an oligopoly of at least two producers of a homogeneous good with cost functions $f_i = f_i(q_i)$, $i = 1, \dots, n$, $n \geq 2$, where $q_i \geq 0$ is the supply by producer i . Consumers' demand is described by a demand function $G = G(p)$, whose argument p is the market price established by a cleared market. An active demand D is nonnegative and does not depend upon the price. The equilibrium between supply and demand for a given price p is guaranteed by the following balance equality:

$$\sum_{i=1}^n q_i = G(p) + D. \quad (2.2)$$

We assume the following properties of the model's data:

A2.1. The demand function $G = G(p) \geq 0$ is defined for $p > 0$, being strictly decreasing and continuously differentiable.

A2.2. For each $i = 1, \dots, n$, the function $f_i = f_i(q_i)$ is defined for every $q_i \geq 0$, is twice continuously differentiable, and in addition, the following inequalities hold:

$$f'_i(0) > 0 \text{ and } f''_i(q_i) > 0, \quad \forall q_i \geq 0. \quad (2.3)$$

Next, every producer $i = 1, \dots, n$ chooses its output volume $q_i \geq 0$ so as to maximize its net profit function:

$$\pi_i(p, q_i) = pq_i - f_i(q_i). \quad (2.4)$$

Now we postulate that the producers admit that their perturbations in production volumes may affect the price value p . Thus, the first order maximum condition to describe the equilibrium will have the form:

$$\frac{\partial \pi_i}{\partial q_i} = p + q_i \frac{\partial p}{\partial q_i} - f'_i(q_i) \begin{cases} = 0, & \text{if } q_i > 0, \\ \leq 0, & \text{if } q_i = 0, \end{cases} \quad i = 1, \dots, n. \quad (2.5)$$

Therefore, to describe the (infinitesimal) behavior of producer i , it is enough to conjecture the first order derivative

$$\frac{\partial p}{\partial q_i} \equiv -\nu_i. \quad (2.6)$$

The conjectured first-order dependence of p on q_i must provide, at least locally, concavity of the i -th producer's conjectured profit as a function of its output. As we suppose that the cost functions f_i are strictly convex and strictly increasing, by inequalities (2.3), then, for all $i = 1, \dots, n$, concavity of the product pq_i with respect to q_i would suffice. Here, it is enough to assume the coefficient ν_i to be nonnegative and constant. Then, the conjectured dependence of the profit's variations upon the production output $q_i + \eta_i$ has the form:

$$\widehat{\pi}_i(q_i + \eta_i) = (p - \nu_i \eta_i)(q_i + \eta_i) - f_i(q_i + \eta_i), \quad (2.7)$$

which is a concave function on η_i .

Here, it is worthwhile to mention that relation (2.7) does *not* mean that producer i exercises its market power. In fact, vice versa, player i is a price-taker: it accepts its *conjectured* influence coefficient ν_i and calculates the expected variation in the market-

clearing price p (and hence in its net profit) under the infinitesimal variation of its produce η_i .

Therefore, the maximum necessary condition at $\eta_i = 0$ is provided by the relationships

$$\begin{cases} p = \nu_i q_i + f'_i(q_i), & \text{if } q_i > 0, \\ p \leq f'_i(0), & \text{if } q_i = 0, \end{cases} \quad i = 1, \dots, n, \quad (2.8)$$

and it is the sufficient condition, too.

Here we use, again, the approach from Bulavsky (1997) and Kalashnykova et al. (2012), where the conjectured parameters for the equilibrium are determined simultaneously with the price p and the output values q_i in the *interior* equilibrium state described by the combined vector $(p, q_1, \dots, q_n, \nu_1, \dots, \nu_n)$. Nevertheless, in order to define the interior equilibrium, we first need to introduce the notion of *exterior* equilibrium (cf., Bulavsky (1997)) with the parameters ν_i assigned in the exogenous form.

2.3 EXTERIOR EQUILIBRIUM

We define the concept of exterior equilibrium as follows:

Definition 2.1. *A vector (p, q_1, \dots, q_n) is called exterior equilibrium for the given influence coefficients $\nu_i \geq 0$, $i = 1, \dots, n$, if the market is balanced, i.e., equality (2.2) holds, and for each $i = 1, \dots, n$, the maximum conditions (2.8) are valid.*

From now onward, we are going to consider only the case when the set of really producing participants is fixed. To guarantee this feature, we make the assumption listed below.

A2.3. *For $p_0 = \max_{1 \leq i \leq n} \{f'_i(0)\}$ and any $i = 1, \dots, n$, there exists a unique (due to A2.2) supply volume $q_i^0 \geq 0$ such that*

$$p_0 = f'_i(q_i^0), \text{ and in addition, } \sum_{i=1}^n q_i^0 < G(p_0). \quad (2.9)$$

Lemma 2.1. *Assumptions A2.1-A2.3 imply that for all nonnegative values of ν_i , $i = 1, \dots, n$, any exterior equilibrium has its supply values q_i strictly positive if and only if $p > p_0$.*

Proof. cf., Kalashnikov, Bulavsky, Kalashnykova and López-Ramos (2017).

The existence and uniqueness of the exterior equilibrium for any set of (nonnegative) conjectures (influence coefficients) were established in Bulavsky (1997). However, in the latter paper, only differentiability of the equilibrium clearing price p with respect to the active demand D was proven, while in this work, we also need to show that the same equilibrium price function $p = p(D, \nu_1, \dots, \nu_n)$ is differentiable by the influence coefficients, too. Therefore, the following theorem has been proved.

Theorem 2.1. *Under assumptions **A2.1-A2.3**, for any $D \geq 0$, $\nu_i \geq 0$, $i = 1, \dots, n$, there exists uniquely the exterior equilibrium (p, q_1, \dots, q_n) that depends continuously on the parameters $D \geq 0$, $\nu_i \geq 0$, $i = 1, \dots, n$. The equilibrium price $p = p(D, \nu_1, \dots, \nu_n)$, as a function of these parameters, is differentiable with respect to both D and ν_i , $i = 1, \dots, n$. Moreover, $p(D, \nu_1, \dots, \nu_n) > p_0$, and*

$$\frac{\partial p}{\partial D} = \frac{1}{\sum_{i=1}^n \frac{1}{\nu_i + f_i''(q_i)} - G'(p)}, \quad (2.10)$$

while

$$\frac{\partial p}{\partial \nu_i} = \frac{\frac{q_i}{\nu_i + f_i''(q_i)}}{\sum_{k=1}^n \frac{1}{\nu_k + f_k''(q_k)} - G'(p)} > 0, \quad i = 1, \dots, n. \quad (2.11)$$

Similarly, the equilibrium supply $q_i = q_i(D, \nu_1, \dots, \nu_n)$, $i = 1, \dots, n$, is differentiable with respect to the influence coefficients ν_k , $k = 1, \dots, n$, with the partial derivatives having the forms:

$$\frac{\partial q_i}{\partial \nu_i} = -\frac{q_i}{\nu_i + f_i''(q_i)} \frac{\left[\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k + f_k''(q_k)} - G'(p) \right]}{\sum_{k=1}^n \frac{1}{\nu_k + f_k''(q_k)} - G'(p)} < 0, \quad i = 1, \dots, n, \quad (2.12)$$

and

$$\frac{\partial q_i}{\partial \nu_j} = \frac{1}{\nu_i + f_i''(q_i)} \left[\frac{\frac{q_j}{\nu_j + f_j''(q_j)}}{\sum_{k=1}^n \frac{1}{\nu_k + f_k''(q_k)} - G'(p)} \right] > 0, \quad i, j = 1, \dots, n, \quad j \neq i. \quad (2.13)$$

Proof. cf., Kalashnikov, Bulavsky, Kalashnykova and López-Ramos (2017).

2.4 INTERIOR EQUILIBRIUM

Now we are in a position to define the concept of interior equilibrium. To do that, we make use of the verification procedure for the influence coefficients ν_i introduced in Bulavsky (1997). Assume that the system is in the exterior equilibrium $(p, q_1 \dots, q_n)$ that occurs for some given D and $\nu_i, i = 1, \dots, n$. Now, producer $k, 1 \leq k \leq n$, temporarily changes its behavior by abstaining from maximizing its conjectured profit, subtracts its production q_k from the total demand and makes infinitesimal fluctuations around the latter. This is tantamount to restricting the model's producers to the subset $I_{-k} := \{i \mid 1 \leq i \leq n, i \neq k\}$ with the output q_k subtracted from the active demand with the balance equality restated in the form:

$$\sum_{\substack{i=0 \\ i \neq k}}^n q_i = G(p) + D - q_k. \quad (2.14)$$

Variations of the production output by producer k are, then, equivalent to the active demand fluctuation in the form $dD_k := d(D - q_k) = -dq_k$. If we treat these variations as infinitesimal, producer k can evaluate the derivative of the equilibrium price with respect to the active demand, i.e., their influence coefficient.

Applying formula (2.10) from theorem 2.1 to calculate the derivatives, producer k is temporarily absent from the equilibrium model. Hence one has to exclude the term with number $i = k$ from the sum. This yields the consistency criterion.

Definition 2.2 (Consistency criterion). *In the exterior equilibrium $(p, q_1 \dots, q_n)$, the influence coefficients $\nu_i, i = 1, \dots, n$, are called consistent if the following equalities hold:*

$$\nu_i = \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{\nu_j + f_j''(q_j)} - G'(p)}, \quad i = 1, \dots, n. \quad (2.15)$$

Now, we define the consistent (interior) equilibrium.

Definition 2.3. *A collection $(p, q_1 \dots, q_n, \nu_1, \dots, \nu_n)$ is referred to as interior equilibrium if, for the influence coefficients $\nu_i \geq 0, i = 1, \dots, n$, the vector $(p, q_1 \dots, q_n)$ is the exterior equilibrium, and the consistency criterion 2.2 is valid for all those ν_i .*

Theorem 2.2. *Let the number of oligopoly producers be at least three, i.e., $n \geq 3$, then, under assumptions **A2.1-A2.3**, there exists an interior equilibrium. Moreover, if the number of producers is two, i.e., $n = 2$, in addition to assumptions **A2.1-A2.3**, suppose that there exists an $\varepsilon > 0$ such that $G'(p) \leq -\varepsilon$ for all $p > 0$, then, there exists interior equilibrium.*

Proof. See appendix B.

Without additional assumptions or simplifications of the model, uniqueness of the interior equilibrium is not guaranteed.

In our future research, we are going to extend the obtained results to the case of not necessarily differentiable demand functions. However, some of the essential techniques can be developed now, in the differentiable case but under slightly stronger assumptions about the structure of the producers' cost functions. Namely, let us introduce the following assumption instead of **A2.2**. Moreover, this new assumption is used in the extension of the existence theorem 2.2 for the case of duopoly, i.e., when $n = 2$.

A2.4. *For every $i = 1, \dots, n$, the cost function f_i is quadratic (and strictly convex) with $f_i(0) = 0$, $f'_i(0) > 0$ and $f''_i > 0$, i.e.,*

$$f_i(q_i) = \frac{1}{2}a_i q_i^2 + b_i q_i, \quad (2.16)$$

where $a_i > 0$, $b_i > 0$, $i = 1, \dots, n$.

Theorem 2.3. *Let $n = 2$ (duopoly), and assumptions **A2.1**, **A2.3** and **A2.4** hold true. If in addition there exists $\varepsilon > 0$ such that $G'(p) \leq -\varepsilon$ for all $p > 0$, then, there exists the unique interior equilibrium.*

Proof. cf., Kalashnikov, Bulavsky, Kalashnykova and López-Ramos (2017).

Now denote the value of the demand function's derivative by $\tau = G'(p)$ and rewrite the consistency equations (2.15) in the form:

$$\nu_i = \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{\nu_j + f''_j(q_j)} - \tau}, \quad i = 1, \dots, n, \quad (2.17)$$

where $\tau \in (-\infty, 0]$.

When $\tau \rightarrow -\infty$, the system (2.17) converges to the solution $\nu_i = 0$, $i = 1, \dots, n$. For all finite values of τ we establish the following proposition.

Theorem 2.4. *Let assumptions **A2.1**, **A2.3** and **A2.4** be valid. Then, for any $\tau \in (-\infty, 0]$ there exists a unique solution $\nu_i = \nu_i(\tau)$, $i = 1, \dots, n$, of system (2.17), continuously depending upon τ . Furthermore, $\nu_i \rightarrow 0$ when $\tau \rightarrow -\infty$, and $\nu_i(\tau)$ strictly increases and tends to $\nu_i(0)$ as $\tau \rightarrow 0$, $i = 1, \dots, n$.*

Proof. The proof is easily deduced from that of theorem 3 in Kalashnikov, Bulavsky, Kalashnykova and Castillo-Pérez (2011).

2.5 CONSISTENT CONJECTURES AS OPTIMAL NASH STRATEGIES IN THE META-GAME

This section establishes the three most important results of this chapter. Indeed, under certain rather mild conditions, we prove the equivalence of the consistent conjectural variations equilibrium (CCVE) in the oligopoly model to the classical Cournot-Nash equilibrium in, what we call the meta-game. The latter comprises the same producers of the oligopoly but with their conjectures (about the possible price variations) as their strategies.

These results seem to be very interesting in two aspects. First, they could be considered as a good justification of the CVE concept as being tightly related to the classical Nash equilibrium. Second, this equivalence occurring in the oligopoly can help one develop a concept similar to the CVE but in application to other kinds of economic and financial models that lack some attributes of the oligopoly and thus do not allow one to introduce the consistent CVE directly. In other words, one could define the consistent CVE in such a model via the Nash equilibrium in the corresponding meta-game.

To begin with, theorem 2.1 allows us to define the following many-person game $\Gamma = (N, V, \Pi, D)$, which will be referred to as the *meta-game*. Here, $D \geq 0$ is the fixed value of the active demand, $N = \{1, \dots, n\}$ is the set of the same producers (the players) as in the model described above, $V = \mathbb{R}_+^n$ represents the set of possible strategies, i.e., the vectors of conjectures $\nu = (\nu_1, \dots, \nu_n)$ accepted by the producers. Finally, $\Pi = \Pi(\nu) = (\pi_1 \dots \pi_n)$ is the collection of payoff values defined (uniquely, according to theorem 2.1) by the strategy vector ν . Indeed, the payoff values $\pi_i = \pi_i(\nu)$, $i = 1, \dots, n$, are defined by formula (2.4), where the equilibrium outputs $q_i \geq 0$, $i = 1, \dots, n$, as well as the equilibrium price p , are the elements of the exterior equilibrium whose existence and uniqueness is guaranteed by theorem 2.1 from section 2.3.

Now, the main results of this chapter are as follows. As was mentioned in the introduction, the Cournot-Nash conjectures $\omega_i = 1$ are usually inconsistent (in sense of criterion 2.2) in our single commodity market model. In other words, the Cournot-Nash conjectures $\nu_i = -1/G'(p)$ in general *do not* satisfy the consistency system (2.15). However, in the meta-game introduced above, the consistent conjectures ν_i , $i = 1, \dots, n$, determined by (2.15) provide the Nash equilibrium. This curious fact could be considered as an extra argument supporting the concept of interior equilibrium introduced in section 2.4.

Theorem 2.5. *Suppose that assumptions **A2.1**-**A2.3** hold. Then, any Nash equilibrium in the meta-game $\Gamma = (N, V, \Pi, D)$ generates interior equilibrium in the original oligopoly.*

Proof. cf., Kalashnikov, Bulavsky, Kalashnykova and López-Ramos (2017).

Since the meta-game strategies set $V = \mathbb{R}_+^n$ is unbounded, the existence of at least one Nash equilibrium state in this game is by no means easy to check. The following three results (under some extra assumptions) guarantee that the existence of interior equilibrium in the original oligopoly implies the existence of Nash equilibrium in the meta-game.

Theorem 2.6. *Suppose that the stronger assumption **A2.4** is true, together with **A2.1** and **A2.3**, and suppose that the function G is concave. Then, the consistency criterion for the original oligopoly is a necessary and sufficient condition for the collection of influence conjectures $\nu = (\nu_1, \dots, \nu_n)$ to produce Nash equilibrium in the meta-game.*

Proof. See appendix B.

Corollary 2.1. *In addition to assumptions **A2.1**, **A2.3** and **A2.4**, if the demand function is affine, that is,*

$$G(p) := -Kp + T, \quad (2.18)$$

where $K > 0$ and $T > 0$, then, the consistency criterion for the original oligopoly is a necessary and sufficient condition for the collection of influence conjectures $\nu = (\nu_1, \dots, \nu_n)$ to form the Nash equilibrium in the meta-game.

Since the concavity of the demand function may be a much too restrictive requirement, the next theorem relaxes it by replacing it with the Lipschitz continuity of the derivative $G'(p)$.

Theorem 2.7. *Suppose that apart from assumptions **A2.1**, **A2.3** and **A2.4**, the regular demand function's derivative is Lipschitz continuous. In more detail, for $n \geq 3$ assume*

that for any $p_1 > 0$ and $p_2 > 0$ the following inequality holds:

$$|G'(p_1) - G'(p_2)| \leq \frac{1}{2s^2 G(p_0)} |p_1 - p_2|, \quad (2.19)$$

where $s = \max\{a_1, \dots, a_n\}$, and the price p_0 is the one defined in the assumption **A2.3**. Next, if $n = 2$ (duopoly), we again suppose that there exists $\varepsilon > 0$ such that $G'(p) \leq -\varepsilon$ for all $p > 0$, and the Lipschitz continuity of the demand function is described in the form:

$$|G'(p_1) - G'(p_2)| \leq \frac{2}{\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\}\right)^2 G(p_0)} |p_1 - p_2|, \quad \forall p_1, p_2 > 0. \quad (2.20)$$

Then, the consistency criterion for the original oligopoly is a necessary and sufficient condition for the collection of influence conjectures $\nu = (\nu_1 \dots \nu_n)$ to be the Nash equilibrium in the meta-game.

Proof. See appendix B.

2.6 NUMERICAL EXPERIMENTS

Now we illustrate our main results from section 2.5. For the numerical experiments, we consider the inverse demand and costs function from an electricity market presented in Liu et al. (2007).

The inverse demand function is given by:

$$p(G, D) = 50 - 0.02(G + D), \quad (2.21)$$

thus, the demand function has the form:

$$G(p) + D = -50p + 2500. \quad (2.22)$$

There are $n = 6$ firms with quadratic costs functions, i.e., $f_i(q_i) = \frac{1}{2}a_i q_i^2 + b_i q_i$, $i = 1, \dots, n$, where the coefficients a_i and b_i are given in table 2.1.

In addition, as a kind of dual concept developed in our previous papers, we are going to consider the influence of the producers over the total output G (cf., Isac et al. (2002)), i.e, the influence coefficients $\omega_i := \frac{\partial G}{\partial q_i}$, $i = 1, \dots, n$.

i	a_i	b_i
1	0.02	2.0
2	0.0175	1.75
3	0.025	3.0
4	0.025	3.0
5	0.0625	1.0
6	0.00834	3.25

Table 2.1: Quadratic costs function's coefficients

In this sense, the conjectures $\omega_i = 1, \forall i = 1, \dots, n$, correspond to the Cournot-Nash conjecture, while the zero-conjectures $\omega_i = 0, \forall i = 1, \dots, n$, lead to the perfect competition model.

Moreover, by the chain rule, one can easily verify the relationship

$$\omega_i = -G'(p)\nu_i, \quad \forall i = 1, \dots, n. \quad (2.23)$$

Experiment 2.1. *For the electricity market described above, the producers' influence coefficients ω_i , supplies q_i and profits π_i , $i = 1, \dots, n$, along with the market's price p and demand G , of the interior equilibrium results are shown in table 2.2.*

i	1	2	3	4	5	6
ω_i	0.19275	0.19635	0.18759	0.18759	0.17472	0.22391
q_i	353.40	405.12	258.44	258.44	142.90	560.18
π_i	1730.4	2080.6	1085.4	1085.4	709.48	2713.8
p	10.431					
G	1978.5					

Table 2.2: Interior equilibrium for the electricity market

Next, we vary the influence coefficient of one of the producers and compute the corresponding exterior equilibrium to see how their profits change.

i	1	2	3	4	5	6
ω_i	0.19275	0.19635	0.18759	0.18759	0.17472	0.22391
	0.39165					
π_i	1730.4	2163.4	1136.0	1136.0	735.45	2844.6
	1703.2					

Table 2.3: Net profits when producer 1 (unilaterally) changes its consistent influence coefficient

In tables 2.3-2.8 we see that whenever one of the producers (unilaterally) increases or decreases its consistent influence coefficient, its profit drops. This, as proved in section

i	1	2	3	4	5	6
ω_i	0.19275	0.19635 0.12000	0.18759	0.18759	0.17472	0.22391
π_i	1688.8	2080.6 2072.0	1055.9	1055.9	694.22	2637.3

Table 2.4: Net profits when producer 2 (unilaterally) changes its consistent influence coefficient

i	1	2	3	4	5	6
ω_i	0.19275	0.19635	0.18759 0.43615	0.18759	0.17472	0.22391
π_i	1783.6	2142.7	1085.4 1066.3	1123.3	728.97	2811.9

Table 2.5: Net profits when producer 3 (unilaterally) changes its consistent influence coefficient

i	1	2	3	4	5	6
ω_i	0.19275	0.19635	0.18759	0.18759 0.061570	0.17472	0.22391
π_i	1697.1	2041.7	1061.8	1085.4 1077.8	697.27	2652.6

Table 2.6: Net profits when producer 4 (unilaterally) changes its consistent influence coefficient

i	1	2	3	4	5	6
ω_i	0.19275	0.19635	0.18759	0.18759	0.17472 0.14870	0.22391
π_i	1728.9	2078.8	1084.3	1084.3	709.48 709.44	2711.0

Table 2.7: Net profits when producer 5 (unilaterally) changes its consistent influence coefficient

i	1	2	3	4	5	6
ω_i	0.19275	0.19635	0.18759	0.18759	0.17472	0.22391 0.47305
π_i	1968.4	2358.2	1255.6	1255.6	796.41	2713.8 2578.1

Table 2.8: Net profits when producer 6 (unilaterally) changes its consistent influence coefficient

2.5, is due to the fact that the consistent influence coefficients describing the interior equilibrium form the Nash equilibrium in the meta-game.

Experiment 2.2. *Here, we are going to find the interior equilibrium for an electricity market (as the base model) that does not meet the conditions of theorems 2.6 and 2.7, and test if it serves as the Nash equilibrium in the meta game.*

Now, we consider the following demand function:

$$G(p) + D = 2400p^{-1.2} + 1600, \quad (2.24)$$

along with the cost functions described above.

The function (2.24) is *not* concave and its derivative is *not* Lipschitz, but we can still compute the corresponding interior equilibrium. The corresponding data is shown in table 2.9.

i	1	2	3	4	5	6
ω_i	0.086234	0.088004	0.083674	0.083674	0.077221	0.10112
q_i	317.77	363.63	231.17	231.17	132.16	479.30
π_i	1467.8	1769.1	903.22	903.22	616.76	2179.9
p	9.7968					
G	1755.2					

Table 2.9: Interior equilibrium for the electricity market with the demand function from (2.24)

Finally, we vary the consistent influence coefficient of one of the producers and compute the corresponding exterior equilibrium to see how their profits change. These results are shown in table 2.10.

i	1	2	3	4	5	6
ω_i	0.060438	0.076765	0.035183	0.079306	0.074729	0.093522
	0.086234	0.088004	0.083674	0.083674	0.077221	0.10112
	0.11277	0.10719	0.095928	0.12058	0.079999	0.1033
π_i	1464.0	1768.1	897.04	903.18	616.76	2178.5
	1467.8	1769.1	903.22	903.22	616.76	2179.9
	1464.3	1766.4	902.88	900.35	616.76	2179.8

Table 2.10: Net profits when producers change unilaterally their consistent strategy with the demand function from (2.24)

In table 2.10 one can see that each time one of the producers changes (unilaterally) its consistent influence coefficient, its profit decreases (each column i , $i \in \{1, \dots, 6\}$, shows

the profit of producer i for the respective influence coefficient while the other producers remain stuck to their consistent conjectures). Thus, the results from theorems 2.6 and 2.7 can hold true for a wider set of functions than the ones satisfying the theorems' requirements.

2.7 CONCLUSIONS AND FUTURE RESEARCH

This chapter logically completes the previous papers Kalashnikov, Bulavsky, Kalashnykova and Castillo-Pérez (2011) and Kalashnikov, Bulavsky, Kalashnykova and López-Ramos (2017) by providing in section 2.5 three results establishing (under certain rather mild conditions) the equivalence of the consistent conjectural variations equilibrium (CCVE) in an original oligopoly model to the classical Cournot-Nash equilibrium in the meta-model. The latter comprises the same agents of the original oligopoly but with their conjectures (about the possible price variations) as their strategies.

These results seem to be very interesting in two aspects. First, they could be considered as a good justification of the CVE concept as being tightly related to the classical Nash equilibrium. Second, this equivalence occurring in the oligopoly can help one develop a concept similar to the CVE but in application to other kinds of economic and financial models that lack some attributes of the oligopoly and thus do not allow one to introduce the (consistent) CVE directly. In other words, one could define the (consistent) CVE in such a model via the Nash equilibrium in the corresponding meta-model.

In our future research, we plan to implement the above-mentioned ideas, as well as extend the developed constructions to the cases of mixed oligopolies, where at least one agent endeavors to maximize not its net profit but some other function related to the social surplus. Another important extension could be related to more general economic models with not necessarily differentiable (even discontinuous) inverse demand and/or cost functions.

CHAPTER 3

BILEVEL TOLLS OPTIMIZATION PROBLEM WITH QUADRATIC COSTS

3.1 INTRODUCTION

One of the problems affecting megapolises is the traffic congestion in highway networks, the said congestion being a direct consequence of all drivers trying to run the “shortest” path. One can think that a simple solution to this problem might be the construction of more roads in the network, which, however, is usually very expensive. Moreover, this often has the opposite effect of increasing traffic congestion even more (cf., the well-known Braess’s paradox published first in Braess (1969) and then republished in English in Braess et al. (2005)). On the other hand, highway toll pricing has proved to be a convenient tool for decreasing traffic congestion since the drivers now minimize not only their travel time but also its cost. The fees charged on the roads help maintain the highway networks in good conditions. The Tolls Optimization Problem (TOP) deals with selecting and assigning the optimal tolls to the toll arcs in the highway graph. The nature of this problem is generically bilevel because of the existence of multiple decision-makers: (at least) one at the upper-level making decisions concerning the tolls selected with aim to maximize its net profit, and the lower-level highway users (followers) each trying to find the best way along its origin-destination (O-D) path that can include both toll and free arcs. As all the drivers (followers) have to share the same resources (highways) with probably limited capacities, and moreover, their (quadratic) transportation costs might involve flows of the other agents, too, then the lower-level program is, in fact, a typical case of the general Nash equilibrium problem. Therefore, the problem of the leader is to find equilibrium among the toll values that provide high revenues while being attractive enough to the users (followers). The Tolls Optimization Problem has attracted the attention of numerous prominent researchers. In this Introduction, we mention only a few publications that have dealt with the problem in question.

Magnanti and Wong (1984) were the first who provided a theoretical structure for the decision-makers at both levels, based on integer programming. They also presented a scheme for uniforming similar network design models and the ways of developing network design algorithms.

Marcotte (1986) pointed out that the Network Design Problem deals with the optimal balance of the transportation, the investment and the maintenance costs of the network subject to the congestion, where the network's users behave according to Wardrop's first principle of traffic equilibrium. Also, in Marcotte (1986), the Network Design Problem is described for the first time as a bilevel programming problem.

Dempe and Lohse (2005) handled the Network Design Problem by the analysis of another optimization problem being reverse to the Network Design Problem.

Didi-Biha et al. (2006) proposed a primal-dual algorithm generating lower and upper bounds for the maximum profit collected from tolls of the highway network.

Dempe and Starostina (2009) developed fuzzy-set-theory methods to approximate the solution to the Tolls Optimization Problem.

Labbé et al. (2000) also treated the Tolls Optimization Problem by means of bilevel optimization, considering the firm in charge of the tolls as the leader and the drivers as the followers. This problem was also studied by Brotcorne (1998) for deciding tariffs on cargo trucks running the highways. In this case, the leader is a group of competing companies whose profit is yielded from the tolls they establish, while the (unique) follower is a carrier minimizing its travel expenses.

Another instance of the TOP was examined in Kalashnikov, Kalashnykova and Herrera (2011), where the leader is a highway administrator maximizing the turnover from the tolls on a subset of the arcs of a network with the drivers as the followers seeking for the "fastest" route (taking into account both the running time and the expenses on the tolls) uniting their origin and destination (O-D) points.

The Tolls Optimization Problem is also set as a combinatorial program which implies it belongs to the class of NP-hard problems (cf. Labbé et al. (1998)). Apart from the already known NP-hardness proofs, Roch et al. (2005) obtained new results about the computational complexity of some popular algorithms.

Brotcorne et al. (2011) treated the TOP with the premise that the network could be supplied with a subsidy thus permitting the tolls to have unlimited values. In this paper, they develop an algorithm first constructing paths and then forming columns to reveal the optimal values of the tolls for the current path, which then serve as lower bounds.

After that, the upper bound for the leader's profit is updated and finally, a diversification step is applied. The authors continued working on this problem in Brotcorne et al. (2012) where they apply a tabu search procedure to conclude that their heuristics produced better results than other combinatorial algorithms.

Dempe and Zemkoho (2012) restated the TOP by using of the optimal-value-function technique which works better than the Karush-Kuhn-Tucker optimality conditions. The optimality conditions for this recasting were deduced and other theoretical properties were exploited.

An innovative approach to solving the TOP was proposed in Kalashnikov, Herrera, Camacho and Kalashnykova (2016): as the lower-level equilibrium problem was easily reducible to a (large) linear program, the prospective direction of increase for the upper-level objective functions can be determined with the efficient tool of Sensitivity Analysis. This technique allowed the authors Kalashnikov, Herrera, Camacho and Kalashnykova (2016) to significantly accelerate the solution of the highly nonconvex TOP in comparison to the previous well-known algorithm.

In the works mentioned above, all the bilevel Tolls Optimization Problems treat linear problems at the lower-level. Thus, the main objective of this thesis' chapter is to extend the previous formulations and the promising results from Kalashnikov, Herrera, Camacho and Kalashnykova (2016) by considering *quadratic* problems at the lower-level (the latter reflect more accurately the traffic congestion on the roads). We also present an efficient algorithm to solve this new form of the TOP making again use of a method based on the Sensitivity Analysis in convex quadratic optimization presented by Hadigheh et al. (2007). This sensitivity analysis, similar to the technique described in Kalashnikov, Herrera, Camacho and Kalashnykova (2016) helps determine the allowable variations of the tolls values that do not mess up the optimality of a solution.

In addition to this method based on Sensitivity Analysis, the proposed algorithm also applies a filled function (FF) technique adapted to our case from Renpu (1990), Wan et al. (2012), and Wu et al. (2007). This method is quite efficient when a local optimum is found. The filled function procedure allows to either “jump” to another region of the feasible set which could lead to a better local optimum, or conclude that a good approximation of the global optimum has been found.

To show the efficiency of our proposed heuristic algorithm, its performance is compared with that of other well-known heuristic algorithms from Kalashnikov, Camacho, Askin and Kalashnykova (2010) in a series of numerical experiments with several instances for the bilevel formulation of the TOP with a quadratic lower-level. Moreover, since the algorithms from Kalashnikov, Camacho, Askin and Kalashnykova (2010) were designed for

the bilevel formulation of the Tolls Optimization Problem with a linear lower-level, about half of the instances have almost zero coefficients of the quadratic terms of the lower-level, thus, imitating the linear lower-level formulation from the previous works.

3.2 THE BILEVEL TOLLS OPTIMIZATION PROBLEM'S FORMULATION

As usual, we formulate the Tolls Optimization Problem (TOP) as a single-leader-multi-follower game that occurs in a multi-commodity highway network. The usual parameters for this formulation are the following:

- The multi-commodity network is defined by a set of directed arcs $A = \{1, 2, \dots, M\}$ representing the roads, a set of nodes $N = \{1, 2, \dots, \eta\}$ selected as the origins, transit points, and final destinations, as well as a set of commodities $K = \{1, 2, \dots, \kappa\}$ reflecting the groups of drivers sharing the same origins and destinations. The corresponding set powers are denoted as $|A| = M$, $|N| = \eta$, and $|K| = \kappa$.
- The set of arcs A is split into a nonempty proper subset $A_1 \subset A$ of toll arcs and its complementary subset $A_2 = A \setminus A_1$ of toll-free arcs, where $|A_1| = M_1$, $|A_2| = M_2$ and $M_1 + M_2 = M$.
- The subset $i^+ \subset A$ of arcs having the node $i \in N$ as their head and the subset $i^- \subset A$ of arcs having the node $i \in N$ as their tail.
- Every arc $a \in A$ has a fixed travel delay cost c_a and a capacity upper bound q_a .
- Each toll arc $a \in A_1$ has also a maximum toll value t_a^{\max} that can be charged being still attractive enough for the drivers.
- The commodity group $k \in K$ is assigned a demand value of n_k and arranges transportation from the origin node $o(k) \in N$ to the destination node $\delta(k) \in N$.
- Thus, the demand for each commodity $k \in K$ at every node $i \in N$ of the network is given by:

$$b_i^k = \begin{cases} -n_k & \text{if } i = o(k), \\ n_k & \text{if } i = \delta(k), \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

The decision variables for the bilevel TOP are as follows:

- The toll values t_a for each toll-arc $a \in A_1$ which has to be decided by the leader at the upper-level. These variables are stored in the vector $t = \{t_a \mid a \in A_1\}$.
- The flows x_a^k along every arc $a \in A$ which has to be decided by the commodity $k \in K$ at the lower-level. Each commodity $k \in K$ stores its variables in the vector $x^k = \{x_a^k \mid a \in A\}$ and all the variables of the lower-level are combined into the vector $x = \{x_a^k \mid a \in A, k \in K\}$. We also define the vectors $x^{-k} = \{x^\ell \mid \ell \in K \setminus \{k\}\}$ and $x_a^{-k} = \{x_a^\ell \mid \ell \in K \setminus \{k\}\}$ as it is usually done in Game Theory.

In Didi-Biha et al. (2006) and Kalashnikov, Herrera, Camacho and Kalashnykova (2016) the delay cost for a single unit to travel through arc $a \in A$ is the constant c_a , then, the travel delay cost for each commodity $k \in K$ to travel through arc a is given by the linear function $\bar{c}_a(x_a^k) = c_a x_a^k$.

However, the travel delay cost (for a single driver) being a fixed constant value, is unrealistic in the presence of congestion. In fact, the travel delay for each driver increases with the total amount of drivers sharing the same road, so the next approach would be to use a linear approximation for the travel delay depending upon the total amount of drivers traveling along the same arc.

For our new formulation of the TOP, the following parameter is introduced:

- For a single arc $a \in A$ the travel delay generated due to the traffic congestion is given by a nonnegative factor d_a .

Then, for a single unit to travel through arc a , the travel delay cost will be given by the linear function $c_a + d_a \xi$, where ξ is the total amount of drivers traveling through the same arc a . Hence, for each arc $a \in A$, the travel delay cost for each commodity $k \in K$ traveling through arc a is given by the (nonlinear) function

$$\bar{c}_a(x_a^k; x_a^{-k}) = \left(c_a + d_a \sum_{\ell \in K} x_a^\ell \right) x_a^k = c_a x_a^k + \sum_{\ell \in K \setminus \{k\}} d_a x_a^k x_a^\ell + d_a (x_a^k)^2, \quad (3.2)$$

which leads to the quadratic structure of the lower-level objective functions (instead of the linear structure as it was assumed in all previous papers mentioned in the introduction).

The travel delay costs, the tolls values, and the congestion factors are all measured in some equivalent units for the sake of consistency.

Using the notation listed above, our extended bilevel formulation for the Tolls Optimization Problem is given by the following single-leader-multi-follower game:

$$\text{Maximize}_t \quad F(t) = \sum_{k \in K} \sum_{a \in A_1} t_a x_a^k, \quad (3.3)$$

$$\text{subject to} \quad t_a \leq t_a^{\max}, \quad \forall a \in A_1, \quad (3.4)$$

$$t_a \geq 0, \quad \forall a \in A_1, \quad (3.5)$$

$$x^k \in \Psi_k(t, x^{-k}), \quad \forall k \in K, \quad (3.6)$$

where

$$\begin{aligned} \Psi_k(t, x^{-k}) = \text{Argmin}_{x^k} \quad f_k(x^k) = & \sum_{a \in A_1} t_a x_a^k + \sum_{a \in A} c_a x_a^k \\ & + \sum_{\ell \in K \setminus \{k\}} \sum_{a \in A} d_a x_a^k x_a^\ell + \sum_{a \in A} d_a (x_a^k)^2, \end{aligned} \quad (3.7)$$

$$\text{subject to} \quad \sum_{a \in i^+} x_a^k - \sum_{a \in i^-} x_a^k = b_i^k, \quad \forall i \in N, \quad (3.8)$$

$$x_a^k \leq q_a - \sum_{\ell \in K \setminus \{k\}} x_a^\ell, \quad \forall a \in A, \quad (3.9)$$

$$x_a^k \geq 0, \quad \forall a \in A. \quad (3.10)$$

Now we describe in more details the bilevel program (3.3)-(3.10):

- The leader's objective function (3.3) reflects its goal of maximizing its profit which is the sum of every toll charged times the total flow of drivers in the respective arc.
- The constraints (3.4) and (3.5) bound the tolls values to be nonnegative and not greater than the maximum tolls that can be charged to the drivers.
- The constraints (3.6) represents that the follower's variable x^k must be a solution for its respective quadratic programming problem (3.7)-(3.10) for all commodities $k \in K$. This means that the lower-level variables x^k , $k \in K$, must provide a Nash equilibrium for the non-cooperative game (3.6)-(3.10).
- The objective or payoff functions (3.7) of the followers represent their desire to minimize the total travel cost given by the tolls charges, the fixed travel delays of the roads and the travel delays caused by the traffic congestion produced by the transported commodities. It is also important to notice that in the objective function (3.7) of commodity $k \in K$, while the terms corresponding to $(x_a^k)^2$ are quadratic, the terms corresponding to $x_a^k x_a^\ell$ are in fact *linear* since the values x_a^ℓ are fixed when $\ell \neq k$.

- The constraints (3.8) are the flow conservation constraints for each commodity through the network.
- The constraints (3.9) and (3.10) are the nonnegativity and capacity constraints for each of the commodities flows.

Finally, in order to exclude trivial solutions, we make the following assumptions as in Didi-Biha et al. (2006):

1. There is no profitable vector that induces a negative cost cycle in the network. This condition is satisfied if all the travel costs and congestion factors are nonnegative.
2. For every commodity, there is always at least one path composed solely of toll-free arcs connecting its origin and destination nodes.

3.3 LINEAR-QUADRATIC BILEVEL PROGRAM REFORMULATION

The bilevel formulation of the Tolls Optimization Problem described in the previous section might be inconsistent since the lower-level's Nash equilibrium required in constraints (3.6) might not be unique, thus, making the upper-level's feasible region ill-defined.

In the previous works which consider linear travel delay costs $\bar{c}_a(x_a^k) = c_a x_a^k$, this situation was handled taking advantage that the lower-level game was separable (if $d_a = 0$ for all a , then, the objective function f_k doesn't have variables from the other commodities $\ell \neq k$), which allowed them to replace the lower-level's problem with a linear programming problem where the objective function to minimize was the sum of all the followers' objective functions and the constraints were all the constraints of the followers gathered together. The latter linear programming problem always provided the Nash equilibrium for the original lower-level game since the followers' problems were separable. Moreover, if the linear lower-level still had multiple solutions, considering the optimistic approach guaranteed the leader's feasible region to be well-defined.

In our formulation (3.3)-(3.10) the followers' problems are *no longer separable*, thus, minimizing the sum of their objective functions subject to all their constraints might *not* provide the Nash equilibrium for the lower-level game (3.6)-(3.10). However, we were able to show that a solution for the lower-level problem (3.6)-(3.10) can be obtained as a

solution for the following quadratic programming problem:

$$x \in \Psi(t), \quad (3.11)$$

where

$$\begin{aligned} \Psi(t) = \underset{x}{\text{Argmin}} \quad f(x) = & \sum_{k \in K} \sum_{a \in A_1} t_a x_a^k + \sum_{k \in K} \sum_{a \in A} c_a x_a^k \\ & + \sum_{k \in K} \sum_{\ell \in K \setminus \{k\}} \sum_{a \in A} \frac{1}{2} d_a x_a^k x_a^\ell + \sum_{k \in K} \sum_{a \in A} d_a (x_a^k)^2, \end{aligned} \quad (3.12)$$

$$\text{subject to} \quad \sum_{a \in i^+} x_a^k - \sum_{a \in i^-} x_a^k = b_i^k, \quad \forall i \in N, \quad \forall k \in K, \quad (3.13)$$

$$\sum_{k \in K} x_a^k \leq q_a, \quad \forall a \in A, \quad (3.14)$$

$$x_a^k \geq 0, \quad \forall a \in A, \quad \forall k \in K. \quad (3.15)$$

In the latter reformulation, the constraints (3.13)-(3.15) are all the constraints (3.8)-(3.10) of the followers combined together, but the objective function (3.12) is *not* the sum of all the followers' objective functions (3.7) since the coefficient corresponding to the term $x_a^k x_a^\ell$ in (3.12) is $\frac{1}{2}d_a$ while the corresponding coefficient in (3.7) is d_a whenever $k \neq \ell$.

This result is stated in the following theorem:

Theorem 3.1. *The quadratic programming problem (3.11)-(3.15) is convex and any of its solutions provides the Nash equilibrium for the non-cooperative game (3.6)-(3.10).*

Proof. See appendix C.

From the proof of theorem 3.1 we also have the following corollary:

Corollary 3.1. *If the capacity constraints (3.9) and (3.14) are removed, then, the problems (3.6)-(3.10) and (3.11)-(3.15) are equivalent.*

If all the congestion factors d_a , $a \in A$, are strictly positive, then, the quadratic programming problem (3.12)-(3.15) is strictly convex and its solution is unique; otherwise, multiple solutions might appear. If that's the case, as in the previous works, the optimistic solution is accepted in order to evaluate the leader's objective function.

Then, the bilevel single-leader-multi-follower game describing the TOP is reformulated as the following linear-quadratic bilevel programming problem:

$$\underset{t, x}{\text{Maximize}} \quad F(t, x) = \sum_{k \in K} \sum_{a \in A_1} t_a x_a^k, \quad (3.16)$$

$$\text{subject to } t_a \leq t_a^{\max}, \forall a \in A_1, \quad (3.17)$$

$$t_a \geq 0, \forall a \in A_1, \quad (3.18)$$

$$x \in \Psi(t), \quad (3.19)$$

where

$$\begin{aligned} \Psi(t) = \underset{x}{\text{Argmin}} \quad f(x) = & \sum_{k \in K} \sum_{a \in A_1} t_a x_a^k + \sum_{k \in K} \sum_{a \in A} c_a x_a^k \\ & + \sum_{k \in K} \sum_{\ell \in K \setminus \{k\}} \sum_{a \in A} \frac{1}{2} d_a x_a^k x_a^\ell + \sum_{k \in K} \sum_{a \in A} d_a (x_a^k)^2, \end{aligned} \quad (3.20)$$

$$\text{subject to } \sum_{a \in i^+} x_a^k - \sum_{a \in i^-} x_a^k = b_i^k, \forall i \in N, \forall k \in K, \quad (3.21)$$

$$\sum_{k \in K} x_a^k \leq q_a, \forall a \in A, \quad (3.22)$$

$$x_a^k \geq 0, \forall a \in A, \forall k \in K. \quad (3.23)$$

Thus, theorem 3.1 allows us to find a solution for the bilevel single-leader-multi-follower game (3.3)-(3.10), which requires solving an equilibrium problem to compute the leader's objective function, by solving the simpler linear-quadratic bilevel programming problem (3.16)-(3.23), thus reducing the complexity to compute the leader's objective function.

3.4 THE SOLUTION METHODOLOGY

To find a solution for the Tolls Optimization Problem, we propose a heuristic algorithm processing the linear-quadratic bilevel programming problem (3.16)-(3.23). The main idea behind this heuristic is to obtain the allowable increases or decreases in the coefficients corresponding to the linear terms in the objective function such that the sets of basic variables for the solution of the original problem and the solution of the perturbed problem are the same. The procedure to compute these *allowable ranges to stay basic* (ARSB) is described in the sensitivity analysis for convex quadratic programming from Hadigheh et al. (2007). An adaptation of the procedure to find the ARSB for the bilevel TOP is presented in section 3.5.

Similar to the heuristic based on the allowable ranges to stay optimal (ARSO) described in Kalashnikov, Herrera, Camacho and Kalashnykova (2016), given an upper-level feasible solution t , we solve the lower-level quadratic problem with the aid of the Wolfe-Dual to compute the allowable ranges to stay basic $\{\Delta_a^-, \Delta_a^+\}$ for the leader's

decision variables t_a , $a \in A_1$, which are the coefficients corresponding to the linear terms for the lower-level objective function. If the flow along an arc $a \in A_1$ is positive we increase the toll value t_a slightly less than the allowable increase Δ_a^+ (which may decrease the flow in the arc a but not drop to zero since the basic variables x_a^k , $k \in K$, will stay basic) in an attempt to increase the profit generated by the flow along the arc a . Otherwise, if the flow along the arc $a \in A_1$ is zero, it means that the toll assigned t_a is too high. Thus, we decrease its value by slightly more than the allowable decrease Δ_a^- so the toll can become attractive to the users again. In order to maximize the leader's objective function, we implement three different rules to decide which tolls will increase and which tolls will decrease; they are described in section 3.7.

If the use of the allowable ranges to stay basic does *not* provide a better feasible solution, it could be because the current solution is a local maximum for the leader's objective function. In this case, we make use of the filled function technique first proposed in Renpu (1990), then, developed in Wan et al. (2012) and Wu et al. (2007) and widely discussed in Kalashnikov, Kreinovich, Flores-Muñiz and Kalashnykova (2016) and Flores-Muñiz et al. (2017). However, since the filled function method was developed for minimization, we adapted the methodology from Wu et al. (2007) to the problem of maximization. The theory corresponding to this adaptation is presented in section 3.6

The filled function method smooths the original upper-level function (3.16) by transforming the current local maximum into a local minimum. Moreover, the modified function's structure is such that when it is maximized again, the algorithm makes a jump to a different region of the feasible set which has a good chance to lead to a better local maximum (if such exists) for the upper-level.

Once we have a new toll vector t generated by the filled function method, we proceed to maximize the leader's objective function again with the help of the allowable ranges to stay basic. When a local maximum is found again, we proceed once again with the filled function method. After several fruitless attempts in a row, say 5 to 10, the algorithm is stopped, and the best local maximum found is accepted as a good approximation of the global maximum solution.

3.5 THE ALLOWABLE RANGES TO STAY BASIC PROCEDURE

First, for the current solution t find the set $\Psi(t)$ of minimum solutions $x = \{x_a^k \mid a \in A, k \in K\}$ of the quadratic program (3.20)-(3.23). For each $x \in \Psi(t)$ define $x_a^0 = q_a - \sum_{k \in K} x_a^k \geq 0, a \in A$.

Second, find the set $\Phi(t)$ of complementary solutions (x, y, s) for the Wolfe-Dual of the quadratic program (3.20) – (3.23), which is given by:

$$\begin{aligned} \Phi(t) = \operatorname{Argmax}_{x, y, s} \quad & \phi(x, y, s) = \sum_{k \in K} \sum_{i \in N} b_i^k y_i^k - \sum_{a \in A} q_a s_a^0 \\ & - \sum_{k \in K} \sum_{\ell \in K \setminus \{k\}} \sum_{a \in A} \frac{1}{2} d_a x_a^k x_a^\ell - \sum_{k \in K} \sum_{a \in A} d_a (x_a^k)^2, \end{aligned} \quad (3.24)$$

$$\begin{aligned} \text{subject to} \quad & \sum_{i \in a^+} y_i^k - \sum_{i \in a^-} y_i^k - s_a^0 + s_a^k - \sum_{\ell \in K \setminus \{k\}} d_a x_a^\ell \\ & - 2d_a x_a^k = t_a + c_a, \quad \forall a \in A_1, \quad \forall k \in K, \end{aligned} \quad (3.25)$$

$$\begin{aligned} & \sum_{i \in a^+} y_i^k - \sum_{i \in a^-} y_i^k - s_a^0 + s_a^k - \sum_{\ell \in K \setminus \{k\}} d_a x_a^\ell \\ & - 2d_a x_a^k = c_a, \quad \forall a \in A_2, \quad \forall k \in K, \end{aligned} \quad (3.26)$$

$$x \in \Psi(t), \quad (3.27)$$

$$y_i^k \in \mathbb{R}, \quad \forall i \in N, \quad \forall k \in K, \quad (3.28)$$

$$s_a^k \geq 0, \quad \forall a \in A, \quad \forall k \in K \cup \{0\}, \quad (3.29)$$

where $y = \{y_i^k \mid i \in N, k \in K\}$, $s = \{s_a^k \mid a \in A, k \in K \cup \{0\}\}$, $a^+ = \{i \in N \mid a \in i^+\}$, $a^- = \{i \in N \mid a \in i^-\}$, $a \in A$.

Next, for the index set $\mathcal{I} = \{(a, k) \mid a \in A, k \in K \cup \{0\}\}$, define the partition:

$$\mathcal{B} = \left\{ (a, k) \in \mathcal{I} \mid \sup_{x \in \Psi(t)} \{x_a^k\} > 0 \right\}, \quad (3.30)$$

$$\mathcal{N} = \left\{ (a, k) \in \mathcal{I} \mid \sup_{(x, y, s) \in \Phi(t)} \{s_a^k\} > 0 \right\}, \quad (3.31)$$

$$\mathcal{T} = \mathcal{I} \setminus (\mathcal{B} \cup \mathcal{N}). \quad (3.32)$$

Then, for every $\hat{a} \in A_1$, find the maximum $\lambda_{\hat{a}}^u$ and minimum $\lambda_{\hat{a}}^{\ell}$ of the following linear programming problem:

$$\begin{array}{ll} \text{Maximize/Minimize} & \lambda^{\hat{a}}(\lambda) = \lambda, \\ \lambda, x, y, s & \end{array} \quad (3.33)$$

$$\text{subject to} \quad \sum_{a \in i^+} x_a^k - \sum_{a \in i^-} x_a^k = b_i^k, \quad \forall i \in N, \quad \forall k \in K, \quad (3.34)$$

$$\sum_{k \in K} x_a^k + x_a^0 = q_a, \quad \forall a \in A, \quad (3.35)$$

$$\begin{aligned} \sum_{i \in \hat{a}^+} y_i^k - \sum_{i \in \hat{a}^-} y_i^k - s_{\hat{a}}^0 + s_{\hat{a}}^k - \sum_{\ell \in K \setminus \{k\}} d_{\hat{a}} x_a^{\ell} \\ - 2d_{\hat{a}} x_a^k - \lambda = t_{\hat{a}} + c_{\hat{a}}, \quad \forall k \in K, \end{aligned} \quad (3.36)$$

$$\begin{aligned} \sum_{i \in a^+} y_i^k - \sum_{i \in a^-} y_i^k - s_a^0 + s_a^k - \sum_{\ell \in K \setminus \{k\}} d_a x_a^{\ell} \\ - 2d_a x_a^k = t_a + c_a, \quad \forall a \in A_1 \setminus \{\hat{a}\}, \quad \forall k \in K, \end{aligned} \quad (3.37)$$

$$\begin{aligned} \sum_{i \in a^+} y_i^k - \sum_{i \in a^-} y_i^k - s_a^0 + s_a^k - \sum_{\ell \in K \setminus \{k\}} d_a x_a^{\ell} \\ - 2d_a x_a^k = c_a, \quad \forall a \in A_2, \quad \forall k \in K, \end{aligned} \quad (3.38)$$

$$x_a^k \geq 0, \quad \forall (a, k) \in \mathcal{B}, \quad (3.39)$$

$$x_a^k = 0, \quad \forall (a, k) \in \mathcal{N} \cup \mathcal{T}, \quad (3.40)$$

$$s_a^k \geq 0, \quad \forall (a, k) \in \mathcal{N}, \quad (3.41)$$

$$s_a^k = 0, \quad \forall (a, k) \in \mathcal{B} \cup \mathcal{T}, \quad (3.42)$$

$$-t_{\hat{a}}^{\max} \leq \lambda \leq t_{\hat{a}}^{\max}. \quad (3.43)$$

Finally, the allowable ranges to stay basic are given by $\Delta_a^+ = \lambda_u^a$ and $\Delta_a^- = -\lambda_{\ell}^a$, $a \in A_1$.

3.6 THE FILLED FUNCTION ADAPTATION

Let $u = u(t)$ be a differentiable function defined over a box $T \subset \mathbb{R}^n$ such that any local maximum point of the latter function is strictly positive.

Definition 3.1. Let $\beta > \alpha > 1$. A continuously differentiable function $Q_{t^*}(t)$ is said to be a filled function for the maximization problem

$$\text{maximize}_t \quad u(t), \quad (3.44)$$

$$\text{subject to} \quad t \in T, \quad (3.45)$$

at the point $t^* \in T$ with $u(t^*) > 0$, if:

1. t^* is a strict local minimizer of $Q_{t^*}(t)$ on T .
2. Any local maximizer \bar{t} of $Q_{t^*}(t)$ on T satisfies $u(\bar{t}) > \alpha u(t^*)$ or \bar{t} is a vertex of T .
3. Any local maximizer \bar{t} of the optimization problem (3.44)-(3.45) with $u(\bar{t}) > \beta u(t^*)$ is a local maximizer of $Q_{t^*}(t)$ on T .
4. Any $\bar{t} \in T \setminus \{t^*\}$ with $\nabla Q_{t^*}(\bar{t}) = 0$ implies $u(\bar{t}) > \alpha u(t^*)$.

Now, to construct a filled function in the sense of definition 3.1, define two auxiliary functions as follows. For arbitrary t and $t^* \in T$, denote $b = u(t^*) > 0$ and $v = u(t)$, define:

$$g_b(v) = \begin{cases} 1, & \text{if } v \leq 0, \\ 2\frac{v^3}{b^3} - 3\frac{v^2}{b^2} + 1, & \text{if } 0 \leq v \leq b, \\ 0, & \text{if } v \geq b, \end{cases} \quad (3.46)$$

and

$$s_b(v) = \begin{cases} 2, & \text{if } v \leq 0, \\ \frac{125v^3}{4b^2} - \frac{75v^2}{4b^2} + 2, & \text{if } 0 \leq v \leq \frac{2}{5}b, \\ 1, & \text{if } \frac{2}{5}b \leq v \leq \frac{4}{5}b, \\ (250 - 25b)\frac{v^3}{b^2} + (65b - 675)\frac{v^2}{b^2} \\ \quad + (600 - 56b)\frac{v}{b} + (16b + 175), & \text{if } \frac{4}{5}b \leq v \leq b, \\ b - v, & \text{if } v \geq b. \end{cases} \quad (3.47)$$

Now, given a point $t^* \in T$ such that $u(t^*) > 0$, we define the following filled function:

$$\begin{aligned} Q_{\rho, \alpha, \beta, t^*}(t) = & -\exp(-\|t - t^*\|^2)g_{(\beta-\alpha)u(t^*)}(u(t) - \alpha u(t^*)) \\ & - \rho s_{(\beta-\alpha)u(t^*)}(u(t) - \alpha u(t^*)), \end{aligned} \quad (3.48)$$

where $\rho > 0$ is a parameter.

Based on Wu et al. (2007) we have the following theorem:

Theorem 3.2. Assume that the function $u(t)$ is continuously differentiable in the box $T \subset \mathbb{R}^n$ and t^* is a local maximum with $u(t^*) > 0$. Then, for any $\beta > \alpha > 1$ and $\rho > 0$, the function $Q_{\rho, \alpha, \beta, t^*}(t)$ from (3.48) is a filled function for the maximization problem (3.44)-(3.45) at the point t^* .

Demostración. The proof of this theorem is analogous to the one from Wu et al. (2007).

3.7 THE ALGORITHM'S DESCRIPTION

In this section, we describe the proposed heuristic algorithm in detail.

3.7.1 THE MAIN ALGORITHM

The main procedure of the heuristic algorithm is presented below as algorithm 3.1.

Algorithm 3.1 The Main Algorithm

Step 0: Set $m = 1$, $t_a^m = 0$, $a \in A_1$, $\rho = 1$, $\alpha = 1.5$, $\beta = 2$, $m_{FF} = 0$, $\tilde{m} = 1$ and $FF_{max} = 5$.

Step 1: Update the current iteration m with the *ARSB version 1* algorithm.

Step 2: Set $\hat{m} = m$ and update the current iteration m with the *ARSB version 2* algorithm. If $F(t^m, x(t^m)) > F(t^{\hat{m}}, x(t^{\hat{m}}))$, return to *Step 1*.

Step 3: Set $\hat{m} = m$ and update the current iteration m with the *ARSB version 3* algorithm. If $F(t^m, x(t^m)) > F(t^{\hat{m}}, x(t^{\hat{m}}))$, return to *Step 1*.

Step 4: If $F(t^m, x(t^m)) > F(t^{\tilde{m}}, x(t^{\tilde{m}}))$, set $\alpha = 1.5$, $\beta = 2$, $m_{FF} = 0$, and $\tilde{m} = m$. Otherwise, set $\beta = \alpha$, $\alpha = (\alpha + 1)/2$, $m_{FF} = m_{FF} + 1$, and $m = \tilde{m}$.

Step 5: If $m_{FF} \leq FF_{max}$, update the current iteration m with the *FF method* algorithm using the current values of the parameters ρ , α , β , and return to *Step 1*.

Step 6: Return the toll vector t^m as an approximate maximum solution.

In *Step 0* of the main algorithm, we start by defining the initial solution with all tolls set to zero, the initial iteration counter $m = 1$ and the initial values for the parameters ρ , α and β required for the *FF method* algorithm. We also define another counter m_{FF} to track how many times in a row we have applied the *FF method* algorithm without improving the best solution and FF_{max} is the maximum number of fruitless attempts in a row allowed. The iteration where the best solution was found is stored in \tilde{m} , so we can retrieve it when needed.

In *Step 1*, we update the current best solution t^m with the first version of the algorithms based on the allowable ranges to stay basic (which are described later).

When making *Step 2* and *Step 3*, we update the current best solution t^m with the *ARSB version 2* and *ARSB version 3* algorithms, respectively. In both cases, if a better solution is found, we try to update the solution again with the *ARSB version 1* algorithm, otherwise, we proceed to the next step. The three algorithms based on the allowable ranges to stay basic only update a previous iteration if the objective function is improved and they stop when they can't find a better solution.

In *Step 4*, we check if the current solution t^m is better than the up-to-now best solution $t^{\tilde{m}}$. If so, the parameters α and β , and the counter m_{FF} (corresponding to the filled function algorithm used in *Step 5*) are reset to its original values and the iteration of the new best current solution is stored ($\tilde{m} = m$). Otherwise, the parameters α and β are adjusted, the filled function counter m_{FF} increases by 1, and the iteration from the best current solution is restored ($m = \tilde{m}$).

In *Step 5*, we update the current best solution t^m with the *FF method* algorithm and return to *Step 1* even if the new solution is worse, since the objective is to jump to another region of the feasible set (this is the reason why we need to store the iteration corresponding to the best current solution found). However, if the filled function algorithm does not find a better solution in a while (when $m_{FF} > FF_{max}$), then, the last solution found t^m (which is the best current solution found) is accepted as a good approximation of the global maximum solution (*Step 6*).

It is also important to mention that in order to compute the objective function value of a solution t^m , we first have to solve the quadratic program (3.20)-(3.23) to find the (optimistic) solution $x(t^m)$ for the lower-level and, then, evaluate $F(t^m, x(t^m))$ in the upper-level function given by (3.16).

3.7.2 THE ARSB ALGORITHMS

The three algorithms based on the allowable ranges to stay basic are described in algorithms 3.2-3.4.

Algorithm 3.2 ARSB version 1

Step 0: Set $\bar{m} = m$, $\hat{t}_a = 0$, $a \in A_1$, $G = \emptyset$ and $\varepsilon = 1 \times 10^{-3}$.

Step 1: For the current solution t^m find the allowable ranges to stay basic $\{\Delta_a^-, \Delta_a^+ \mid a \in A_1\}$. Define the set

$$A_1^+ = \left\{ a \in A_1 \mid \sum_{k \in K} x_a^k(t^m) > 0 \right\},$$

the values $\bar{\Delta}_a^+ = \min\{t_a^m + \Delta_a^+, t_a^{\max}\} - t_a^m$, $a \in A_1^+$, and find $e \in \operatorname{Argmax}_{a \in A_1^+} \left\{ \bar{\Delta}_a^+ \cdot \sum_{k \in K} x_a^k(t^m) \right\}$.

Step 2: If $A_1^+ = \emptyset$ or $\Delta_e^+ \leq \varepsilon$, go to *Step 4*. Otherwise, set

$$t_a^{m+1} = \begin{cases} \min\{t_a^m + \Delta_a^+ - \varepsilon, t_a^{\max}\} & \text{if } a = e, \\ t_a^m & \text{if } a \neq e, \end{cases} \quad \forall a \in A_1.$$

Step 3: If $F(t^{m+1}, x(t^{m+1})) > F(t^m, x(t^m))$, set $m = m + 1$ and return to *Step 1*.

Step 4: If $F(t^m, x(t^m)) > F(t^{\bar{m}}, x(t^{\bar{m}}))$, set $\bar{m} = m$ and go to *Step 6*.

Step 5: If $m = \bar{m}$, go to *Step 14*. Otherwise, set $m = \bar{m}$ and return to *Step 1*.

Step 6: Define the set

$$E_1^+ = \left\{ a \in A_1 \mid \sum_{k \in K} x_a^k(t^m) > 0, t_a^m < t_a^{\max} \right\}$$

and find $e \in \operatorname{Argmax}_{a \in E_1^+} \left\{ t_a^m \cdot \sum_{k \in K} x_a^k(t^m) \right\}$.

Step 7: If $E_1^+ = \emptyset$, go to *Step 9*. Otherwise, set

$$t_a^{m+1} = \begin{cases} t_a^{\max} & \text{if } a = e, \\ t_a^m & \text{if } a \neq e, \end{cases} \quad \forall a \in A_1,$$

$\hat{t}_e = t_e^m$, $G = G \cup \{e\}$ and $m = m + 1$.

Step 8: If $F(t^m, x(t^m)) > F(t^{\bar{m}}, x(t^{\bar{m}}))$, set $\bar{m} = m$. Go to *Step 11*.

Step 9: Set

$$t_a^{m+1} = \begin{cases} \hat{t}_a & \text{if } a \in G, \\ t_a^m & \text{if } a \notin G, \end{cases} \quad \forall a \in A_1,$$

$m = m + 1$, $\hat{t}_a = 0$, $a \in A_1$, and $G = \emptyset$.

Step 10: If $F(t^m, x(t^m)) > F(t^{\bar{m}}, x(t^{\bar{m}}))$, set $\bar{m} = m$. Return to *Step 1*.

Step 11: For the current solution t^m find the allowable ranges to stay basic $\{\Delta_a^-, \Delta_a^+ \mid a \in A_1\}$. Define the set

$$A_1^+ = \left\{ a \in A_1 \mid \sum_{k \in K} x_a^k(t^m) > 0 \right\},$$

the values $\bar{\Delta}_a^+ = \min\{t_a^m + \Delta_a^+, t_a^{\max}\} - t_a^m$, $a \in A_1^+$, and find

$$e \in \operatorname{Argmax}_{a \in A_1^+} \left\{ \bar{\Delta}_a^+ \cdot \sum_{k \in K} x_a^k(t^m) \right\}.$$

Step 12: If $A_1^+ = \emptyset$ or $\Delta_e^+ \leq \varepsilon$, return to *Step 6*. Otherwise, set

$$t_a^{m+1} = \begin{cases} \min\{t_a^m + \Delta_a^+ - \varepsilon, t_a^{\max}\} & \text{if } a = e, \\ t_a^m & \text{if } a \neq e, \end{cases} \quad \forall a \in A_1.$$

Step 13: If $F(t^{m+1}, x(t^{m+1})) > F(t^m, x(t^m))$, set $m = m + 1$ and return to *Step 8*. Otherwise, return to *Step 6*.

Step 14: Return the last iteration m and the best solution found t^m to the main algorithm.

For this first version of the ARSB algorithms, in *Step 0*, we first store the initial iteration ($\bar{m} = m$), an auxiliary vector $\hat{t} = \{\hat{t}_a \mid a \in A_1\}$, an auxiliary set G , and a small tolerance value ε .

From *Step 1* to *Step 5*, we find the allowable ranges to stay basic and try to improve the current solution t^m by increasing the tolls $a \in A_1$ a little less than the allowable increase ($\Delta_a^+ - \varepsilon$) so that the flow on the arc a don't drop to zero. The tolls are increased one by one, choosing first those with the highest expected profit increase which is given by $\bar{\Delta}_a^+ \sum_{k \in K} x_a^k(t^m)$. If an improvement can't be done with this method, we go to *Step 6*. Also, we keep track of the iteration \bar{m} corresponding to the best solution that has been found.

From *Step 6* to *Step 10*, we increase the tolls to its maximum values, one by one, in each of the roads that have a nonzero flow, so that the drivers search for another path that will or will not include toll arcs. If, after a single toll increases to its maximum value, the drivers' new path includes a toll arc, then, after improving the new tolls vector with the idea of *Step 1* to *Step 5*, the profit will likely be higher. This time, the tolls are increased choosing first those that have the highest profit which is given by $t_a^m \sum_{k \in K} x_a^k(t^m)$. If there is any improvement, we proceed to *Step 1* if the current solution t^m is better than the best solution found $t^{\bar{m}}$. Otherwise, we go to *Step 11*. We continue this procedure until no improvement is done. By this time, it may happen that there are no drivers in any of the toll-arcs because of the expensive costs. If the latter occurs, we reset every toll to its value before being maximized (these values are stored in the variables \hat{t}_a , $a \in G$) which should generate a better solution than the previous one. Then, we return to *Step 1* to repeat the process.

Step 11 to *Step 13* are auxiliary steps that mirror the procedure of *Step 1* to *Step 5* but adapted to return to *Step 6*, instead of *Step 1*, when an improvement is reached.

The final *Step 14* returns the iteration m , corresponding to the best solution t^m that was found, to be used in the main algorithm.

Algorithm 3.3 ARSB version 2

Step 1: For the current solution t^m , define

$$A_1^+ = \left\{ a \in A_1 \mid \sum_{k \in K} x_a^k(t^m) > 0 \right\},$$

$M_a^+ = |A_1^+|$, and enumerate its elements $A_1^+ = \{a_1, a_2, \dots, a_{M_1^+}\}$.

Step 2: If $A_1^+ = \emptyset$, go to *Step 6*. Otherwise, set $j = 1$.

Step 3: If $j > M_a^+$, go to *Step 6*.

Step 4: Set

$$t_a^{m+1} = \begin{cases} t_a^{\max} & \text{if } a = a_j, \\ t_a^m & \text{if } a \neq a_j, \end{cases} \quad \forall a \in A_1.$$

Step 5: If $F(t^{m+1}, x(t^{m+1})) > F(t^m, x(t^m))$, set $m = m + 1$ and return to *Step 1*. Otherwise, set $j = j + 1$ and return to *Step 3*.

Step 6: Return the last iteration m and the best solution found t^m to the main algorithm.

For the second version of the ARSB algorithms, we use the same idea from *ARSB version 1* algorithm's *Step 6* to *Step 10*. We first identify the toll-arcs with positive flows and increase their tolls to the maximum one by one, but this time, only if this move provides a better solution than the previous one. After all the arcs have been tried, we return to the main algorithm with the updated solution.

Algorithm 3.4 ARSB version 3

Step 1: For the current solution t^m , define

$$E_1^- = \left\{ a \in A_1 \mid \sum_{k \in K} x_a^k(t^m) = 0, t_a^m > 0 \right\},$$

$M_a^- = |E_1^-|$, and enumerate its elements $E_1^- = \{a_1, a_2, \dots, a_{M_1^-}\}$.

Step 2: If $E_1^- = \emptyset$, go to *Step 6*. Otherwise, set

$$t_a^{m+1} = \begin{cases} t_a^{\max} & \text{if } a \in E_1^-, \\ t_a^m & \text{if } a \notin E_1^-, \end{cases} \quad \forall a \in A_1,$$

$m = m + 1$ and $i = 1$.

Step 3: If $i > M_a^-$, go to *Step 6*. Otherwise, for the current solution t^m find the allowable ranges to stay basic $\{\Delta_a^-, \Delta_a^+ \mid a \in A_1\}$.

Step 4: Set

$$t_a^{m+1} = \begin{cases} \max\{t_a^m - \Delta_a^- - \varepsilon, 0\} & \text{if } a = a_i, \\ t_a^m & \text{if } a \neq a_i, \end{cases} \quad \forall a \in A_1.$$

Step 5: If $F(t^{m+1}, x(t^{m+1})) > F(t^m, x(t^m))$, set $m = m + 1$. Set $i = i + 1$ and return to *Step 3*.

Step 6: Return the last iteration m and the best solution found t^m to the main algorithm.

For the third version of the ARSB algorithms, we first identify the set E_1^- of toll-arcs with zero flow, which is the consequence of the toll being too high. Then, we maximize all these tolls (which will not change the solution since the toll will be even higher) and find the allowable ranges to stay basic. Next, we select each arc of this set E_1^- one by one to decrease its value by a little more than the allowable decrease, so that the flow becomes positive. If this procedure increases the profit, we update the iteration m and the solution t^m . Otherwise, we undo this change and try with the next arc. This procedure ends when all the arcs have been tested, then, we return to the main algorithm with the updated solution.

3.7.3 THE FF METHOD ALGORITHM

Finally, we present the FF method's procedures as algorithm 3.5. The auxiliary function that will be used are the ones defined in section 3.6.

Algorithm 3.5 The FF Method

Step 0: Define the function $u(t) = F(t, x(t))$ and set $t_0 = t^m$.

Step 1: Find a local maximum t^* of the auxiliary problem:

$$\begin{aligned} \text{Maximize}_t \quad & Q_{\rho, \alpha, \beta, t_0}(t) = -\exp(-\|t - t_0\|^2) g_{(\beta - \alpha)u(t_0)}(u(t) - \alpha u(t_0)) \\ & - \rho S_{(\beta - \alpha)u(t_0)}(u(t) - \alpha u(t_0)), \end{aligned} \quad (3.49)$$

$$\text{subject to} \quad t_a \leq t_a^{\max}, \quad \forall a \in A_1, \quad (3.50)$$

$$t_a \geq 0, \quad \forall a \in A_1. \quad (3.51)$$

Step 2: Set $t^{m+1} = t^*$ and $m = m + 1$.

Step 3: Return the iteration m and the solution found t^m to the main algorithm.

The *FF method* is very simple but it helps the main algorithm not to get stuck at a local maximum. Moreover, the auxiliary function $Q_{\rho,\alpha,\beta,t_0}(t)$ was designed in such a way that if an interior local maximum is found, then, $u(\bar{t}) > \alpha u(t_0)$, where $\alpha > 1$.

3.8 NUMERICAL RESULTS

To test the efficiency of our heuristic algorithm, its performance is compared with two algorithms from Kalashnikov, Camacho, Askin and Kalashnykova (2010), and the function `fmincon` from Matlab. The algorithms from Kalashnikov, Camacho, Askin and Kalashnykova (2010) are the quasi-Newton algorithm and the sharpest ascent method. Although these two algorithms were designed for the linear bilevel formulation of the Tolls Optimization Problem, they can still be used for our linear-quadratic bilevel formulation. In addition, for this comparison to be as natural as possible, half of the instances are tested with the congestion coefficients near zero to mirror the linear behavior of the lower-level from the previous formulations. The function `fmincon` from Matlab was designed to solve optimization problems with nonlinear objective functions and constraints in general, so it is worthwhile to compare its performance with other algorithms. Moreover, since the quasi-Newton, sharpest ascent and `fmincon` algorithms are designed for local optimization, the filled function method was added as an extra step to all these algorithms not to get stuck at a local maximum.

The numerical experiments were run on a computer with an Intel(R) Core(TM) i7-8750H CPU @ 2.20 GHz processor and 8.00 GB of RAM memory. The codes were compiled in Matlab R2017b making use of the linear and quadratic programming tools in the “Optimization Toolbox”.

For this comparison, we used two different networks, which are shown in figures 3.1 and 3.2. In each of the networks, the toll arcs are represented with regular straight lines and the toll-free arcs are depicted with dotted lines.

Each of these networks was tested with 5 different instances and each of them with 2 different sets of congestion parameters; the first of them mirroring the linear behavior of the Tolls Optimization Problem formulation. The sets of commodities $K = \{1, 2, \dots, \kappa\}$ along with the origin and destination nodes $o(k)$ and $\delta(k)$ are represented together as: $\Omega = \{(o(1), \delta(1)), (o(2), \delta(2)), \dots, (o(\kappa), \delta(\kappa))\}$. The parameters for network 1 are shown in table 3.1 and the parameters for network 2 are shown in table 3.2.

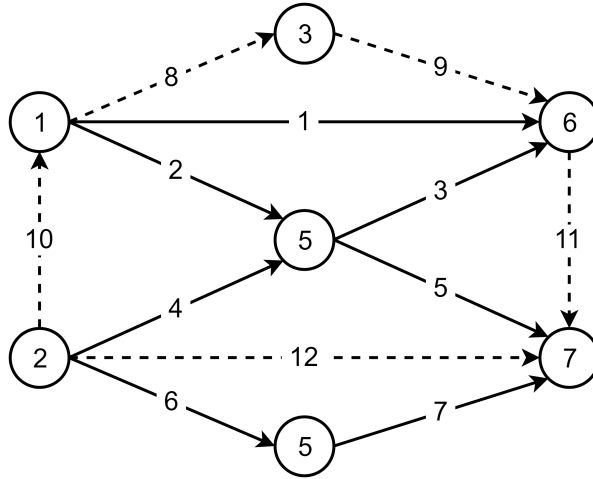


Figure 3.1: Network 1 with 7 nodes and 12 arcs where 7 are toll arcs

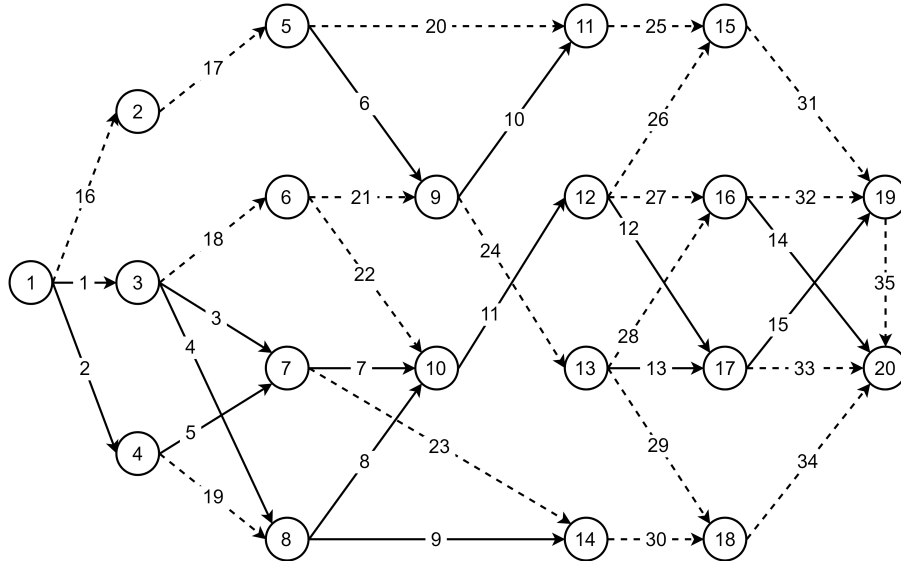


Figure 3.2: Network 2 with 20 nodes and 35 arcs where 15 are toll arcs

Instance	Parameters
1	$c = (1, 2, 5, 4, 3, 3, 2, 7, 4, 3, 8, 12)$, $\Omega = \{(1, 6), (2, 7)\}$, $n = (10, 9)$.
2	$c = (3, 4, 2, 2, 3, 3, 4, 9, 9, 5, 6, 15)$, $\Omega = \{(1, 6), (2, 7)\}$, $n = (15, 5)$.
3	$c = (4, 3, 2, 1, 1, 3, 2, 5, 6, 3, 1, 5)$, $\Omega = \{(1, 6), (2, 7)\}$, $n = (5, 8)$.
4	$c = (1, 3, 1, 2, 3, 1, 1, 5, 4, 2, 4, 13)$, $\Omega = \{(1, 6), (2, 7)\}$, $n = (5, 12)$.
5	$c = (3, 4, 5, 3, 3, 6, 2, 7, 7, 8, 10, 9)$, $\Omega = \{(1, 6), (2, 7)\}$, $n = (10, 9)$.
Vector of maximum toll values that can be charged	
$t^{\max} = (20, 20, 20, 20, 20, 20, 20, 20)$	

Table 3.1: Parameters for network 1

the quasi-Newton algorithm, *SA* for the sharpest ascent algorithm, and *fmincon* for the Matlab function.

For the first comparison, we took the best values found by each algorithm and compute the relative error with the best value found overall. This is shown in tables 3.4-3.11.

N1D1	ARSB	QN	SA	fmincon	Best
1	162.74	144.74	162.74	162.74	162.74
2	274.72	264.72	264.72	274.72	274.72
3	58.85	58.85	34.97	53.82	58.85
4	171.69	171.69	131.56	95.71	171.69
5	136.74	110.25	109.80	109.90	136.74

Table 3.4: Best upper-level objective function found for network 1 with the congestion parameters from set 1

N1D1	ARSB	QN	SA	fmincon
1	0.00 %	11.06 %	0.00 %	0.00 %
2	0.00 %	3.64 %	3.64 %	0.00 %
3	0.00 %	0.00 %	40.57 %	8.54 %
4	0.00 %	0.00 %	23.37 %	44.25 %
5	0.00 %	19.37 %	19.70 %	19.63 %

Table 3.5: Relative error for network 1 with the congestion parameters from set 1

N1D2	ARSB	QN	SA	fmincon	Best
1	97.89	91.72	22.09	83.33	97.89
2	197.22	152.83	152.87	191.96	197.22
3	35.38	25.57	15.76	35.38	35.38
4	97.45	34.27	69.23	97.45	97.45
5	86.10	75.88	75.88	80.12	86.10

Table 3.6: Best upper-level objective function found for network 1 with the congestion parameters from set 2

N1D2	ARSB	QN	SA	fmincon
1	0.00 %	6.31 %	77.44 %	14.88 %
2	0.00 %	22.51 %	22.49 %	2.67 %
3	0.00 %	27.73 %	55.46 %	0.00 %
4	0.00 %	64.83 %	28.96 %	0.00 %
5	0.00 %	11.87 %	11.87 %	6.94 %

Table 3.7: Relative error for network 1 with the congestion parameters from set 2

N2D1	ARSB	QN	SA	fmincon	Best
1	1760.57	1673.51	1673.28	1673.51	1760.57
2	2088.29	1532.96	1430.99	2088.29	2088.29
3	1122.81	1048.14	999.23	1122.84	1122.84
4	2166.00	2085.39	2085.26	2085.53	2166.00
5	345.91	193.96	189.60	189.97	345.91

Table 3.8: The best upper-level objective function value found for network 2 with the congestion parameters from set 1

N2D1	ARSB	QN	SA	fmincon
1	0.00 %	4.95 %	4.96 %	4.95 %
2	0.00 %	26.59 %	31.48 %	0.00 %
3	0.00 %	6.65 %	11.01 %	0.00 %
4	0.00 %	3.72 %	3.73 %	3.72 %
5	0.00 %	43.93 %	45.19 %	45.08 %

Table 3.9: Relative errors for network 2 with the congestion parameters from set 1

N2D2	ARSB	QN	SA	fmincon	Best
1	1307.87	503.88	503.89	823.04	1307.87
2	2660.24	1730.59	1820.78	1976.87	2660.24
3	803.81	402.43	402.58	419.32	803.81
4	2183.01	274.04	455.14	1845.23	2183.01
5	262.53	110.49	110.80	225.92	262.53

Table 3.10: The best upper-level objective function value for network 2 with the congestion parameters from set 2

N2D2	ARSB	QN	SA	fmincon
1	0.00 %	61.47 %	61.47 %	37.07 %
2	0.00 %	34.95 %	31.56 %	25.69 %
3	0.00 %	49.93 %	49.92 %	47.83 %
4	0.00 %	87.45 %	79.15 %	15.47 %
5	0.00 %	57.91 %	57.80 %	13.95 %

Table 3.11: Relative errors for network 2 with the congestion parameters from set 2

In tables 3.4 and 3.11, we can see that the *best* values of the objective functions in all the instances were provided by our algorithm. In some cases, the same best solutions was also found by some of the other algorithms.

Also, from table 3.4, we can see that the best maximum values for the objective functions found are very similar to those found in the numerical results in Kalashnikov, Herrera, Camacho and Kalashnykova (2016) for the previous linear formulation of the TOP, which confirms that the linear behavior is modelled correctly, too. This also means that our quadratic formulation illustrates the continuity of the solution with respect to the parameters, i.e., when the congestion parameters tend to zero, the solution of our quadratic formulation tends to the solution of the linear setup.

In tables 3.4 and 3.5 corresponding to the first network and the congestion coefficients near to zero, we see that in most of the instances, the other algorithms found an objective value near the best found. However, in tables 3.6 and 3.7 corresponding to the first network and greater congestion coefficients, we see that the performance of the other algorithms is better than ours only in a few cases.

The same behavior can be seen in tables 3.8-3.11 corresponding to network 2. When the congestion coefficients are near to zero the performance of the other algorithms is good, but when the congestion coefficients are greater, the best solution found by the other algorithms has a relative error greater than 13% compared with the best solution found by our algorithm.

However, finding better solutions is not enough for an algorithm: we also have to see if the time required to find that solution is feasible. Then, for our next comparison, we show the average solution found and the average time (in seconds) required by all the algorithms in each instance for the 10 tests. These results are shown in tables 3.12-3.19.

N1D1	ARSB	QN	SA	fmincon
1	19.34	4.66	6.35	10.89
2	12.06	4.68	5.48	6.22
3	12.98	7.39	4.90	8.45
4	12.82	7.14	4.19	4.69
5	14.20	4.16	4.03	5.91

Table 3.12: The average execution time (seconds) for network 1 with the congestion parameters from set 1

In tables 3.12-3.15 corresponding to network 1, we can see that in most of the cases the average objective function value found is very similar for all the algorithms with set

N1D1	ARSB	QN	SA	fmincon
1	160.72	126.69	138.12	162.71
2	274.72	229.06	240.07	231.60
3	51.44	48.06	34.97	53.82
4	170.96	160.48	131.43	95.71
5	130.99	109.00	109.57	109.90

Table 3.13: The average objective function value found for network 1 with the congestion parameters from set 1

N1D2	ARSB	QN	SA	fmincon
1	6.59	7.28	7.45	5.18
2	6.54	7.33	6.15	18.96
3	8.86	4.18	7.90	8.58
4	7.61	4.51	5.73	22.63
5	6.83	4.35	8.78	3.60

Table 3.14: The average execution time (seconds) for network 1 with the congestion parameters from set 2

N1D2	ARSB	QN	SA	fmincon
1	95.81	61.10	22.09	83.33
2	196.70	149.76	152.87	191.96
3	35.01	15.59	15.76	35.38
4	97.45	34.26	69.22	91.81
5	85.84	69.33	75.88	80.12

Table 3.15: The average objective function value found for network 1 with the congestion parameters from set 2

N2D1	ARSB	QN	SA	fmincon
1	92.20	14.07	15.59	15.92
2	84.96	18.57	8.08	33.92
3	88.16	14.32	11.68	28.27
4	110.81	10.07	7.35	19.78
5	130.88	21.92	10.54	25.55

Table 3.16: The average execution time (seconds) for network 2 with the congestion parameters from set 1

N2D1	ARSB	QN	SA	fmincon
1	1538.35	1279.04	1656.39	1673.23
2	2022.19	1114.04	1430.99	2004.43
3	989.02	1001.25	963.29	1122.77
4	2003.80	2081.96	2083.69	1953.31
5	288.22	138.45	158.51	189.97

Table 3.17: The average objective function value found for network 2 with the congestion parameters from set 1

N2D2	ARSB	QN	SA	fmincon
1	105.20	11.82	18.82	52.27
2	67.94	17.88	41.67	17.54
3	44.60	10.13	21.73	19.17
4	107.85	10.72	19.09	18.05
5	161.30	9.65	16.89	68.72

Table 3.18: The average execution time (seconds) for network 2 with the congestion parameters from set 2

N2D2	ARSB	QN	SA	fmincon
1	1270.42	503.79	503.89	823.04
2	2346.52	1229.51	1820.78	1976.87
3	739.64	401.11	402.58	419.32
4	1287.49	274.02	455.14	1845.23
5	204.53	102.38	110.79	225.92

Table 3.19: The average objective function value found for network 2 with the congestion parameters from set 2

1 of the congestion parameters, whereas, with set 2 of the congestion parameters, the average objective function value found is slightly better with our algorithm than with the other methods.

On the other hand, we also see that for set 1 of congestion parameters, the average execution time of our algorithm is almost the double of the average execution time of the other 3 algorithms. However, with set 2 of the congestion parameters, this average execution time decreases with our algorithm, while the execution time of the other 3 algorithms slightly increases.

The better performance of our algorithm with the second set of congestion parameters can be explained by the fact that it was designed specifically for the case when the lower-level is quadratic.

The same behavior can be seen in tables 3.16-3.19 for network 2. The average objective function value is similar in most of the instances with set 1 of the congestion parameters, while with set 2 of those parameters, the average objective function value is better in almost all the cases than the same found with the other algorithms. Also, we can see that for set 1 of the congestion parameters, the average time of our algorithms it's about 5 to 10 times higher than the time required for the other algorithms, but these times are similar for set 2 of the congestion parameters.

Thus, we can notice that the extra time taken by our algorithm results in an improvement in the objective function. However, even though our algorithm can be 10 times slower, the maximum execution time does not exceed 3 minutes which is still very fast.

For the last comparison, we calculate how many times each algorithm found a solution with less than 10 % relative error with respect to the best solution found by all the algorithms. Next, we do the same comparison as to the average time required by each algorithm to find the solution with less than 10 % relative error with respect to the best solution found by all the algorithms. These results are shown in tables 3.20-3.27.

N1D1	ARSB	QN	SA	fmincon
1	10	0	3	10
2	10	1	4	2
3	3	5	0	10
4	10	7	0	0
5	10	0	0	0

Table 3.20: Number of final objective functions found with less than 10 % relative error for network 1 with the congestion parameters in set 1

N1D1	ARSB	QN	SA	fmincon
1	9.99	N/A	6.90	7.39
2	1.27	5.93	4.40	6.17
3	12.19	5.15	N/A	5.89
4	2.06	5.60	N/A	N/A
5	1.30	N/A	N/A	N/A

Table 3.21: Average time to find the first objective function with less than 10 % relative error for network 1 with the congestion parameters in set 1

N1D2	ARSB	QN	SA	fmincon
1	9	1	0	0
2	10	0	0	10
3	10	0	0	10
4	10	0	0	8
5	10	0	0	10

Table 3.22: Number of final objective functions found with less than 10 % relative error for network 1 with the congestion parameters in set 2

N1D2	ARSB	QN	SA	fmincon
1	1.03	4.64	N/A	N/A
2	2.18	N/A	N/A	2.48
3	2.40	N/A	N/A	2.56
4	1.95	N/A	N/A	9.44
5	1.14	N/A	N/A	0.42

Table 3.23: Average time to find the first objective function with less than 10 % relative error for network 1 with the congestion parameters in set 2

N2D1	ARSB	QN	SA	fmincon
1	4	4	10	10
2	8	0	0	8
3	5	1	0	10
4	7	10	10	4
5	3	0	0	0

Table 3.24: Number of final objective functions found with less than 10 % relative error for network 2 with the congestion parameters in set 1

N2D1	ARSB	QN	SA	fmincon
1	18.77	8.96	7.97	6.84
2	10.06	N/A	N/A	23.20
3	6.93	13.13	N/A	15.93
4	52.41	0.94	0.57	9.75
5	19.91	N/A	N/A	N/A

Table 3.25: Average time to find the first objective function with less than 10 % relative error for network 2 with the congestion parameters in set 1

N2D2	ARSB	QN	SA	fmincon
1	10	0	0	0
2	7	0	0	0
3	8	0	0	0
4	2	0	0	0
5	1	0	0	0

Table 3.26: Number of final objective functions found with less than 10 % relative error for network 2 with the congestion parameters in set 2

N2D2	ARSB	QN	SA	fmincon
1	28.07	N/A	N/A	N/A
2	14.34	N/A	N/A	N/A
3	4.10	N/A	N/A	N/A
4	5.30	N/A	N/A	N/A
5	129.66	N/A	N/A	N/A

Table 3.27: Average time to find the first objective function with less than 10 % relative error for network 2 with the congestion parameters in set 2

From tables 3.20-3.27, we can see again that the performance of our algorithm is better for set 2 of the congestion parameters since the other algorithms most of the times cannot find a solution with less than 10 % relative error. We also can see that the average time required by our algorithm is reduced drastically for almost all the instances in network 2. This means that our algorithm is able to find a good solution within the average time of the other algorithms, and the rest is invested to find an even better solution.

3.9 CONCLUSIONS AND FUTURE RESEARCH

In this chapter, we propose a new formulation of the Tolls Optimization Problem with quadratic congestion terms and develop an efficient algorithm for its solution based on Sensitivity Analysis for quadratic programming. This algorithm also makes use of the filled function technique adapted for maximization in order to prevent the algorithm from getting stuck at a local maximum.

A series of numerical experiments were conducted with two different networks and two different sets of congestion parameters (one of them imitating the linear structures of previous formulations). These numerical results lead to two conclusions.

The first is that while our algorithm performs well along with the other 3 algorithms tested for small instances, our algorithm shows better results as the size of the instance increases and the congestion of the network is comparatively large (which is usually the case).

The second is that our model presents a natural extension of the linear formulation because when the congestion parameters tend to zero, the solution of our quadratic formulation tends to the solution of the linear model.

For our future research, we plan to develop efficient numerical algorithms for the cases in which the network's toll-free arcs capacity limits prevent finding the purely toll-free paths competing with the other feasible routes. These new restrictions may give birth to some theoretically novel results as well.

Finally, as an important link between this and the previous chapter, the idea inside the proof of theorem 3.1 can be useful to reduce the many-person game from the lower level of the meta-game into a single optimization problem in future works.

APPENDIX A

PROOFS OF CHAPTER 1

Lemma 1.1. *Let assumptions **A1.1-A1.3** be valid. Then, for all nonnegative values of ν_i , $i = 0, 1$, supply values q_i are strictly positive (i.e., $q_i > 0$, $i = 0, 1$) at any exterior equilibrium if and only if $p > p_0$.*

Proof. If $p > p_0 = b_1$, then, the inequalities $p \leq -\beta\nu_0q_1 + b_0$ and $p \leq b_1$, from the optimality conditions (1.11) and (1.13), respectively, never apply, which implies that no (equilibrium) value q_i , $i = 0, 1$, can vanish. Conversely, if all the equilibrium outputs are positive, i.e., $q_i > 0$, $i = 0, 1$, then, the optimality condition (1.13) directly entails $p = q_1\nu_1 + a_1q_1 + b_1 > b_1$. Hence, $p > p_0 = b_1$, and the proof is complete ■

Theorem 1.1. *Under assumptions **A1.1-A1.3**, for any $\beta \in (0, 1]$, $D \geq 0$ and $\nu_i \geq 0$, $i = 0, 1$, there exists uniquely the exterior equilibrium (p, q_0, q_1) depending continuously upon the parameters (D, ν_0, ν_1) . The equilibrium price p as a function of these parameters is continuously differentiable with respect to D and ν_i , $i = 0, 1$. Moreover $p(D, \nu_0, \nu_1) > p_0$ and*

$$\frac{\partial p}{\partial D} = \frac{1}{\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) - G'(p)}. \quad (1.15)$$

Proof. Let $\nu_0, \nu_1 \geq 0$. By using the optimality conditions (1.11) and (1.13), we can find the output volume functions $q_i = q_i(p, \nu_0, \nu_1)$, $i = 0, 1$, defined on the interval $[p_0, +\infty)$. These functions are differentiable with respect to p and ν_i , $i = 0, 1$, and they are given by:

$$q_0 = \frac{p - b_0}{(1-\beta)\nu_0 + a_0} + \frac{\beta\nu_0}{(1-\beta)\nu_0 + a_0} \left(\frac{p - b_1}{\nu_1 + a_1} \right), \quad (A.1)$$

$$q_1 = \frac{p - b_1}{\nu_1 + a_1}. \quad (A.2)$$

Now we introduce the following function:

$$\begin{aligned}
Q(p, \nu_0, \nu_1) &= q_0(p, \nu_0, \nu_1) + q_1(p, \nu_0, \nu_1) \\
&= \frac{p - b_0}{(1 - \beta)\nu_0 + a_0} + \frac{\beta\nu_0}{(1 - \beta)\nu_0 + a_0} \left(\frac{p - b_1}{\nu_1 + a_1} \right) + \frac{p - b_1}{\nu_1 + a_1} \\
&= p \left[\frac{1}{(1 - \beta)\nu_0 + a_0} + \frac{\beta\nu_0}{(1 - \beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) + \frac{1}{\nu_1 + a_1} \right] \\
&\quad - \left[\frac{b_0}{(1 - \beta)\nu_0 + a_0} + \frac{\beta\nu_0}{(1 - \beta)\nu_0 + a_0} \left(\frac{b_1}{\nu_1 + a_1} \right) + \frac{b_1}{\nu_1 + a_1} \right] \\
&= p \left[\frac{1}{(1 - \beta)\nu_0 + a_0} + \left(\frac{\beta\nu_0}{(1 - \beta)\nu_0 + a_0} + 1 \right) \left(\frac{1}{\nu_1 + a_1} \right) \right] \\
&\quad - \left[\frac{b_0}{(1 - \beta)\nu_0 + a_0} + \left(\frac{\beta\nu_0}{(1 - \beta)\nu_0 + a_0} + 1 \right) \left(\frac{b_1}{\nu_1 + a_1} \right) \right] \\
&= p \left[\frac{1}{(1 - \beta)\nu_0 + a_0} + \frac{\beta\nu_0 + (1 - \beta)\nu_0 + a_0}{(1 - \beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) \right] \\
&\quad - \left[\frac{b_0}{(1 - \beta)\nu_0 + a_0} + \frac{\beta\nu_0 + (1 - \beta)\nu_0 + a_0}{(1 - \beta)\nu_0 + a_0} \left(\frac{b_1}{\nu_1 + a_1} \right) \right] \\
&= p \left[\frac{1}{(1 - \beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1 - \beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) \right] \\
&\quad - \left[\frac{b_0}{(1 - \beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1 - \beta)\nu_0 + a_0} \left(\frac{b_1}{\nu_1 + a_1} \right) \right].
\end{aligned} \tag{A.3}$$

As we can see from (A.3), the function Q is linear in p with positive slope. Therefore, $Q(p, \nu_0, \nu_1)$ strictly increases with respect to p , and tends to $+\infty$ when $p \rightarrow +\infty$. By assumption **A1.3**, one has that for all $\nu_i \geq 0$, $i = 0, 1$,

$$\begin{aligned}
Q(p_0, \nu_0, \nu_1) &= q_0(p_0, \nu_0, \nu_1) + q_1(p_0, \nu_0, \nu_1) \\
&= \frac{p_0 - b_0}{(1 - \beta)\nu_0 + a_0} \leq \frac{p_0 - b_0}{a_0} < G(p_0) \leq G(p_0) + D.
\end{aligned} \tag{A.4}$$

Hence, $Q(p, \nu_0, \nu_1)$ strictly increases with respect to p , the function $G(p)$ is non-increasing by p and D is constant, so by inequality (A.4), there exists a unique value $p^* > p_0$ such that

$$Q(p^*, \nu_0, \nu_1) = G(p^*) + D. \tag{A.5}$$

For this value p^* , using (A.1) and (A.2), we compute uniquely the equilibrium output volumes $q_i^* = q_i(p^*, \nu_0, \nu_1)$, $i = 0, 1$. So we have established the existence and uniqueness of the exterior equilibrium (p^*, q_0^*, q_1^*) for any $D \geq 0$ and $\nu_i \geq 0$, $i = 0, 1$.

Now we are going to show that the equilibrium price p^* of the exterior equilibrium is differentiable with respect to the parameters (D, ν_0, ν_1) . From (A.5) we get the following relationships:

$$Q(p^*, \nu_0, \nu_1) - G(p^*) - D = 0, \quad (\text{A.6})$$

and we introduce the following function:

$$\begin{aligned} \Gamma(p^*, D, \nu_0, \nu_1) &= Q(p^*, \nu_0, \nu_1) - G(p^*) - D \\ &= p^* \left[\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) \right] \\ &\quad - \left[\frac{b_0}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{b_1}{\nu_1 + a_1} \right) \right] \\ &\quad - G(p^*) - D. \end{aligned} \quad (\text{A.7})$$

Thus, we can rewrite (A.6) as a functional equation

$$\Gamma(p^*, D, \nu_0, \nu_1) = 0 \quad (\text{A.8})$$

and compute its partial derivative with respect to p^* :

$$\begin{aligned} \frac{\partial \Gamma}{\partial p^*} &= \frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) - G'(p^*) \\ &\geq \frac{1}{(1-\beta)\nu_0 + a_0} > 0. \end{aligned} \quad (\text{A.9})$$

From (A.9) we can see that the partial derivative of Γ with respect to p^* is positive. Because of that, Implicit Function Theorem implies that the equilibrium price p^* can be considered as a function $p^* = p^*(D, \nu_0, \nu_1)$, which is differentiable with respect to D and ν_i , $i = 0, 1$. Moreover, the partial derivative of the price p^* with respect to D can be found from the equation

$$\frac{\partial \Gamma}{\partial p^*} \frac{\partial p^*}{\partial D} + \frac{\partial \Gamma}{\partial D} = 0. \quad (\text{A.10})$$

The latter leads to

$$\frac{\partial p^*}{\partial D} = - \frac{\frac{\partial \Gamma}{\partial D}}{\frac{\partial \Gamma}{\partial p^*}} = \frac{1}{\frac{1}{(1-\beta)\nu_0 + a_0} + \frac{\nu_0 + a_0}{(1-\beta)\nu_0 + a_0} \left(\frac{1}{\nu_1 + a_1} \right) - G'(p^*)}. \quad (\text{A.11})$$

Finally, since the function p^* depends upon (D, ν_0, ν_1) and is differentiable with respect to D and ν_i , $i = 0, 1$, the functions q_i^* , $i = 0, 1$, also depend on (D, ν_0, ν_1) and are differentiable with respect to D and ν_i , $i = 0, 1$. Therefore, the equilibrium (p^*, q_0^*, q_1^*) continuously depends on the parameters (D, ν_0, ν_1) . The proof of the theorem is complete ■

Proposition 1.1. *For all $\tau \leq 0$, there exists a unique solution $\nu_i = \nu_i(\tau)$, $i = 0, 1$, of system (1.20) and (1.21), which continuously depends upon τ . In addition, $\nu_i(\tau) \rightarrow 0$ whenever $\tau \rightarrow -\infty$, and $\nu_i(\tau)$ strictly grows and tends to $\nu_i(0)$ as $\tau \rightarrow 0$, $i = 0, 1$.*

Proof. The variables ν_i , $i = 0, 1$, given by (1.20) and (1.21) are considered on their domains: $\nu_i \geq 0$, $a_i > 0$, $i = 0, 1$, $\beta \in (0, 1]$, and $\tau \in (-\infty, 0]$.

Substituting (1.21) in (1.20) we get the following equation:

$$\begin{aligned}
 \nu_0 &= \frac{1}{\frac{1}{\left(\frac{1}{\frac{1}{(1-\beta)\nu_0 + a_0} - \tau} \right) + a_1} - \tau} \\
 &= \frac{1}{\frac{1}{\left(\frac{(1-\beta)\nu_0 + a_0}{1 - [(1-\beta)\nu_0 + a_0]\tau} \right) + a_1} - \tau} \\
 &= \frac{1}{\frac{1}{\left(\frac{(1-\beta)\nu_0 + a_0}{-(1-\beta)\tau\nu_0 + (1-a_0\tau)} \right) + a_1} - \tau} \tag{A.12} \\
 &= \frac{1}{\frac{-(1-\beta)\tau\nu_0 + (1-a_0\tau)}{(1-\beta)\nu_0 + a_0 + a_1[-(1-\beta)\tau\nu_0 + (1-a_0\tau)]} - \tau} \\
 &= \frac{1}{\frac{-(1-\beta)\tau\nu_0 + (1-a_0\tau)}{(1-\beta)(1-a_1\tau)\nu_0 + (a_0 + a_1 - a_0a_1\tau)} - \tau} \\
 &= \frac{(1-\beta)(1-a_1\tau)\nu_0 + (a_0 + a_1 - a_0a_1\tau)}{-(1-\beta)\tau\nu_0 + (1-a_0\tau) - [(1-\beta)(1-a_1\tau)\nu_0 + (a_0 + a_1 - a_0a_1\tau)]\tau} \\
 &= \frac{(1-\beta)(1-a_1\tau)\nu_0 + (a_0 + a_1 - a_0a_1\tau)}{(1-\beta)(-2\tau + a_1\tau^2)\nu_0 + (1 - 2a_0\tau - a_1\tau + a_0a_1\tau^2)}.
 \end{aligned}$$

Then, we can multiply (A.12) by $[(1 - \beta)(-2\tau + a_1\tau^2)\nu_0 + (1 - 2a_0\tau - a_1\tau + a_0a_1\tau^2)]$ to obtain

$$\begin{aligned} & [(1 - \beta)(-2\tau + a_1\tau^2)\nu_0 + (1 - 2a_0\tau - a_1\tau + a_0a_1\tau^2)]\nu_0 \\ & = (1 - \beta)(1 - a_1\tau)\nu_0 + (a_0 + a_1 - a_0a_1\tau). \end{aligned} \quad (\text{A.13})$$

Move all the terms of (A.13) to the left-hand side and get

$$\begin{aligned} & [(1 - \beta)(-2\tau + a_1\tau^2)\nu_0 + (1 - 2a_0\tau - a_1\tau + a_0a_1\tau^2)]\nu_0 \\ & - (1 - \beta)(1 - a_1\tau)\nu_0 - (a_0 + a_1 - a_0a_1\tau) = 0. \end{aligned} \quad (\text{A.14})$$

By extracting a common factors in (A.14) we obtain the following quadratic equation for ν_0 :

$$(1 - \beta)(-2\tau + a_1\tau^2)\nu_0^2 + (\beta - 2a_0\tau - \beta a_1\tau + a_0a_1\tau^2)\nu_0 - (a_0 + a_1 - a_0a_1\tau) = 0. \quad (\text{A.15})$$

Now, in order to simplify the notation, we rewrite (A.15) as follows:

$$A\nu_0^2 + B\nu_0 - C = 0, \quad (\text{A.16})$$

where

$$A = A(\tau) = (1 - \beta)(-2\tau + a_1\tau^2) \geq 0, \quad (\text{A.17})$$

$$B = B(\tau) = \beta - 2a_0\tau - \beta a_1\tau + a_0a_1\tau^2 > 0, \quad (\text{A.18})$$

$$C = C(\tau) = a_0 + a_1 - a_0a_1\tau > 0. \quad (\text{A.19})$$

If $\tau = 0$ or $\beta = 1$, then, $A = 0$ and (A.16) is linear, so we can find the unique solution for ν_0 given by:

$$\nu_0(\tau) = \frac{C}{B} = \begin{cases} \frac{a_0 + a_1}{\beta} & \text{if } \tau = 0, \\ \frac{a_0 + a_1 - a_0a_1\tau}{1 - 2a_0\tau - a_1\tau + a_0a_1\tau^2} & \text{if } \beta = 1. \end{cases} \quad (\text{A.20})$$

If $\beta \in (0, 1)$ and $\tau < 0$, then, $A \neq 0$ and we can find both roots of (A.16), which are:

$$\nu_0(\tau) = \frac{-B + \sqrt{B^2 + 4AC}}{2A}, \quad (\text{A.21})$$

$$\nu_0(\tau) = \frac{-B - \sqrt{B^2 + 4AC}}{2A}. \quad (\text{A.22})$$

However, since $\nu_0 \geq 0$, the root (A.22) is impossible; that is, (A.21) is the unique solution of (A.16).

Moreover, (A.20) and (A.21) can be combined in a single equation for all $\beta \in (0, 1]$ and $\tau \in (-\infty, 0]$ as follows:

$$\begin{aligned} \nu_0(\tau) = \nu_0 &= \frac{2C}{B + \sqrt{B^2 + 4AC}} \\ &= \frac{2(a_0 + a_1 - a_0 a_1 \tau)}{(\beta - 2a_0\tau - \beta a_1\tau + a_0 a_1 \tau^2) + \sqrt{(\beta - 2a_0\tau - \beta a_1\tau + a_0 a_1 \tau^2)^2 + 4(1-\beta)(-2\tau + a_1\tau^2)(a_0 + a_1 - a_0 a_1 \tau)}}, \end{aligned} \quad (\text{A.23})$$

where

$$B + \sqrt{B^2 + 4AC} > 0, \quad (\text{A.24})$$

and so (A.23) is the unique solution for ν_0 .

We can see that the solution (A.23) for any parameter $\beta \in (0, 1]$ satisfies the condition $\nu_0 \rightarrow 0$ as $\tau \rightarrow -\infty$. Hence, there exists a positive value $\overline{\nu_0}(\beta)$ such that $\nu_0(\tau) \leq \overline{\nu_0}(\beta)$ for all $\tau \leq 0$.

From (1.21) and (A.23), we can see that ν_1 also has a unique solution, which is given by

$$\nu_1(\tau) = \nu_1 = \frac{1}{\frac{1}{(1-\beta)\nu_0(\tau) + a_0} - \tau}. \quad (\text{A.25})$$

For any parameter $\beta \in (0, 1]$, the conditions $\nu_1 \rightarrow 0$ as $\tau \rightarrow -\infty$ and $\nu_1(\tau) \leq a_0 + (1-\beta)\overline{\nu_0}(\beta)$ for all $\tau \leq 0$, are satisfied.

Now, it is apparent that the functions (A.17)-(A.19) are continuously differentiable with respect to τ , $\tau \in (-\infty, 0]$, and

$$A' = (1-\beta)(-2 + 2a_1\tau) \leq 0, \quad (\text{A.26})$$

$$B' = -2a_0 - \beta a_1 + 2a_0 a_1 \tau < 0, \quad (\text{A.27})$$

$$C' = -a_0 a_1 < 0. \quad (\text{A.28})$$

Thus, from (A.23), we have that $\nu_0(\tau)$ is continuously differentiable and

$$\begin{aligned}
\nu'_0 &= \frac{2C' \left(B + \sqrt{B^2 + 4AC} \right) - 2C \left(B' + \frac{2BB' + 4A'C + 4AC'}{2\sqrt{B^2 + 4AC}} \right)}{\left(B + \sqrt{B^2 + 4AC} \right)^2} \\
&= \frac{2C' \left(B + \sqrt{B^2 + 4AC} \right) \sqrt{B^2 + 4AC} - 2C \left(B' \sqrt{B^2 + 4AC} + \frac{2BB' + 4A'C + 4AC'}{2} \right)}{\left(B + \sqrt{B^2 + 4AC} \right)^2 \sqrt{B^2 + 4AC}} \\
&= \frac{2C' \left(B \sqrt{B^2 + 4AC} + B^2 + 4AC \right) - 2C \left(B' \sqrt{B^2 + 4AC} + BB' + 2A'C + 2AC' \right)}{\left(B + \sqrt{B^2 + 4AC} \right)^2 \sqrt{B^2 + 4AC}} \tag{A.29} \\
&= \frac{2C' B \sqrt{B^2 + 4AC} + 2C' B^2 + 4ACC' - 2CB' \sqrt{B^2 + 4AC} - 2CBB' - 4A'C^2}{\left(B + \sqrt{B^2 + 4AC} \right)^2 \sqrt{B^2 + 4AC}} \\
&= \frac{2(C'B - CB') \sqrt{B^2 + 4AC} + 2(C'B - CB') B + 4(AC' - A'C) C}{\left(B + \sqrt{B^2 + 4AC} \right)^2 \sqrt{B^2 + 4AC}} \\
&= \frac{2(C'B - CB') \left(B + \sqrt{B^2 + 4AC} \right) + 4(AC' - A'C) C}{\left(B + \sqrt{B^2 + 4AC} \right)^2 \sqrt{B^2 + 4AC}}.
\end{aligned}$$

Now, we estimate the values of (A.29) in order to reveal the behavior of $\nu_0(\tau)$ as the function of τ .

From (A.17)-(A.19), it is evident that the denominator of (A.29) is positive:

$$\left(B + \sqrt{B^2 + 4AC} \right)^2 \sqrt{B^2 + 4AC} > 0. \tag{A.30}$$

Thus, plugging (A.17)-(A.19) and (A.26)-(A.28) in (A.29), we can find that

$$\begin{aligned}
C'B - CB' &= (-a_0 a_1) (\beta - 2a_0 \tau - \beta a_1 \tau + a_0 a_1 \tau^2) \\
&\quad - (a_0 + a_1 - a_0 a_1 \tau) (-2a_0 - \beta a_1 + 2a_0 a_1 \tau) \\
&= (-a_0 a_1) (\beta - a_0 a_1 \tau^2) + (-a_0 a_1) (-2a_0 \tau - \beta a_1 \tau + 2a_0 a_1 \tau^2) \\
&\quad - [(a_0 + a_1) (-2a_0 - \beta a_1 + 2a_0 a_1 \tau) + (-a_0 a_1 \tau) (-2a_0 - \beta a_1 + 2a_0 a_1 \tau)] \\
&= a_0 a_1 (-\beta + a_0 a_1 \tau^2) - a_0 a_1 (-2a_0 \tau - \beta a_1 \tau + 2a_0 a_1 \tau^2) \\
&\quad + a_0 a_1 (-2a_0 \tau - \beta a_1 \tau + 2a_0 a_1 \tau^2) + (a_0 + a_1) (2a_0 + \beta a_1 - 2a_0 a_1 \tau) \\
&= a_0 a_1 (-\beta + a_0 a_1 \tau^2) + (a_0 + a_1) (2a_0 + \beta a_1 - 2a_0 a_1 \tau) \\
&= a_0 a_1 (-\beta) + a_0 a_1 (a_0 a_1 \tau^2) + (a_0 + a_1) (2a_0 - 2a_0 a_1 \tau) + (a_0 + a_1) (\beta a_1) \\
&= -a_0 (\beta a_1) + (a_0 + a_1) (\beta a_1) + a_0 a_1 (a_0 a_1 \tau^2) + (a_0 + a_1) (2a_0 - 2a_0 a_1 \tau) \\
&= \beta a_1^2 + a_0^2 a_1^2 \tau^2 + (a_0 + a_1) (2a_0 - 2a_0 a_1 \tau) > 0,
\end{aligned} \tag{A.31}$$

and

$$\begin{aligned}
AC' - A'C &= (1 - \beta) (-2\tau + a_1\tau^2) (-a_0a_1) \\
&\quad - (1 - \beta) (-2 + 2a_1\tau) (a_0 + a_1 - a_0a_1\tau) \\
&= (1 - \beta) (-2\tau + 2a_1\tau^2) (-a_0a_1) + (1 - \beta) (-a_1\tau^2) (-a_0a_1) \\
&\quad - (1 - \beta) (-2 + 2a_1\tau) (a_0 + a_1) - (1 - \beta) (-2 + 2a_1\tau) (-a_0a_1\tau) \quad (\text{A.32}) \\
&= (1 - \beta) a_0a_1^2\tau^2 - (1 - \beta) (-2\tau + 2a_1\tau^2) (a_0a_1) \\
&\quad + (1 - \beta) (-2\tau + 2a_1\tau^2) (a_0a_1) + (1 - \beta) (2 - 2a_1\tau) (a_0 + a_1) \\
&= (1 - \beta) a_0a_1^2\tau^2 + (1 - \beta) (2 - 2a_1\tau) (a_0 + a_1) \geq 0.
\end{aligned}$$

Therefore, given the values of (A.17)-(A.19), (A.24) and (A.30)-(A.32), we can conclude that

$$\begin{aligned}
\nu'_0 &= \frac{2(C'B - CB') (B + \sqrt{B^2 + 4AC}) + 4(AC' - A'C)C}{(B + \sqrt{B^2 + 4AC})^2 \sqrt{B^2 + 4AC}} \\
&\geq \frac{2(C'B - CB') (B + \sqrt{B^2 + 4AC})}{(B + \sqrt{B^2 + 4AC})^2 \sqrt{B^2 + 4AC}} > 0.
\end{aligned} \quad (\text{A.33})$$

Therefore, $\nu_0(\tau)$ is strictly increasing with respect to $\tau, \tau \in (-\infty, 0]$. Since the function $\nu_0 = \nu_0(\tau)$ is continuous, it tends to $\nu_0(0)$ as τ goes to 0.

Now, from (A.25) we have

$$\nu_1 = \frac{1}{\frac{1}{(1 - \beta)\nu_0 + a_0} - \tau}. \quad (\text{A.34})$$

Since, $\nu_0(\tau)$ is continuously differentiable with respect to τ , the same is true for $\nu_1(\tau)$, and

$$\begin{aligned}
\nu'_1 &= - \frac{1}{\left(\frac{1}{(1 - \beta)\nu_0 + a_0} - \tau\right)^2} \left(-\frac{1}{[(1 - \beta)\nu_0 + a_0]^2} (1 - \beta)\nu'_0 - 1\right) \\
&= \nu_1^2 \left(\frac{(1 - \beta)\nu'_0}{[(1 - \beta)\nu_0 + a_0]^2} + 1\right),
\end{aligned} \quad (\text{A.35})$$

where $\nu'_0 > 0$, On account of that,

$$\nu'_1 = \nu_1^2 \left(\frac{(1 - \beta)\nu'_0}{[(1 - \beta)\nu_0 + a_0]^2} + 1\right) \geq \nu_1^2 > 0. \quad (\text{A.36})$$

Therefore, $\nu_1(\tau)$ strictly increases with respect to τ , $\tau \in (-\infty, 0]$. Since the function $\nu_1 = \nu_1(\tau)$ is continuous, it tends to $\nu_1(0)$ as τ goes to 0. The proof of the theorem is complete. ■

Theorem 1.2. *Under assumptions **A1.1-A1.3**, there exists the interior equilibrium.*

Proof. We are going to show that there exist $\nu_i^* \geq 0$, $q_i^* \geq 0$, $i = 0, 1$, and $p^* > p_0$ such that the vector (p^*, q_0^*, q_1^*) is the exterior equilibrium, and the influence coefficients (ν_0^*, ν_1^*) are consistent, i.e., equations (1.18) and (1.19) hold.

As it was proved in proposition 1.1, ν_0 and ν_1 solve uniquely equations (1.20) and (1.21), and continuously depend on $\tau = G'(p)$. Moreover, $G'(p)$ continuously depends on p , hence, the functions ν_0 and ν_1 are continuous with respect to p .

Recall the function (A.3) introduced when proving theorem 1.1:

$$\begin{aligned} Q(p, \nu_0(p), \nu_1(p)) &= Q(p) = q_0(p, \nu_0(p), \nu_1(p)) + q_1(p, \nu_0(p), \nu_1(p)) \\ &= \frac{p - b_0}{(1 - \beta)\nu_0(p) + a_0} + \frac{\beta\nu_0(p)}{(1 - \beta)\nu_0(p) + a_0} \left(\frac{p - b_1}{\nu_1(p) + a_1} \right) + \frac{p - b_1}{\nu_1(p) + a_1} \\ &= p \left[\frac{1}{(1 - \beta)\nu_0(p) + a_0} + \frac{\nu_0(p) + a_0}{(1 - \beta)\nu_0(p) + a_0} \left(\frac{1}{\nu_1(p) + a_1} \right) \right] \\ &\quad - \left[\frac{b_0}{(1 - \beta)\nu_0(p) + a_0} + \frac{\nu_0(p) + a_0}{(1 - \beta)\nu_0(p) + a_0} \left(\frac{b_1}{\nu_1(p) + a_1} \right) \right]. \end{aligned} \quad (\text{A.37})$$

which continuously depends on p and tends to $+\infty$ as $p \rightarrow +\infty$ since $\nu_0(p)$ and $\nu_1(p)$ are bounded. Thus, by assumption **A1.3**, we have that

$$\begin{aligned} Q(p_0) &= q_0(p_0, \nu_0(p_0), \nu_1(p_0)) + q_1(p_0, \nu_0(p_0), \nu_1(p_0)) \\ &= \frac{p_0 - b_0}{(1 - \beta)\nu_0(p_0) + a_0} \leq \frac{p_0 - b_0}{a_0} < G(p_0) \leq G(p_0) + D. \end{aligned} \quad (\text{A.38})$$

Therefore, there exists the value $p^* > p_0$ such that

$$Q(p^*) = G(p^*) + D. \quad (\text{A.39})$$

For this value p^* , we compute the influence coefficients $\nu_i^* = \nu_i(G'(p^*))$, $i = 0, 1$, using (A.23) and (A.25), as well as the output volumes $q_i^* = q_i(p^*, \nu_0^*, \nu_1^*)$, $i = 0, 1$, given by (A.1) and (A.2). Thus, ν_0^* and ν_1^* satisfy (1.18) and (1.19), whereas the vector (p^*, q_0^*, q_1^*) is the exterior equilibrium. As a consequence, the extended vector $(p^*, q_0^*, q_1^*, \nu_0^*, \nu_1^*)$ is the interior equilibrium. The proof of the theorem is complete. ■

Corollary 1.1. *Under assumptions **A1.1-A1.3**, for all $\beta \in (0, 1]$, the demand function of type (1.22) implies the uniqueness of the interior equilibrium.*

Proof. Consider an arbitrary $\beta \in (0, 1]$. Since $G'(p) = -K$, then, by proposition 1.1, for $\tau = -K$ there exists a unique solution (ν_0^*, ν_1^*) of equations (1.20) and (1.21):

$$\nu_0^* = \frac{2(a_0 + a_1 + a_0 a_1 K)}{(\beta + 2a_0 K + \beta a_1 K + a_0 a_1 K^2) + \sqrt{(\beta + 2a_0 K + \beta a_1 K + a_0 a_1 K^2)^2 + 4(1 - \beta)(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}}, \quad (\text{A.40})$$

and

$$\nu_1^* = \frac{1}{\frac{1}{(1 - \beta)\nu_0^* + a_0} + K} = \frac{(1 - \beta)\nu_0^* + a_0}{1 + [(1 - \beta)\nu_0^* + a_0]K}. \quad (\text{A.41})$$

Moreover, from (1.20), we can rewrite (A.40) as follows:

$$\nu_0^* = \frac{1}{\frac{1}{\nu_1^* + a_1} + K}. \quad (\text{A.42})$$

It is not difficult to see that the influence coefficients ν_0^* and ν_1^* don't depend on p , therefore, by theorem 1.1, there exists the unique exterior equilibrium (p^*, q_0^*, q_1^*) with the influence coefficients (ν_0^*, ν_1^*) . Hence, the vector

$$(p^*, q_0^*, q_1^*, \nu_0^*, \nu_1^*) = (p^*(\beta), q_0^*(\beta), q_1^*(\beta), \nu_0^*(\beta), \nu_1^*(\beta))$$

is the unique interior equilibrium for $\beta \in (0, 1]$. The proof of the corollary is complete. ■

Theorem 1.3. *For the affine demand function $G(p)$ from (1.22), the price $p^*(\beta)$, the supply outputs $q_i^*(\beta)$, $i = 0, 1$, and the influence coefficients $\nu_i^*(\beta)$, $i = 0, 1$, characterizing the interior equilibrium, together with total market supply $G^*(\beta) = q_0^*(\beta) + q_1^*(\beta)$, are continuously differentiable by $\beta \in (0, 1]$. Furthermore, $q_0^*(\beta)$ and $G^*(\beta)$ strictly increase, whereas $p^*(\beta)$, $\nu_0^*(\beta)$, $\nu_1^*(\beta)$ and $q_1^*(\beta)$ strictly decrease.*

Proof. First, we are going to show that the functions $\nu_i^*(\beta)$, $i = 0, 1$, are continuously differentiable and strictly decreasing with respect to β . Let us consider the functions

$$\mathcal{A} = \mathcal{A}(\beta) = (1 - \beta)(2K + a_1 K^2) \geq 0, \quad (\text{A.43})$$

$$\mathcal{B} = \mathcal{B}(\beta) = \beta + 2a_0 K + \beta a_1 K + a_0 a_1 K^2 > 0, \quad (\text{A.44})$$

$$\mathcal{C} = \mathcal{C}(\beta) = a_0 + a_1 + a_0 a_1 K > 0, \quad (\text{A.45})$$

which are continuously differentiable with respect to β , with

$$\mathcal{A}' = -(2K + a_1 K^2) < 0, \quad (\text{A.46})$$

$$\mathcal{B}' = 1 + a_1 K > 0, \quad (\text{A.47})$$

$$\mathcal{C}' = 0. \quad (\text{A.48})$$

Using (A.43)-(A.45) we rewrite (A.40) as follows:

$$\nu_0^*(\beta) = \nu_0^* = \frac{2\mathcal{C}}{\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}}, \quad (\text{A.49})$$

where

$$\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} > 0. \quad (\text{A.50})$$

Then, $\nu_0^*(\beta)$ is continuously differentiable with respect to β and, similarly to (A.29),

$$\nu_0^{*'} = \frac{2(\mathcal{C}'\mathcal{B} - \mathcal{C}\mathcal{B}')(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}) + 4(\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C})\mathcal{C}}{(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}})^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}}. \quad (\text{A.51})$$

Since $\mathcal{C}' = 0$, then,

$$\begin{aligned} \nu_0^{*'} &= \frac{2(-\mathcal{C}\mathcal{B}')(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}) + 4(-\mathcal{A}'\mathcal{C})\mathcal{C}}{(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}})^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}} \\ &= \frac{-2\mathcal{C}[\mathcal{B}'(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}) + 2\mathcal{A}'\mathcal{C}]}{(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}})^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}}. \end{aligned} \quad (\text{A.52})$$

Now we are going to estimate the value of (A.52) in order to describe the behavior of $\nu_0^*(\beta)$ as a function of β .

From (A.43)-(A.45), it is evident that the denominator of (A.52) is positive:

$$(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}})^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} > 0. \quad (\text{A.53})$$

Suppose that the numerator of (A.52) is nonnegative for some $\beta_0 \in (0, 1]$, i.e,

$$-2\mathcal{C}[\mathcal{B}'(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}) + 2\mathcal{A}'\mathcal{C}] \geq 0. \quad (\text{A.54})$$

Since $\mathcal{C} > 0$, by (A.45), we have that

$$\mathcal{B}' \left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) + 2\mathcal{A}'\mathcal{C} \leq 0. \quad (\text{A.55})$$

Moreover, $\mathcal{B}' > 0$, by (A.47), therefore,

$$\sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \leq \frac{-2\mathcal{A}'\mathcal{C}}{\mathcal{B}'} - \mathcal{B} \quad (\text{A.56})$$

where $\sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} > 0$. Now squaring both sides of (A.56) we have

$$\mathcal{B}^2 + 4\mathcal{A}\mathcal{C} \leq \frac{4\mathcal{A}'^2\mathcal{C}^2}{\mathcal{B}'^2} + \frac{4\mathcal{A}'\mathcal{C}\mathcal{B}}{\mathcal{B}'} + \mathcal{B}^2. \quad (\text{A.57})$$

Solving (A.57) for \mathcal{A} we get

$$\mathcal{A} \leq \frac{\mathcal{A}'^2\mathcal{C}}{\mathcal{B}'^2} + \frac{\mathcal{A}'\mathcal{B}}{\mathcal{B}'}. \quad (\text{A.58})$$

Multiplying both sides of (A.58) by \mathcal{B}^2 we deduce

$$\mathcal{A}\mathcal{B}'^2 \leq \mathcal{A}'^2\mathcal{C} + \mathcal{A}'\mathcal{B}\mathcal{B}' = \mathcal{A}'(\mathcal{A}'\mathcal{C} + \mathcal{B}\mathcal{B}'). \quad (\text{A.59})$$

Now, we substitute the values of \mathcal{A} and \mathcal{A}' given by (A.43) and (A.46) in (A.59) to obtain:

$$(1 - \beta)(2K + a_1K^2)\mathcal{B}'^2 \leq -(2K + a_1K^2) \left[-(2K + a_1K^2)\mathcal{C} + \mathcal{B}\mathcal{B}' \right], \quad (\text{A.60})$$

and since $(2K + a_1K^2) > 0$, we have that

$$(1 - \beta)\mathcal{B}'^2 \leq - \left[-(2K + a_1K^2)\mathcal{C} + \mathcal{B}\mathcal{B}' \right] = (2K + a_1K^2)\mathcal{C} - \mathcal{B}\mathcal{B}'. \quad (\text{A.61})$$

The latter implies

$$(1 - \beta)\mathcal{B}'^2 + \mathcal{B}\mathcal{B}' - (2K + a_1K^2)\mathcal{C} = [(1 - \beta)\mathcal{B}' + \mathcal{B}]\mathcal{B}' - (2K + a_1K^2)\mathcal{C} \leq 0. \quad (\text{A.62})$$

Plugging equations (A.44), (A.45) and (A.47) in (A.62) we yield

$$\begin{aligned}
& [(1 - \beta)\mathcal{B}' + \mathcal{B}]\mathcal{B}' - (2K + a_1K^2)\mathcal{C} = \\
& = [(1 - \beta)(1 + a_1K) + (\beta + 2a_0K + \beta a_1K + a_0a_1K^2)](1 + a_1K) \\
& \quad - (2K + a_1K^2)(a_0 + a_1 + a_0a_1K) \\
& = (1 + 2a_0K + a_1K + a_0a_1K^2)(1 + a_1K) \\
& \quad - (2 + a_1K)(a_0K + a_1K + a_0a_1K^2) \\
& = 1 + (a_1K) + (2a_0K + a_1K + a_0a_1K^2) + (2a_0K + a_1K + a_0a_1K^2)(a_1K) \\
& \quad - (2 + a_1K)(a_0K + a_1K + a_0a_1K^2) \tag{A.63} \\
& = 1 + 2(a_0K + a_1K + a_0a_1K^2) + (a_0K + a_1K + a_0a_1K^2)(a_1K) \\
& \quad - (2 + a_1K)(a_0K + a_1K + a_0a_1K^2) \\
& = 1 + (2 + a_1K)(a_0K + a_1K + a_0a_1K^2) \\
& \quad - (2 + a_1K)(a_0K + a_1K + a_0a_1K^2) \\
& = 1 > 0,
\end{aligned}$$

which contradicts (A.62). Hence, (A.54) cannot hold for any $\beta \in (0, 1]$, which implies

$$-2\mathcal{C} \left[\mathcal{B}' \left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) + 2\mathcal{A}'\mathcal{C} \right] < 0 \tag{A.64}$$

for all $\beta \in (0, 1]$.

Therefore, from (A.53) and (A.64), we conclude that

$$\nu_0^{*'} = \frac{-2\mathcal{C} \left[\mathcal{B}' \left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) + 2\mathcal{A}'\mathcal{C} \right]}{(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}})^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}} < 0 \tag{A.65}$$

for all $\beta \in (0, 1]$. On account of the latter, $\nu_0^*(\beta)$ is continuously differentiable and strictly decreasing with respect β , $\beta \in (0, 1]$.

From (A.41), it is clear that ν_1^* is continuously differentiable with respect to ν_0^* and, since $\nu_0^*(\beta)$, on its own, is also smooth as a function of β , then, $\nu_1^*(\beta)$, is continuously differentiable by β .

Differentiating (A.42) with respect to β we get

$$\nu_0^{*'} = \frac{1}{\left(\frac{1}{\nu_1^* + a_1} + K \right)^2} \left(\frac{1}{(\nu_1^* + a_1)^2} \nu_1^{*'} \right) = \nu_0^{*2} \left(\frac{\nu_1^{*'}}{(\nu_1^* + a_1)^2} \right) = \left(\frac{\nu_0^*}{\nu_1^* + a_1} \right)^2 \nu_1^{*'} \tag{A.66}$$

Since $\nu_0^{*'} < 0$, then, (A.66) implies that $\nu_1^{*'} < 0$, for all $\beta \in (0, 1]$. Thus, $\nu_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β , $\beta \in (0, 1]$. Before continuing the proof, we are going to establish the following inequality:

$$\nu_0^* + \beta \nu_0^{*'} > 0. \quad (\text{A.67})$$

Substituting (A.49) and (A.52) in (A.67), we get

$$\begin{aligned} \nu_0^* + \beta \nu_0^{*'} &= \frac{2\mathcal{C}}{\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}} + \beta \frac{-2\mathcal{C} \left[\mathcal{B}' \left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) + 2\mathcal{A}'\mathcal{C} \right]}{\left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right)^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}} \\ &= \frac{2\mathcal{C}}{\left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right)^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}} \\ &\quad \cdot \left\{ \left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} - \beta \left[\mathcal{B}' \left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) + 2\mathcal{A}'\mathcal{C} \right] \right\} \\ &= \frac{2\mathcal{C}}{\left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right)^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}} \\ &\quad \cdot \left[\left(-\beta \mathcal{B}' + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) \left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) - 2\beta \mathcal{A}'\mathcal{C} \right]. \end{aligned} \quad (\text{A.68})$$

By (A.45), (A.46) and (A.53),

$$\frac{2\mathcal{C}}{\left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right)^2 \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}}} > 0 \quad (\text{A.69})$$

and

$$-2\beta \mathcal{A}'\mathcal{C} > 0. \quad (\text{A.70})$$

Then, to prove inequality (A.67), it suffices to show that

$$\left(-\beta \mathcal{B}' + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) \left(\mathcal{B} + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \right) > 0, \quad (\text{A.71})$$

which, by (A.50), is equivalent to show that

$$-\beta \mathcal{B}' + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} > 0. \quad (\text{A.72})$$

Suppose, on the contrary, that

$$-\beta \mathcal{B}' + \sqrt{\mathcal{B}^2 + 4\mathcal{A}\mathcal{C}} \leq 0. \quad (\text{A.73})$$

Then,

$$\sqrt{\mathcal{B}^2 + 4\mathcal{AC}} \leq \beta\mathcal{B}' \quad (\text{A.74})$$

where $\sqrt{\mathcal{B}^2 + 4\mathcal{AC}} > 0$. Hence, by squaring both sides of (A.74) we have

$$\mathcal{B}^2 + 4\mathcal{AC} \leq \beta^2 \mathcal{B}'^2. \quad (\text{A.75})$$

Plugging (A.44) and (A.47) in (A.75) yields

$$(\beta + 2a_0K + \beta a_1K + a_0a_1K^2)^2 + 4\mathcal{AC} \leq \beta^2 (1 + a_1K)^2, \quad (\text{A.76})$$

which implies

$$[(\beta + \beta a_1K) + 2a_0K + a_0a_1K^2]^2 + 4\mathcal{AC} \leq (\beta + \beta a_1K)^2. \quad (\text{A.77})$$

However, by (A.43) and (A.45),

$$4\mathcal{AC} \geq 0, \quad (\text{A.78})$$

that is,

$$[(\beta + \beta a_1K) + 2a_0K + a_0a_1K^2]^2 \leq (\beta + \beta a_1K)^2. \quad (\text{A.79})$$

On the other hand,

$$2a_0K + a_0a_1K^2 > 0, \quad (\text{A.80})$$

whence

$$(\beta + \beta a_1K) < (\beta + \beta a_1K) + 2a_0K + a_0a_1K^2 \quad (\text{A.81})$$

where $(\beta + \beta a_1K) > 0$. Now by squaring both sides of (A.74) we have

$$(\beta + \beta a_1K)^2 < [(\beta + \beta a_1K) + 2a_0K + a_0a_1K^2]^2. \quad (\text{A.82})$$

Nevertheless, inequality (A.82) contradicts (A.79), which means that (A.72) must hold and thus prove (A.67).

Now, coming back to the proof of the theorem, we are going to show that the equilibrium price $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β . Consider again the function (A.3) and by plugging it in $G(p^*) = -Kp^* + T$ get the

following relationships:

$$\begin{aligned}
& Q(p^*, \nu_0^*, \nu_1^*) - G(p^*) - D = \\
& = p^* \left[\frac{1}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0} \left(\frac{1}{\nu_1^* + a_1} \right) \right] \\
& \quad - \left[\frac{b_0}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0} \left(\frac{b_1}{\nu_1^* + a_1} \right) \right] \\
& \quad + Kp^* - T - D = 0.
\end{aligned} \tag{A.83}$$

Consider the function

$$\begin{aligned}
\mathcal{F}(p^*, \beta) = & p^* \left[\frac{1}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0} \left(\frac{1}{\nu_1^* + a_1} \right) \right] \\
& - \left[\frac{b_0}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0} \left(\frac{b_1}{\nu_1^* + a_1} \right) \right] \\
& + Kp^* - T - D,
\end{aligned} \tag{A.84}$$

having in mind that ν_0^* and ν_1^* depend on β , but not on p^* . Now, we rewrite (A.83) using (A.84) as a functional equation:

$$\mathcal{F}(p^*, \beta) = 0. \tag{A.85}$$

Now we are in a position to estimate the value of the partial derivative of the function $\mathcal{F}(p^*, \beta)$ with respect to p^* :

$$\frac{\partial \mathcal{F}}{\partial p^*} = \frac{1}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0} \left(\frac{1}{\nu_1^* + a_1} \right) + K \geq K > 0. \tag{A.86}$$

We observe that the partial derivative \mathcal{F} with respect to p^* is positive. Hence, by the Implicit Function Theorem, the function $p^* = p^*(\beta)$ is differentiable with respect to β , and its partial derivative with respect to β can be found from the equation

$$\frac{\partial \mathcal{F}}{\partial p^*} \frac{dp^*}{d\beta} + \frac{\partial \mathcal{F}}{\partial \beta} = 0, \tag{A.87}$$

which leads to

$$\frac{dp^*}{d\beta} = - \frac{\frac{\partial \mathcal{F}}{\partial \beta}}{\frac{\partial \mathcal{F}}{\partial p^*}}. \tag{A.88}$$

From (A.86), we have

$$\frac{\partial \mathcal{F}}{\partial p^*} > 0. \quad (\text{A.89})$$

Therefore, to prove that p^* is strictly increasing, we have to show that

$$\frac{\partial \mathcal{F}}{\partial \beta} > 0. \quad (\text{A.90})$$

Indeed,

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial \beta} &= \frac{\partial}{\partial \beta} (Q(p^*, \nu_0^*, \nu_1^*) - G(p^*) - D) = \frac{\partial}{\partial \beta} Q(p^*, \nu_0^*, \nu_1^*) \\ &= \frac{\partial}{\partial \beta} (q_0(p^*, \nu_0^*, \nu_1^*) + q_1(p^*, \nu_0^*, \nu_1^*)) \\ &= \frac{\partial}{\partial \beta} \left[\frac{p^* - b_0}{(1 - \beta)\nu_0^* + a_0} + \frac{\beta\nu_0^*}{(1 - \beta)\nu_0^* + a_0} \left(\frac{p^* - b_1}{\nu_1^* + a_1} \right) + \frac{p^* - b_1}{\nu_1^* + a_1} \right] \\ &= - \frac{p^* - b_0}{[(1 - \beta)\nu_0^* + a_0]^2} [-\nu_0^* + (1 - \beta)\nu_0^{*'}] \\ &\quad + \frac{(\nu_0^* + \beta\nu_0^{*'}) [(1 - \beta)\nu_0^* + a_0] - \beta\nu_0^* [-\nu_0^* + (1 - \beta)\nu_0^{*'}]}{[(1 - \beta)\nu_0^* + a_0]^2} \left(\frac{p^* - b_1}{\nu_1^* + a_1} \right) \\ &\quad + \frac{\beta\nu_0^*}{(1 - \beta)\nu_0^* + a_0} \left(-\frac{p^* - b_1}{(\nu_1^* + a_1)^2} \nu_1^{*'} \right) + \left(-\frac{p^* - b_1}{(\nu_1^* + a_1)^2} \nu_1^{*'} \right) \\ &= \frac{p^* - b_0}{[(1 - \beta)\nu_0^* + a_0]^2} [\nu_0^* + (1 - \beta)(-\nu_0^{*'})] \\ &\quad + \frac{(\nu_0^* + \beta\nu_0^{*'}) [(1 - \beta)\nu_0^* + a_0] + \beta\nu_0^* [\nu_0^* + (1 - \beta)(-\nu_0^{*'})]}{[(1 - \beta)\nu_0^* + a_0]^2} \left(\frac{p^* - b_1}{\nu_1^* + a_1} \right) \\ &\quad + \frac{\beta\nu_0^*}{(1 - \beta)\nu_0^* + a_0} \left[\frac{p^* - b_1}{(\nu_1^* + a_1)^2} (-\nu_1^{*'}) \right] + \left[\frac{p^* - b_1}{(\nu_1^* + a_1)^2} (-\nu_1^{*'}) \right]. \end{aligned} \quad (\text{A.91})$$

Given the values of $a_0, a_1, b_0, b_1, \beta, \nu_0^*, \nu_1^*, \nu_0^{*'}, \nu_1^{*'}, p^*$ and equation (A.67), it isn't difficult to see that (A.91) is nonnegative. Moreover,

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial \beta} &= \frac{p^* - b_0}{[(1 - \beta)\nu_0^* + a_0]^2} [\nu_0^* + (1 - \beta)(-\nu_0^{*'})] \\ &\quad + \frac{(\nu_0^* + \beta\nu_0^{*'}) [(1 - \beta)\nu_0^* + a_0] + \beta\nu_0^* [\nu_0^* + (1 - \beta)(-\nu_0^{*'})]}{[(1 - \beta)\nu_0^* + a_0]^2} \left(\frac{p^* - b_1}{\nu_1^* + a_1} \right) \\ &\quad + \frac{\beta\nu_0^*}{(1 - \beta)\nu_0^* + a_0} \left[\frac{p^* - b_1}{(\nu_1^* + a_1)^2} (-\nu_1^{*'}) \right] + \left[\frac{p^* - b_1}{(\nu_1^* + a_1)^2} (-\nu_1^{*'}) \right] \\ &\geq \frac{p^* - b_1}{(\nu_1^* + a_1)^2} (-\nu_1^{*'}) > 0, \end{aligned} \quad (\text{A.92})$$

which proves (A.90). On account of that,

$$\frac{dp^*}{d\beta} = -\frac{\frac{\partial \mathcal{F}}{\partial \beta}}{\frac{\partial \mathcal{F}}{\partial p^*}} < 0, \quad (\text{A.93})$$

where $\frac{\partial \mathcal{F}}{\partial \beta}$ and $\frac{\partial \mathcal{F}}{\partial p^*}$ are continuous with respect to β . Hence $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β , $\beta \in (0, 1]$.

Now, since

$$G^*(\beta) = G(p^*(\beta)) = -Kp^*(\beta) + T, \quad (\text{A.94})$$

and $p^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β , and K and T are positive constants, then, $G^*(\beta)$ is continuously differentiable and strictly increasing with respect to β , $\beta \in (0, 1]$.

Now, we are going to show that $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β . To do that, we first solve equation (A.83) for p^* to obtain the following equality:

$$\begin{aligned} p^* &= \frac{\frac{b_0}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0} \left(\frac{b_1}{\nu_1^* + a_1} \right) + T + D}{\frac{1}{(1-\beta)\nu_0^* + a_0} + \frac{\nu_0^* + a_0}{(1-\beta)\nu_0^* + a_0} \left(\frac{1}{\nu_1^* + a_1} \right) + K} \\ &= \frac{b_0 + (\nu_0^* + a_0) \left(\frac{b_1}{\nu_1^* + a_1} \right) + [(1-\beta)\nu_0^* + a_0] (T + D)}{1 + (\nu_0^* + a_0) \left(\frac{1}{\nu_1^* + a_1} \right) + [(1-\beta)\nu_0^* + a_0] K} \\ &= \frac{(\nu_0^* + a_0)b_1 + (\nu_1^* + a_1)b_0 + [(1-\beta)\nu_0^* + a_0](\nu_1^* + a_1)(T + D)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1-\beta)\nu_0^* + a_0](\nu_1^* + a_1)K}. \end{aligned} \quad (\text{A.95})$$

We substitute (A.95) in $q_1^* = q_1(p^*, \nu_0^*, \nu_1^*)$, to deduce

$$\begin{aligned}
q_1^* &= \frac{p^* - b_1}{\nu_1^* + a_1} \\
&= \frac{(\nu_0^* + a_0) b_1 + (\nu_1^* + a_1) b_0 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K} - b_1 \\
&= \frac{\nu_1^* + a_1}{(\nu_0^* + a_0) b_1 + (\nu_1^* + a_1) b_0 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D)} \\
&= \frac{(\nu_0^* + a_0) b_1 + (\nu_1^* + a_1) b_0 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D)}{(\nu_1^* + a_1) \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\}} \\
&\quad - \frac{\{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\} b_1}{(\nu_1^* + a_1) \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\}} \\
&= \frac{(\nu_1^* + a_1) b_0 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D)}{(\nu_1^* + a_1) \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\}} \\
&\quad - \frac{(\nu_1^* + a_1) b_1 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K b_1}{(\nu_1^* + a_1) \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\}} \\
&= \frac{b_0 + [(1 - \beta)\nu_0^* + a_0] (T + D)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K} \\
&\quad - \frac{b_1 + [(1 - \beta)\nu_0^* + a_0] K b_1}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K} \\
&= \frac{-b_1 + b_0 + [(1 - \beta)\nu_0^* + a_0] (-K b_1 + T + D)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K} \\
&= \frac{-(b_1 - b_0) + [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) \{1 + [(1 - \beta)\nu_0^* + a_0] K\}}.
\end{aligned} \tag{A.96}$$

By plugging (A.41) in (A.96) we have that

$$\begin{aligned}
q_1^* &= \frac{-(b_1 - b_0) + [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D)}{(\nu_0^* + a_0) + \left[\frac{(1 - \beta)\nu_0^* + a_0}{1 + [(1 - \beta)\nu_0^* + a_0] K} + a_1 \right] \{1 + [(1 - \beta)\nu_0^* + a_0] K\}} \\
&= \frac{-(b_1 - b_0) + [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D)}{(\nu_0^* + a_0) + [(1 - \beta)\nu_0^* + a_0] + a_1 \{1 + [(1 - \beta)\nu_0^* + a_0] K\}} \\
&= \frac{-(b_1 - b_0) + [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D)}{(\nu_0^* + a_0 + a_1) + [(1 - \beta)\nu_0^* + a_0] (1 + a_1 K)} = \frac{M}{N},
\end{aligned} \tag{A.97}$$

where

$$M = M(\beta) = -(b_1 - b_0) + [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D) \tag{A.98}$$

and

$$N = N(\beta) = (\nu_0^* + a_0 + a_1) + [(1 - \beta)\nu_0^* + a_0] (1 + a_1 K). \tag{A.99}$$

It is easy to see that M and N are continuously differentiable with respect to β with

$$M' = [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D), \quad (\text{A.100})$$

$$N' = \nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K). \quad (\text{A.101})$$

Moreover, $N > 0$, so q_1^* is continuously differentiable with respect to β and

$$q_1^{*'} = \frac{M'N - MN'}{N^2}. \quad (\text{A.102})$$

Thus, to find the value of $q_1^{*'}$ it suffices to estimate the value of the numerator of (A.102).

$$\begin{aligned} M'N - MN' &= \\ &= [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D) \{(\nu_0^* + a_0 + a_1) + [(1 - \beta)\nu_0^* + a_0] (1 + a_1 K)\} \\ &\quad - \{-(b_1 - b_0) + [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D)\} \{\nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K)\} \\ &= (\nu_0^* + a_0 + a_1) [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D) \\ &\quad + [(1 - \beta)\nu_0^* + a_0] [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K) (G(b_1) + D) \\ &\quad + \{\nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K)\} (b_1 - b_0) \\ &\quad - \{\nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K)\} [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D) \\ &= (\nu_0^* + a_0) [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D) \\ &\quad + a_1 [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D) \\ &\quad + [(1 - \beta)\nu_0^* + a_0] [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K) (G(b_1) + D) \\ &\quad + \{\nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K)\} (b_1 - b_0) \\ &\quad - \nu_0^{*'} [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D) \\ &\quad - [(1 - \beta)\nu_0^* + a_0] [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K) (G(b_1) + D) \\ &= (\nu_0^* + a_0) [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D) \\ &\quad + a_1 [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D) \\ &\quad + \{\nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K)\} (b_1 - b_0) \\ &\quad - \nu_0^{*'} [(1 - \beta)\nu_0^* + a_0] (G(b_1) + D) \\ &= \{(\nu_0^* + a_0) [-\nu_0^* + (1 - \beta)\nu_0^{*'}] - \nu_0^{*'} [(1 - \beta)\nu_0^* + a_0]\} (G(b_1) + D) \\ &\quad + a_1 [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D) \\ &\quad + \{\nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K)\} (b_1 - b_0) \\ &= [(\nu_0^* + a_0) (-\nu_0^* - \beta\nu_0^{*'}) + \beta\nu_0^*\nu_0^{*'}] (G(b_1) + D) \\ &\quad + a_1 [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (G(b_1) + D) \\ &\quad + \{\nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}] (1 + a_1 K)\} (b_1 - b_0). \end{aligned} \quad (\text{A.103})$$

Given the values of $a_0, a_1, b_0, b_1, \beta, \nu_0^*, \nu_0^{*'}, G(p), D$ and equation (A.67), it is clear that (A.103) is non-positive. Moreover,

$$\begin{aligned} M'N - MN' &= [(\nu_0^* + a_0)(-\nu_0^* - \beta\nu_0^{*'}) + \beta\nu_0^*\nu_0^{*'}](G(b_1) + D) \\ &\quad + a_1[-\nu_0^* + (1 - \beta)\nu_0^{*'}](G(b_1) + D) \\ &\quad + \{\nu_0^{*'} + [-\nu_0^* + (1 - \beta)\nu_0^{*'}](1 + a_1K)\}(b_1 - b_0) \\ &\leq a_1[-\nu_0^* + (1 - \beta)\nu_0^{*'}](G(b_1) + D) < 0. \end{aligned} \tag{A.104}$$

Thus,

$$M'N - MN' < 0, \tag{A.105}$$

which proves that $q_1^{*'} < 0$, so $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β , $\beta \in (0, 1]$.

Finally, since

$$q_0^*(\beta) + q_1^*(\beta) = G^*(\beta) + D, \tag{A.106}$$

then,

$$q_0^*(\beta) = -q_1^*(\beta) + G^*(\beta) + D. \tag{A.107}$$

And since $G^*(\beta)$ is continuously differentiable and strictly increasing with respect to β , the function $q_1^*(\beta)$ is continuously differentiable and strictly decreasing with respect to β . Because D is constant, we have that $q_0^*(\beta)$ is continuously differentiable and strictly increasing with respect to β , $\beta \in (0, 1]$ The proof of the theorem is complete ■

Theorem 1.4. *For the affine demand function $G(p)$ described in (1.22), the price $p^c(\beta)$ and the supply values $q_i^c(\beta)$, $i = 0, 1$, from the Cournot-Nash equilibrium, are continuously differentiable with respect to $\beta \in (0, 1]$. Moreover, $p^c(\beta)$ and $q_1^c(\beta)$ strictly decrease, whereas $q_0^c(\beta)$ strictly increase.*

Proof. Let's consider the exterior equilibrium (p^c, q_0^c, q_1^c) , i.e., such a vector that the following equalities hold:

$$q_0^c + q_1^c = G(p^c) + D, \tag{A.108}$$

$$q_0^c = \frac{p^c - b_0}{(1 - \beta)\frac{1}{K} + a_0} + \frac{\beta\frac{1}{K}}{(1 - \beta)\frac{1}{K} + a_0} \left(\frac{p^c - b_1}{\frac{1}{K} + a_1} \right), \tag{A.109}$$

$$q_1^c = \frac{p^c - b_1}{\frac{1}{K} + a_1}, \tag{A.110}$$

where

$$G(p^c) = -Kp^c + T. \quad (\text{A.111})$$

From equation (A.108) one has

$$q_0^c + q_1^c - G(p^c) - D = 0. \quad (\text{A.112})$$

By substituting (A.109), (A.110) and (A.111) in (A.112), similarly to (A.3), we have

$$\begin{aligned} q_0^c + q_1^c - G(p^c) - D &= \frac{p^c - b_0}{(1 - \beta)\frac{1}{K} + a_0} + \frac{\beta\frac{1}{K}}{(1 - \beta)\frac{1}{K} + a_0} \left(\frac{p^c - b_1}{\frac{1}{K} + a_1} \right) \\ &\quad + \frac{p^c - b_1}{\frac{1}{K} + a_1} + Kp^c - T - D \\ &= p^c \left[\frac{1}{(1 - \beta)\frac{1}{K} + a_0} + \frac{\frac{1}{K} + a_0}{(1 - \beta)\frac{1}{K} + a_0} \left(\frac{1}{\frac{1}{K} + a_1} \right) \right] \\ &\quad - \left[\frac{b_0}{(1 - \beta)\frac{1}{K} + a_0} + \frac{\frac{1}{K} + a_0}{(1 - \beta)\frac{1}{K} + a_0} \left(\frac{b_1}{\frac{1}{K} + a_1} \right) \right] \\ &\quad + Kp^c - T - D = 0. \end{aligned} \quad (\text{A.113})$$

Solving (A.113) for p^c , similarly to (A.95), we get the equation

$$\begin{aligned} p^c &= \frac{\frac{b_0}{(1 - \beta)\frac{1}{K} + a_0} + \frac{\frac{1}{K} + a_0}{(1 - \beta)\frac{1}{K} + a_0} \left(\frac{b_1}{\frac{1}{K} + a_1} \right) + T + D}{\frac{1}{(1 - \beta)\frac{1}{K} + a_0} + \frac{\frac{1}{K} + a_0}{(1 - \beta)\frac{1}{K} + a_0} \left(\frac{1}{\frac{1}{K} + a_1} \right) + K} \\ &= \frac{\left(\frac{1}{K} + a_0 \right) b_1 + \left(\frac{1}{K} + a_1 \right) b_0 + \left[(1 - \beta)\frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) (T + D)}{\left(\frac{1}{K} + a_0 \right) + \left(\frac{1}{K} + a_1 \right) + \left[(1 - \beta)\frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) K} \\ &= \frac{X}{Y}, \end{aligned} \quad (\text{A.114})$$

where

$$X(\beta) = \left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) (T + D) \quad (\text{A.115})$$

and

$$Y(\beta) = \left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K. \quad (\text{A.116})$$

It's easy to see that X and Y are continuously differentiable with respect to β with

$$X' = -\frac{1}{K} \left(\frac{1}{K} + a_1\right) (T + D), \quad (\text{A.117})$$

$$Y' = -\left(\frac{1}{K} + a_1\right). \quad (\text{A.118})$$

Moreover, $Y > 0$, whence p^c is continuously differentiable with respect to β with

$$p^{c'} = \frac{X'Y - XY'}{Y^2}. \quad (\text{A.119})$$

To compute the value of $p^{c'}$ it is sufficient to calculate the value of the numerator of (A.119):

$$\begin{aligned}
X'Y - XY' &= \\
&= -\frac{1}{K} \left(\frac{1}{K} + a_1 \right) (T + D) \left\{ \left(\frac{1}{K} + a_0 \right) + \left(\frac{1}{K} + a_1 \right) + \left[(1 - \beta) \frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) K \right\} \\
&\quad - \left\{ \left(\frac{1}{K} + a_0 \right) b_1 + \left(\frac{1}{K} + a_1 \right) b_0 + \left[(1 - \beta) \frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) (T + D) \right\} \left[- \left(\frac{1}{K} + a_1 \right) \right] \\
&= -\frac{1}{K} \left(\frac{1}{K} + a_1 \right) (T + D) \left\{ \left(\frac{1}{K} + a_0 \right) + \left(\frac{1}{K} + a_1 \right) + \left[(1 - \beta) \frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) K \right\} \\
&\quad + \left(\frac{1}{K} + a_1 \right) \left\{ \left(\frac{1}{K} + a_0 \right) b_1 + \left(\frac{1}{K} + a_1 \right) b_0 + \left[(1 - \beta) \frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) (T + D) \right\} \\
&= -\frac{1}{K} \left(\frac{1}{K} + a_0 \right) \left(\frac{1}{K} + a_1 \right) (T + D) \\
&\quad - \frac{1}{K} \left(\frac{1}{K} + a_1 \right)^2 (T + D) \\
&\quad - \left[(1 - \beta) \frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right)^2 (T + D) \\
&\quad + \left(\frac{1}{K} + a_0 \right) \left(\frac{1}{K} + a_1 \right) b_1 \\
&\quad + \left(\frac{1}{K} + a_1 \right)^2 b_0 \\
&\quad + \left[(1 - \beta) \frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right)^2 (T + D) \\
&= \left(\frac{1}{K} + a_0 \right) \left(\frac{1}{K} + a_1 \right) b_1 - \frac{1}{K} \left(\frac{1}{K} + a_0 \right) \left(\frac{1}{K} + a_1 \right) (T + D) \\
&\quad + \left(\frac{1}{K} + a_1 \right)^2 b_0 - \frac{1}{K} \left(\frac{1}{K} + a_1 \right)^2 (T + D) \\
&= -\frac{1}{K} \left(\frac{1}{K} + a_0 \right) \left(\frac{1}{K} + a_1 \right) (-Kb_1 + T + D) \\
&\quad - \frac{1}{K} \left(\frac{1}{K} + a_1 \right)^2 (-Kb_0 + T + D) \\
&= -\frac{1}{K} \left(\frac{1}{K} + a_0 \right) \left(\frac{1}{K} + a_1 \right) (G(b_1) + D) \\
&\quad - \frac{1}{K} \left(\frac{1}{K} + a_1 \right)^2 (G(b_0) + D).
\end{aligned} \tag{A.120}$$

Given the values of a_0 , a_1 , K , $G(p)$ and D , it is clear that (A.120) is non-positive. Moreover,

$$\begin{aligned}
X'Y - XY' &= -\frac{1}{K} \left(\frac{1}{K} + a_0 \right) \left(\frac{1}{K} + a_1 \right) (G(b_1) + D) \\
&\quad - \frac{1}{K} \left(\frac{1}{K} + a_1 \right)^2 (G(b_0) + D) \\
&\leq -\frac{1}{K} \left(\frac{1}{K} + a_0 \right) \left(\frac{1}{K} + a_1 \right) (G(b_1) + D) < 0.
\end{aligned} \tag{A.121}$$

Then,

$$X'Y - XY' < 0, \tag{A.122}$$

which proves that $p^{c'} < 0$, so $p^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β , $\beta \in (0, 1]$.

Since

$$q_1^c(\beta) = \frac{p^c(\beta) - b_1}{\frac{1}{K} + a_1}, \quad (\text{A.123})$$

and $p^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β , and a_1 , b_1 and K are positive constants, then, $q_1^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β , $\beta \in (0, 1]$.

Finally, since

$$q_0^c(\beta) + q_1^c(\beta) = G(p^c(\beta)) + D = -Kp^c(\beta) + T + D, \quad (\text{A.124})$$

then,

$$q_0^c(\beta) = -q_1^c(\beta) - Kp^c(\beta) + T + D. \quad (\text{A.125})$$

And as $q_1^c(\beta)$ is continuously differentiable and strictly decreasing with respect to β , the function $p^c(\beta)$ also has the same property, and K , T and D are non-negative constants, then, $q_0^c(\beta)$ is continuously differentiable and strictly increasing with respect to β , $\beta \in (0, 1]$. The proof of the theorem is complete ■

Theorem 1.5. *For the affine demand function $G(p)$ described in (1.22), the price $p^t(\beta)$ and the output volumes $q_i^t(\beta)$, $i = 0, 1$, related to the perfect competition equilibrium, are invariant for all $\beta \in (0, 1]$ and are described by the clear-cut expressions:*

$$p^t = \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)}{a_0 + a_1 + a_0 a_1 K}, \quad (\text{1.27})$$

$$q_0^t = \frac{a_1 (G(b_0) + D) + (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}, \quad (\text{1.28})$$

$$q_1^t = \frac{a_0 (G(b_1) + D) - (b_1 - b_0)}{a_0 + a_1 + a_0 a_1 K}. \quad (\text{1.29})$$

Proof. Let us consider the exterior equilibrium (p^t, q_0^t, q_1^t) , i.e., such a vector that the following equalities hold:

$$q_0^t + q_1^t = G(p^t) + D, \quad (\text{A.126})$$

$$q_0^t = \frac{p^t - b_0}{a_0}, \quad (\text{A.127})$$

$$q_1^t = \frac{p^t - b_1}{a_1}, \quad (\text{A.128})$$

where

$$G(p^t) = -Kp^t + T. \quad (\text{A.129})$$

From (A.108) one gets that

$$q_0^t + q_1^t - G(p^t) - D = 0. \quad (\text{A.130})$$

Next, by plugging (A.127), (A.128) and (A.129) in (A.130), we deduce that

$$\begin{aligned} q_0^t + q_1^t - G(p^t) - D &= \frac{p^t - b_0}{a_0} + \frac{p^t - b_1}{a_1} + Kp^t - T - D \\ &= p^t \left(\frac{1}{a_0} + \frac{1}{a_1} \right) - \left(\frac{b_1}{a_1} + \frac{b_0}{a_0} \right) + Kp^t - T - D = 0. \end{aligned} \quad (\text{A.131})$$

By solving equation (A.131) for p^t , we obtain the equality

$$p^t = \frac{\frac{b_1}{a_1} + \frac{b_0}{a_0} + T + D}{\frac{1}{a_0} + \frac{1}{a_1} + K} = \frac{a_0b_1 + a_1b_0 + a_0a_1(T + D)}{a_0 + a_1 + a_0a_1K}, \quad (\text{A.132})$$

showing that the function $p^t(\beta)$ is constant for all $\beta \in (0, 1]$.

Moreover, since

$$\begin{aligned} q_0^t &= \frac{p^t - b_0}{a_0} = \frac{\frac{a_0b_1 + a_1b_0 + a_0a_1(T + D)}{a_0 + a_1 + a_0a_1K} - b_0}{a_0} \\ &= \frac{a_0b_1 + a_1b_0 + a_0a_1(T + D) - (a_0 + a_1 + a_0a_1K)b_0}{a_0(a_0 + a_1 + a_0a_1K)} \\ &= \frac{a_0(b_1 - b_0) + a_0a_1(-Kb_0 + T + D)}{a_0(a_0 + a_1 + a_0a_1K)} \\ &= \frac{a_1(G(b_0) + D) + (b_1 - b_0)}{a_0 + a_1 + a_0a_1K}, \end{aligned} \quad (\text{A.133})$$

and

$$\begin{aligned} q_1^t &= \frac{p^t - b_1}{a_1} = \frac{\frac{a_0b_1 + a_1b_0 + a_0a_1(T + D)}{a_0 + a_1 + a_0a_1K} - b_1}{a_1} \\ &= \frac{a_0b_1 + a_1b_0 + a_0a_1(T + D) - (a_0 + a_1 + a_0a_1K)b_1}{a_1(a_0 + a_1 + a_0a_1K)} \\ &= \frac{-a_1(b_1 - b_0) + a_0a_1(-Kb_1 + T + D)}{a_1(a_0 + a_1 + a_0a_1K)} \\ &= \frac{a_0(G(b_1) + D) - (b_1 - b_0)}{a_0 + a_1 + a_0a_1K}, \end{aligned} \quad (\text{A.134})$$

the functions $q_0^t(\beta)$ and $q_1^t(\beta)$ are constant for all $\beta \in (0, 1]$, too. The proof of the theorem is complete ■

Theorem 1.6. *For the affine demand function $G(p)$ from (1.22), the price functions in the CCVE, $p^*(\beta)$, the Cournot-Nash equilibrium, $p^c(\beta)$, and the perfect competition equilibrium, p^t , satisfy the following inequalities:*

$$p^t < \lim_{\beta \rightarrow 0} p^*(\beta), \quad (1.30)$$

and

$$p^*(\beta) < p^c(\beta), \quad \forall \beta \in (0, 1]. \quad (1.31)$$

Proof. First, we prove inequality (1.30):

$$p^t < \lim_{\beta \rightarrow 0} p^*(\beta).$$

Introduce the following notation:

$$\begin{aligned} \widehat{\nu}_0^* &= \lim_{\beta \rightarrow 0} \nu_0^*(\beta) \\ &= \frac{2(a_0 + a_1 + a_0 a_1 K)}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} > 0, \end{aligned} \quad (A.135)$$

$$\widehat{\nu}_1^* = \lim_{\beta \rightarrow 0} \nu_1^*(\beta) = \lim_{\beta \rightarrow 0} \frac{(1 - \beta)\nu_0^* + a_0}{1 + [(1 - \beta)\nu_0^* + a_0] K} = \frac{\widehat{\nu}_0^* + a_0}{1 + (\widehat{\nu}_0^* + a_0) K} > 0. \quad (A.136)$$

Therefore,

$$\begin{aligned} \lim_{\beta \rightarrow 0} p^*(\beta) &= \lim_{\beta \rightarrow 0} \frac{(\nu_0^* + a_0) b_1 + (\nu_1^* + a_1) b_0 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K} \\ &= \frac{(\widehat{\nu}_0^* + a_0) b_1 + (\widehat{\nu}_1^* + a_1) b_0 + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) (T + D)}{(\widehat{\nu}_0^* + a_0) + (\widehat{\nu}_1^* + a_1) + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) K}. \end{aligned} \quad (A.137)$$

Now, we compute the difference

$$\begin{aligned}
& \lim_{\beta \rightarrow 0} p^*(\beta) - p^t = \\
&= \frac{(\widehat{\nu}_0^* + a_0) b_1 + (\widehat{\nu}_1^* + a_1) b_0 + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) (T + D)}{(\widehat{\nu}_0^* + a_0) + (\widehat{\nu}_1^* + a_1) + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) K} \\
&\quad - \frac{a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)}{a_0 + a_1 + a_0 a_1 K} \\
&= \frac{\left[(\widehat{\nu}_0^* + a_0) b_1 + (\widehat{\nu}_1^* + a_1) b_0 + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) (T + D) \right] (a_0 + a_1 + a_0 a_1 K)}{\left[(\widehat{\nu}_0^* + a_0) + (\widehat{\nu}_1^* + a_1) + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) K \right] (a_0 + a_1 + a_0 a_1 K)} \quad (\text{A.138}) \\
&\quad - \frac{\left[(\widehat{\nu}_0^* + a_0) + (\widehat{\nu}_1^* + a_1) + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) K \right] [a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)]}{\left[(\widehat{\nu}_0^* + a_0) + (\widehat{\nu}_1^* + a_1) + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) K \right] (a_0 + a_1 + a_0 a_1 K)} \\
&= \frac{R1}{R2},
\end{aligned}$$

where

$$\begin{aligned}
R1 &= \left[(\widehat{\nu}_0^* + a_0) b_1 + (\widehat{\nu}_1^* + a_1) b_0 + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) (T + D) \right] (a_0 + a_1 + a_0 a_1 K) \\
&\quad - \left[(\widehat{\nu}_0^* + a_0) + (\widehat{\nu}_1^* + a_1) + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) K \right] [a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)] \quad (\text{A.139})
\end{aligned}$$

and

$$R2 = \left[(\widehat{\nu}_0^* + a_0) + (\widehat{\nu}_1^* + a_1) + (\widehat{\nu}_0^* + a_0) (\widehat{\nu}_1^* + a_1) K \right] (a_0 + a_1 + a_0 a_1 K). \quad (\text{A.140})$$

Given the values of $a_0, a_1, \widehat{\nu}_0^*, \widehat{\nu}_1^*$, and K , it is easy to see that $R2 > 0$. Hence, to calculate the value of (A.138), it is enough to estimate the value of (A.139). That is,

$$\begin{aligned}
R1 &= \left[\left(\widehat{\nu}_0^* + a_0 \right) b_1 + \left(\widehat{\nu}_1^* + a_1 \right) b_0 + \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) (T + D) \right] (a_0 + a_1 + a_0 a_1 K) \\
&\quad - \left[\left(\widehat{\nu}_0^* + a_0 \right) + \left(\widehat{\nu}_1^* + a_1 \right) + \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) K \right] [a_0 b_1 + a_1 b_0 + a_0 a_1 (T + D)] \\
&= (a_0 + a_1 + a_0 a_1 K) \left(\widehat{\nu}_0^* + a_0 \right) b_1 \\
&\quad + (a_0 + a_1 + a_0 a_1 K) \left(\widehat{\nu}_1^* + a_1 \right) b_0 \\
&\quad + (a_0 + a_1 + a_0 a_1 K) \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) (T + D) \\
&\quad - a_0 \left[\left(\widehat{\nu}_0^* + a_0 \right) + \left(\widehat{\nu}_1^* + a_1 \right) + \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) K \right] b_1 \\
&\quad - a_1 \left[\left(\widehat{\nu}_0^* + a_0 \right) + \left(\widehat{\nu}_1^* + a_1 \right) + \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) K \right] b_0 \\
&\quad - a_0 a_1 \left[\left(\widehat{\nu}_0^* + a_0 \right) + \left(\widehat{\nu}_1^* + a_1 \right) + \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) K \right] (T + D) \\
&= \left\{ (a_1 + a_0 a_1 K) \left(\widehat{\nu}_0^* + a_0 \right) - a_0 \left[\left(\widehat{\nu}_1^* + a_1 \right) + \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) K \right] \right\} b_1 \\
&\quad + \left\{ (a_0 + a_0 a_1 K) \left(\widehat{\nu}_1^* + a_1 \right) - a_1 \left[\left(\widehat{\nu}_0^* + a_0 \right) + \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) K \right] \right\} b_0 \\
&\quad + \left\{ (a_0 + a_1) \left(\widehat{\nu}_0^* + a_0 \right) \left(\widehat{\nu}_1^* + a_1 \right) - a_0 a_1 \left[\left(\widehat{\nu}_0^* + a_0 \right) + \left(\widehat{\nu}_1^* + a_1 \right) \right] \right\} (T + D) \\
&= \left[a_1 \widehat{\nu}_0^* - a_0 \widehat{\nu}_1^* - a_0 \widehat{\nu}_1^* \left(\widehat{\nu}_0^* + a_0 \right) K \right] b_1 \\
&\quad + \left[a_0 \widehat{\nu}_1^* - a_1 \widehat{\nu}_0^* - a_1 \widehat{\nu}_0^* \left(\widehat{\nu}_1^* + a_1 \right) K \right] b_0 \\
&\quad + \left[a_0 \widehat{\nu}_1^* \left(\widehat{\nu}_0^* + a_0 \right) + a_1 \widehat{\nu}_0^* \left(\widehat{\nu}_1^* + a_1 \right) \right] (T + D) \\
&= -a_0 \widehat{\nu}_1^* \left(\widehat{\nu}_0^* + a_0 \right) K b_1 + a_0 \widehat{\nu}_1^* \left(\widehat{\nu}_0^* + a_0 \right) (T + D) \\
&\quad - a_1 \widehat{\nu}_0^* \left(\widehat{\nu}_1^* + a_1 \right) K b_0 + a_1 \widehat{\nu}_0^* \left(\widehat{\nu}_1^* + a_1 \right) (T + D) \\
&\quad + \left(a_1 \widehat{\nu}_0^* - a_0 \widehat{\nu}_1^* \right) b_1 + \left(a_0 \widehat{\nu}_1^* - a_1 \widehat{\nu}_0^* \right) b_0 \\
&= a_0 \widehat{\nu}_1^* \left(\widehat{\nu}_0^* + a_0 \right) (-K b_1 + T + D) \\
&\quad + a_1 \widehat{\nu}_0^* \left(\widehat{\nu}_1^* + a_1 \right) (-K b_0 + T + D) \\
&\quad + \left(a_1 \widehat{\nu}_0^* - a_0 \widehat{\nu}_1^* \right) (b_1 - b_0) \\
&= a_0 \widehat{\nu}_1^* \left(\widehat{\nu}_0^* + a_0 \right) (G(b_1) + D) - a_0 \widehat{\nu}_1^* (b_1 - b_0) \\
&\quad + a_1 \widehat{\nu}_0^* \left(\widehat{\nu}_1^* + a_1 \right) (G(b_0) + D) + a_1 \widehat{\nu}_0^* (b_1 - b_0) \\
&= a_0 \widehat{\nu}_1^* \left[\left(\widehat{\nu}_0^* + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right] \\
&\quad + a_1 \widehat{\nu}_0^* \left[\left(\widehat{\nu}_1^* + a_1 \right) (G(b_0) + D) + (b_1 - b_0) \right].
\end{aligned} \tag{A.141}$$

Given the values of $a_0, a_1, b_0, b_1, \widehat{\nu}_0^*, \widehat{\nu}_1^*, G(p), D$ and assumption **A1.3**, it is trivial that (A.141) is nonnegative. Moreover,

$$\begin{aligned}
 R1 &= a_0 \widehat{\nu}_1^* \left[\left(\widehat{\nu}_0^* + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right] \\
 &\quad + a_1 \widehat{\nu}_0^* \left[\left(\widehat{\nu}_1^* + a_1 \right) (G(b_0) + D) + (b_1 - b_0) \right] \\
 &\geq a_1 \widehat{\nu}_0^* \left[\left(\widehat{\nu}_1^* + a_1 \right) (G(b_0) + D) + (b_1 - b_0) \right] \geq a_1 \widehat{\nu}_0^* \left(\widehat{\nu}_1^* + a_1 \right) (G(b_0) + D) \\
 &\geq a_1^2 \widehat{\nu}_0^* (G(b_0) + D) \geq a_1^2 \widehat{\nu}_0^* G(b_0) > 0.
 \end{aligned} \tag{A.142}$$

And since $R1 > 0$, by (A.142), then,

$$\lim_{\beta \rightarrow 0} p^*(\beta) - p^t > 0, \tag{A.143}$$

which proves inequality (1.30).

Now, we establish inequality (1.31):

$$p^*(\beta) < p^c(\beta) \text{ para todo } \beta \in (0, 1].$$

In order to do that, we introduce the following notation:

$$\nu_i^* = \nu_i^*(\beta), i = 0, 1.$$

From equations (A.41) and (A.42) it's easy to see that the following inequality hold for all $\beta \in (0, 1]$:

$$\nu_i^* < \frac{1}{K}, i = 0, 1. \tag{A.144}$$

Now, we compute the difference

$$\begin{aligned}
& (p^c - p^*) (\beta) = \\
&= \frac{\left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) (T + D)}{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K} \\
&\quad - \frac{(\nu_0^* + a_0) b_1 + (\nu_1^* + a_1) b_0 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K} \\
&= \left\langle \left\{ \left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) (T + D) \right\} \times \right. \\
&\quad \left. \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\} \right. \\
&\quad \left. - \left\{ \left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K \right\} \times \right. \\
&\quad \left. \{(\nu_0^* + a_0) b_1 + (\nu_1^* + a_1) b_0 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D)\} \right\rangle / \\
&\quad \left\langle \left\{ \left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K \right\} \times \right. \\
&\quad \left. \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\} \right\rangle \\
&= \frac{S1}{S2},
\end{aligned} \tag{A.145}$$

where

$$\begin{aligned}
S1 &= \left\{ \left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) (T + D) \right\} \times \\
&\quad \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\} \\
&\quad - \left\{ \left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K \right\} \times \\
&\quad \{(\nu_0^* + a_0) b_1 + (\nu_1^* + a_1) b_0 + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D)\}
\end{aligned} \tag{A.146}$$

and

$$\begin{aligned}
S2 &= \left\{ \left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K \right\} \times \\
&\quad \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) K\}.
\end{aligned} \tag{A.147}$$

For any fixed values of $a_0, a_1, \beta, \nu_0^*, \nu_1^*$, and K , it is apparent that $S2 > 0$. Because of that, in order to find the value of (A.145), it suffices to calculate the value of (A.146). So,

$$\begin{aligned}
S1 &= \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0](\nu_1^* + a_1)K\} \left(\frac{1}{K} + a_0\right) b_1 \\
&\quad + \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0](\nu_1^* + a_1)K\} \left(\frac{1}{K} + a_1\right) b_0 \\
&\quad + \{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0](\nu_1^* + a_1)K\} \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) (T + D) \\
&\quad - \left\{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K\right\} (\nu_0^* + a_0) b_1 \\
&\quad - \left\{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K\right\} (\nu_1^* + a_1) b_0 \\
&\quad - \left\{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + \left[(1 - \beta)\frac{1}{K} + a_0\right] \left(\frac{1}{K} + a_1\right) K\right\} [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) (T + D) \\
&= \left[(1 - \beta)\frac{1}{K} + a_0\right] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1\right) (-Kb_1 + T + D) \\
&\quad - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_0\right) (-Kb_1 + T + D) \\
&\quad + \left[(1 - \beta)\frac{1}{K} + a_0\right] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) (-Kb_0 + T + D) \\
&\quad - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) (-Kb_0 + T + D) \\
&\quad + (\nu_1^* + a_1) \left(\frac{1}{K} + a_0\right) (b_1 - b_0) - (\nu_0^* + a_0) \left(\frac{1}{K} + a_1\right) (b_1 - b_0) \\
&= \{[(1 - \beta)\frac{1}{K} + a_0] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1\right) - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_0\right)\} (-Kb_1 + T + D) \\
&\quad + \{[(1 - \beta)\frac{1}{K} + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right)\} (-Kb_0 + T + D) \\
&\quad + [(\nu_1^* + a_1) \left(\frac{1}{K} + a_0\right) - (\nu_0^* + a_0) \left(\frac{1}{K} + a_1\right)] (b_1 - b_0) \\
&= \{[(1 - \beta)\frac{1}{K} + a_0] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1\right) - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_0\right)\} (G(b_1) + D) \\
&\quad + \{[(1 - \beta)\frac{1}{K} + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right)\} (G(b_0) + D) \\
&\quad + [(\nu_1^* + a_1) \left(\frac{1}{K} + a_0\right) - (\nu_0^* + a_0) \left(\frac{1}{K} + a_1\right)] (b_1 - b_0) \\
&= X_1(G(b_1) + D) + X_2(G(b_0) + D) + X_3(b_1 - b_0),
\end{aligned} \tag{A.148}$$

where

$$X_1 = \left[(1 - \beta)\frac{1}{K} + a_0\right] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1\right) - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_0\right), \tag{A.149}$$

$$X_2 = \left[(1 - \beta)\frac{1}{K} + a_0\right] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) \tag{A.150}$$

and

$$X_3 = (\nu_1^* + a_1) \left(\frac{1}{K} + a_0\right) - (\nu_0^* + a_0) \left(\frac{1}{K} + a_1\right). \tag{A.151}$$

Now, for any given values of $a_0, a_1, \beta, \nu_0^*, \nu_1^*, K$ and (A.144), one finds

$$\begin{aligned}
X_2 &= \left[(1 - \beta)\frac{1}{K} + a_0\right] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) - [(1 - \beta)\nu_0^* + a_0] (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) \\
&= (1 - \beta)pbra\frac{1}{K} - \nu_0^* (\nu_1^* + a_1) \left(\frac{1}{K} + a_1\right) \geq 0
\end{aligned} \tag{A.152}$$

for all $\beta \in (0, 1]$.

Now, we are going to show that $X_1 > 0$ for all $\beta \in (0, 1]$. Plugging (A.41) in X_1 , we obtain

$$\begin{aligned}
X_1 &= \left[(1 - \beta) \frac{1}{K} + a_0 \right] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) \\
&\quad - \left[(1 - \beta) \nu_0^* + a_0 \right] \left(\frac{(1 - \beta) \nu_0^* + a_0}{1 + [(1 - \beta) \nu_0^* + a_0] K} + a_1 \right) \left(\frac{1}{K} + a_0 \right) \\
&= \frac{\left[(1 - \beta) \frac{1}{K} + a_0 \right] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) \{1 + [(1 - \beta) \nu_0^* + a_0] K\}}{1 + [(1 - \beta) \nu_0^* + a_0] K} \\
&\quad - \frac{\left[(1 - \beta) \nu_0^* + a_0 \right] [(1 - \beta) \nu_0^* + a_0 + a_1 \{1 + [(1 - \beta) \nu_0^* + a_0] K\}] \left(\frac{1}{K} + a_0 \right)}{1 + [(1 - \beta) \nu_0^* + a_0] K} \\
&= \frac{T1}{T2},
\end{aligned} \tag{A.153}$$

where

$$\begin{aligned}
T1 &= \left[(1 - \beta) \frac{1}{K} + a_0 \right] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) \{1 + [(1 - \beta) \nu_0^* + a_0] K\} \\
&\quad - \left[(1 - \beta) \nu_0^* + a_0 \right] [(1 - \beta) \nu_0^* + a_0 + a_1 \{1 + [(1 - \beta) \nu_0^* + a_0] K\}] \left(\frac{1}{K} + a_0 \right)
\end{aligned} \tag{A.154}$$

and

$$T2 = 1 + [(1 - \beta) \nu_0^* + a_0] K. \tag{A.155}$$

For any fixed values of a_0 , β , ν_0^* , and K , it is clear that $T2 > 0$. Therefore, to compute the value of (A.153), we need to calculate the value of $T1$.

$$\begin{aligned}
T1 &= \left[(1 - \beta) \frac{1}{K} + a_0 \right] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) \{1 + [(1 - \beta) \nu_0^* + a_0] K\} \\
&\quad - \left[(1 - \beta) \nu_0^* + a_0 \right] [(1 - \beta) \nu_0^* + a_0 + a_1 \{1 + [(1 - \beta) \nu_0^* + a_0] K\}] \left(\frac{1}{K} + a_0 \right) \\
&= \left[(1 - \beta) \frac{1}{K} + a_0 \right] \left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) \\
&\quad + \left[(1 - \beta)^2 \frac{1}{K} \nu_0^* + (1 - \beta) a_0 \left(\frac{1}{K} + \nu_0^* \right) + a_0^2 \right] \left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) K \\
&\quad - a_1 [(1 - \beta) \nu_0^* + a_0] \left(\frac{1}{K} + a_0 \right) \\
&\quad - \left[(1 - \beta)^2 \nu_0^{*2} + 2(1 - \beta) a_0 \nu_0^* + a_0^2 \right] (1 + a_1 K) \left(\frac{1}{K} + a_0 \right) \\
&= (1 - \beta)^2 \nu_0^* \left[\left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) - \left(\frac{1}{K} + a_0 \right) (1 + a_1 K) \nu_0^* \right] \\
&\quad + (1 - \beta) \left[\left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) \frac{1}{K} + \left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) \left(\frac{1}{K} + \nu_0^* \right) a_0 K \right. \\
&\quad \quad \left. - \left(\frac{1}{K} + a_0 \right) a_1 \nu_0^* - 2 \left(\frac{1}{K} + a_0 \right) (1 + a_1 K) a_0 \nu_0^* \right] \\
&\quad + a_0 \left[\left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) a_0 K + \left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) \right. \\
&\quad \quad \left. - \left(\frac{1}{K} + a_0 \right) a_1 - \left(\frac{1}{K} + a_0 \right) (1 + a_1 K) a_0 \right] \\
&= (1 - \beta)^2 \left(\frac{1}{K} + a_1 \right) \left(\frac{1}{K} - \nu_0^* \right) a_0 K \nu_0^* \\
&\quad + (1 - \beta) \left[\left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) \frac{1}{K} + \left(\frac{1}{K} + a_1 \right) (\nu_0^* + a_0) \left(\frac{1}{K} + \nu_0^* \right) a_0 K \right. \\
&\quad \quad \left. - \left(\frac{1}{K} + a_0 \right) a_1 \nu_0^* - 2 \left(\frac{1}{K} + a_0 \right) (1 + a_1 K) a_0 \nu_0^* \right] \\
&\quad + \left(\frac{1}{K} + a_0 \right) a_0 \left[\left(\frac{1}{K} + a_1 \right) K \nu_0^* - a_1 \right] \\
&= (1 - \beta)^2 Y_1 + (1 - \beta) Y_2 + \left(\frac{1}{K} + a_0 \right) a_0 Y_3,
\end{aligned} \tag{A.156}$$

where

$$Y_1 = \left(\frac{1}{K} + a_1\right) \left(\frac{1}{K} - \nu_0^*\right) a_0 K \nu_0^*, \quad (\text{A.157})$$

$$\begin{aligned} Y_2 = & \left(\frac{1}{K} + a_1\right) (\nu_0^* + a_0) \frac{1}{K} + \left(\frac{1}{K} + a_1\right) (\nu_0^* + a_0) \left(\frac{1}{K} + \nu_0^*\right) a_0 K \\ & - \left(\frac{1}{K} + a_0\right) a_1 \nu_0^* - 2 \left(\frac{1}{K} + a_0\right) (1 + a_1 K) a_0 \nu_0^* \end{aligned} \quad (\text{A.158})$$

and

$$Y_3 = \left(\frac{1}{K} + a_1\right) K \nu_0^* - a_1. \quad (\text{A.159})$$

For any fixed values of a_0 , a_1 , ν_0^* , K , and (A.144), one concludes that $Y_1 > 0$, and $Y_3 = Y_3(\beta)$ strictly decreases by β , since $\nu_0^* = \nu_0^*(\beta)$ is strictly decreasing with respect to β , and $\left(\frac{1}{K} + a_1\right) K > 0$. Thus,

$$\begin{aligned} Y_3 = Y_3(\beta) & \geq Y_3(1) = \left(\frac{1}{K} + a_1\right) K \nu_0^*(1) - a_1 \\ & = \left(\frac{1}{K} + a_1\right) K \frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} - a_1 \\ & = \frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} (1 + a_1 K) - a_1 \\ & = \frac{(a_0 + a_1 + a_0 a_1 K) (1 + a_1 K) - (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) a_1}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\ & = \frac{[(1 + a_1 K) a_0 + a_1] (1 + a_1 K) - [(1 + a_1 K) + (2 + a_1 K) a_0 K] a_1}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\ & = \frac{(1 + a_1 K)^2 a_0 + (1 + a_1 K) a_1 - (1 + a_1 K) a_1 - (2 + a_1 K) a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\ & = \frac{[(1 + a_1 K)^2 - (2 + a_1 K) a_1 K] a_0}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\ & = \frac{a_0}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} > 0. \end{aligned} \quad (\text{A.160})$$

Then, $Y_3 > 0$ for all $\beta \in (0, 1]$.

Now, we are going to show that $Y_2 > 0$ for all $\beta \in (0, 1]$:

$$\begin{aligned}
Y_2 &= \left(\frac{1}{K} + a_1\right) (\nu_0^* + a_0) \frac{1}{K} + \left(\frac{1}{K} + a_1\right) (\nu_0^* + a_0) \left(\frac{1}{K} + \nu_0^*\right) a_0 K \\
&\quad - \left(\frac{1}{K} + a_0\right) a_1 \nu_0^* - 2 \left(\frac{1}{K} + a_0\right) (1 + a_1 K) a_0 \nu_0^* \\
&= \left(\frac{1}{K} + a_1\right) \nu_0^* \frac{1}{K} + \left(\frac{1}{K} + a_1\right) a_0 \frac{1}{K} + \left(\frac{1}{K} + a_1\right) a_0 K \left[\nu_0^{*2} + \left(\frac{1}{K} + a_0\right) \nu_0^* + a_0 \frac{1}{K} \right] \\
&\quad - \left(\frac{1}{K} + a_0\right) a_1 \nu_0^* - 2 \left(\frac{1}{K} + a_0\right) (1 + a_1 K) a_0 \nu_0^* \\
&= \left(\frac{1}{K} + a_1\right) \nu_0^* \frac{1}{K} + \left(\frac{1}{K} + a_1\right) a_0 \frac{1}{K} \\
&\quad + \left(\frac{1}{K} + a_1\right) a_0 K \nu_0^{*2} + \left(\frac{1}{K} + a_0\right) \left(\frac{1}{K} + a_1\right) a_0 K \nu_0^* + \left(\frac{1}{K} + a_1\right) a_0^2 \\
&\quad - \left(\frac{1}{K} + a_0\right) a_1 \nu_0^* - 2 \left(\frac{1}{K} + a_0\right) (1 + a_1 K) a_0 \nu_0^* \\
&= \left(\frac{1}{K} + a_1\right) a_0 K \nu_0^{*2} + \left(\frac{1}{K} + a_1\right) \nu_0^* \frac{1}{K} - \left(\frac{1}{K} + a_0\right) a_1 \nu_0^* \\
&\quad + \left(\frac{1}{K} + a_1\right) a_0 \frac{1}{K} + \left(\frac{1}{K} + a_1\right) a_0^2 \\
&\quad + \left(\frac{1}{K} + a_0\right) \left(\frac{1}{K} + a_1\right) a_0 K \nu_0^* - 2 \left(\frac{1}{K} + a_0\right) (1 + a_1 K) a_0 \nu_0^* \\
&= \left(\frac{1}{K} + a_1\right) a_0 K \nu_0^{*2} + \left[\left(\frac{1}{K} + a_1\right) \frac{1}{K} - \left(\frac{1}{K} + a_0\right) a_1\right] \nu_0^* \\
&\quad + \left(\frac{1}{K} + a_0\right) \left(\frac{1}{K} + a_1\right) a_0 - \left(\frac{1}{K} + a_0\right) \left(\frac{1}{K} + a_1\right) a_0 K \nu_0^* \\
&= \left(\frac{1}{K} + a_1\right) a_0 K \nu_0^{*2} + \left(\frac{1}{K^2} - a_0 a_1\right) \nu_0^* + \left(\frac{1}{K} + a_0\right) \left(\frac{1}{K} + a_1\right) a_0 K \left(\frac{1}{K} - \nu_0^*\right) \\
&= \left[\left(\frac{1}{K} + a_1\right) a_0 K \nu_0^* + \left(\frac{1}{K^2} - a_0 a_1\right)\right] \nu_0^* + \left(\frac{1}{K} + a_0\right) \left(\frac{1}{K} + a_1\right) \left(\frac{1}{K} - \nu_0^*\right) a_0 K \\
&= Z_1 \nu_0^* + Z_2,
\end{aligned} \tag{A.161}$$

where

$$Z_1 = \left(\frac{1}{K} + a_1\right) a_0 K \nu_0^* + \left(\frac{1}{K^2} - a_0 a_1\right) \tag{A.162}$$

and

$$Z_2 = \left(\frac{1}{K} + a_0\right) \left(\frac{1}{K} + a_1\right) \left(\frac{1}{K} - \nu_0^*\right) a_0 K. \tag{A.163}$$

Given the values of a_0 , a_1 , ν_0^* , K , and (A.144), one has that $Z_2 > 0$ for all $\beta \in (0, 1]$, and $Z_1 = Z_1(\beta)$ is strictly decreasing with respect to β , because $\nu_0^*(\beta)$ strictly decreases by β , and

$(a_1 + \frac{1}{K})a_0K > 0$. Thus,

$$\begin{aligned}
Z_1 &= Z_1(\beta) \geq Z_1(1) = \left(\frac{1}{K} + a_1\right) a_0K \nu_0^*(1) + \left(\frac{1}{K^2} - a_0a_1\right) \\
&= \left(\frac{1}{K} + a_1\right) a_0K \frac{a_0 + a_1 + a_0a_1K}{1 + 2a_0K + a_1K + a_0a_1K^2} + \left(\frac{1}{K^2} - a_0a_1\right) \\
&= \frac{(a_0 + a_1 + a_0a_1K) \left(\frac{1}{K} + a_1\right) a_0K + (1 + 2a_0K + a_1K + a_0a_1K^2) \left(\frac{1}{K^2} - a_0a_1\right)}{1 + 2a_0K + a_1K + a_0a_1K^2} \\
&= \frac{(a_0 + a_1 + a_0a_1K) (1 + a_1K) a_0 + (1 + 2a_0K + a_1K + a_0a_1K^2) \left(\frac{1}{K^2} - a_0a_1\right)}{1 + 2a_0K + a_1K + a_0a_1K^2} \\
&= \frac{(a_0 + a_1 + 2a_0a_1K + a_1^2K + a_0a_1^2K^2) a_0}{1 + 2a_0K + a_1K + a_0a_1K^2} \\
&\quad + \frac{(1 + 2a_0K + a_1K + a_0a_1K^2) \left(\frac{1}{K^2} - a_0a_1\right)}{1 + 2a_0K + a_1K + a_0a_1K^2} \\
&= \frac{a_0^2 + (1 + 2a_0K + a_1K + a_0a_1K^2) a_0a_1}{1 + 2a_0K + a_1K + a_0a_1K^2} \\
&\quad + \frac{(1 + 2a_0K + a_1K + a_0a_1K^2) \frac{1}{K^2} - (1 + 2a_0K + a_1K + a_0a_1K^2) a_0a_1}{1 + 2a_0K + a_1K + a_0a_1K^2} \\
&= \frac{a_0^2 + (1 + 2a_0K + a_1K + a_0a_1K^2) \frac{1}{K^2}}{1 + 2a_0K + a_1K + a_0a_1K^2} \\
&= \frac{a_0^2}{1 + 2a_0K + a_1K + a_0a_1K^2} + \frac{1}{K^2} > 0.
\end{aligned} \tag{A.164}$$

Then, $Z_1 > 0$ for all $\beta \in (0, 1]$, which proves that $Y_2 = \nu_0^*Z_1 + Z_2 > 0$ for all $\beta \in (0, 1]$.

Now, since $Y_1, Y_2, Y_3 > 0$, we have that

$$T1 = (1 - \beta)^2 Y_1 + (1 - \beta) Y_2 + \left(\frac{1}{K} + a_0\right) a_0 Y_3 > 0, \tag{A.165}$$

which proves that

$$X_1 = \frac{T1}{T2} > 0. \tag{A.166}$$

Since $X_1 > 0$ and $X_2 \geq 0$, then, if $X_3 \geq 0$ for $\beta_0 \in (0, 1]$, we have that

$$S1 = X_1(G(b_1) + D) + X_2(G(b_0) + D) + X_3(b_1 - b_0) > 0, \tag{A.167}$$

for $\beta_0 \in (0, 1]$.

On the other hand, if $X_3 < 0$, for $\beta_0 \in (0, 1]$, then,

$$\begin{aligned}
S1 &= X_1(G(b_1) + D) + X_2(G(b_0) + D) + X_3(b_1 - b_0) \\
&= \left[(1 - \beta) \frac{1}{K} + a_0 \right] (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) (G(b_1) + D) \\
&\quad - \left[(1 - \beta) \nu_0^* + a_0 \right] (\nu_1^* + a_1) \left(\frac{1}{K} + a_0 \right) (G(b_1) + D) \\
&\quad + X_2(G(b_0) + D) + X_3(b_1 - b_0) \\
&= (1 - \beta) \left[\frac{1}{K} (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) - \nu_0^* (\nu_1^* + a_1) \left(\frac{1}{K} + a_0 \right) \right] (G(b_1) + D) \\
&\quad - a_0 \left[(\nu_1^* + a_1) \left(\frac{1}{K} + a_0 \right) - (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) \right] (G(b_1) + D) \\
&\quad + X_2(G(b_0) + D) + X_3(b_1 - b_0) \\
&= (1 - \beta) X_4(G(b_1) + D) - a_0 X_3(G(b_1) + D) + X_2(G(b_0) + D) + X_3(b_1 - b_0) \\
&= (1 - \beta) X_4(G(b_1) + D) - X_3 [a_0(G(b_1) + D) - (b_1 - b_0)] + X_2(G(b_0) + D),
\end{aligned} \tag{A.168}$$

where

$$X_4 = \frac{1}{K} (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) - \nu_0^* (\nu_1^* + a_1) \left(\frac{1}{K} + a_0 \right). \tag{A.169}$$

Applying inequalities (A.144)-(A.169), we see that

$$\begin{aligned}
X_4 &= \frac{1}{K} (\nu_0^* + a_0) \left(\frac{1}{K} + a_1 \right) - \nu_0^* (\nu_1^* + a_1) \left(\frac{1}{K} + a_0 \right) \\
&> \frac{1}{K} (\nu_0^* + a_0) (\nu_1^* + a_1) - \nu_0^* (\nu_1^* + a_1) \left(\frac{1}{K} + a_0 \right) \\
&= (\nu_1^* + a_1) \left[\frac{1}{K} (\nu_0^* + a_0) - \nu_0^* \left(\frac{1}{K} + a_0 \right) \right] \\
&= (\nu_1^* + a_1) \left(a_0 \frac{1}{K} - a_0 \nu_0^* \right) \\
&= a_0 (\nu_1^* + a_1) \left(\frac{1}{K} - \nu_0^* \right) > 0.
\end{aligned} \tag{A.170}$$

Thus, $X_4 > 0$ for $\beta_0 \in (0, 1]$, and since $X_2 \geq 0$, $X_3 < 0$ and assumption **A1.3**, we have that

$$\begin{aligned}
S1 &= (1 - \beta) X_4(G(b_1) + D) - X_3 [a_0(G(b_1) + D) - (b_1 - b_0)] + X_2(G(b_0) + D) \\
&\geq -X_3 [a_0(G(b_1) + D) - (b_1 - b_0)] > 0,
\end{aligned} \tag{A.171}$$

for $\beta_0 \in (0, 1]$.

Therefore, $S1 > 0$ for all $\beta \in (0, 1]$, that is,

$$(p^c - p^*)(\beta) = \frac{S1}{S2} > 0, \tag{A.172}$$

which finally proves (1.31). The proof of the theorem is complete. ■

Theorem 1.7. *The functions $\pi_1^*(\beta)$ and $\pi_1^c(\beta)$ are strictly decreasing with respect to $\beta \in (0, 1]$. Moreover, the following inequalities hold:*

$$\pi_1^*(1) > \pi_1^c(1) \tag{1.34}$$

and

$$\lim_{\beta \rightarrow 0} \pi_1^*(\beta) < \lim_{\beta \rightarrow 0} \pi_1^c(\beta). \quad (1.35)$$

Proof. First, we are going to show that π_1^* and π_1^c strictly decrease by β .

The function π_1^* is differentiable with respect to β and

$$\begin{aligned} \pi_1^{*'} &= \left(p^* q_1^* - \frac{1}{2} a_1 q_1^{*2} - b_1 q_1^* \right)' = p^{*'} q_1^* + p^* q_1^{*'} - a_1 q_1^* q_1^{*'} - b_1 q_1^{*'} \\ &= p^{*'} q_1^* + (p^* - a_1 q_1^* - b_1) q_1^{*'} \\ &= p^{*'} q_1^* + \left(p^* - b_1 - a_1 \frac{p^* - b_1}{\nu_1^* + a_1} \right) q_1^{*'} \\ &= p^{*'} q_1^* + \left(1 - \frac{a_1}{\nu_1^* + a_1} \right) (p^* - b_1) q_1^{*'} \\ &= p^{*'} q_1^* + \frac{\nu_1^*}{\nu_1^* + a_1} (p^* - b_1) q_1^{*'}. \end{aligned} \quad (A.173)$$

Given the values of $a_1, b_1, \nu_1^*, p^*, q_1^*, p^{*'} and $q_1^{*'}$, it's easy to see that$

$$\pi_1^{*'} = p^{*'} q_1^* + \frac{\nu_1^*}{\nu_1^* + a_1} (p^* - b_1) q_1^{*'} < 0. \quad (A.174)$$

Similarly,

$$\pi_1^{c'} = p^{c'} q_1^c + \frac{\frac{1}{K}}{\frac{1}{K} + a_1} (p^c - b_1) q_1^{c'} < 0. \quad (A.175)$$

Because of that, π_1^* and π_1^c strictly decrease with respect to $\beta \in (0, 1]$.

Now consider the difference of the functions π_1^* and π_1^c as follows:

$$\begin{aligned} \pi_1^c - \pi_1^* &= \left(p^c q_1^c - \frac{1}{2} a_1 q_1^{c2} - b_1 q_1^c \right) - \left(p^* q_1^* - \frac{1}{2} a_1 q_1^{*2} - b_1 q_1^* \right) \\ &= \left(p^c - b_1 - \frac{1}{2} a_1 q_1^c \right) q_1^c - \left(p^* - b_1 - \frac{1}{2} a_1 q_1^* \right) q_1^* \\ &= \left[\left(\frac{1}{K} + a_1 \right) \frac{p^c - b_1}{\frac{1}{K} + a_1} - \frac{1}{2} a_1 q_1^c \right] q_1^c - \left[(\nu_1^* + a_1) \frac{p^* - b_1}{\nu_1^* + a_1} - \frac{1}{2} a_1 q_1^* \right] q_1^* \\ &= \left[\left(\frac{1}{K} + a_1 \right) q_1^c - \frac{1}{2} a_1 q_1^c \right] q_1^c - \left[(\nu_1^* + a_1) q_1^* - \frac{1}{2} a_1 q_1^* \right] q_1^* \\ &= \left(\frac{1}{K} + \frac{1}{2} a_1 \right) q_1^{c2} - \left(\nu_1^* + \frac{1}{2} a_1 \right) q_1^{*2}. \end{aligned} \quad (A.176)$$

From (A.96) we have that

$$\begin{aligned} q_1^* &= \frac{-(b_1 - b_0) + [(1 - \beta)\nu_0^* + a_0](G(b_1) + D)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) \{1 + [(1 - \beta)\nu_0^* + a_0] K\}} \\ &= \frac{[(1 - \beta)\nu_0^* + a_0](G(b_1) + D) - (b_1 - b_0)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0](\nu_1^* + a_1) K}, \end{aligned} \quad (\text{A.177})$$

and similarly to (A.96) and (A.177),

$$\begin{aligned} q_1^c &= \frac{p^c - b_1}{\frac{1}{K} + a_1} = \frac{\left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) (T + D)}{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) K} - b_1 \\ &= \frac{\left(\frac{1}{K} + a_0\right) b_1 + \left(\frac{1}{K} + a_1\right) b_0 + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) (T + D)}{\left(\frac{1}{K} + a_1\right) \left[\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) K\right]} \\ &\quad - \frac{\left[\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) K\right] b_1}{\left(\frac{1}{K} + a_1\right) \left[\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) K\right]} \\ &= \frac{-(\frac{1}{K} + a_1)(b_1 - b_0) + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) (-Kb_1 + T + D)}{\left(\frac{1}{K} + a_1\right) \left[\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) K\right]} \\ &= \frac{-(b_1 - b_0) + [(1 - \beta)\frac{1}{K} + a_0] (-Kb_1 + T + D)}{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) K} \\ &= \frac{[(1 - \beta)\frac{1}{K} + a_0](G(b_1) + D) - (b_1 - b_0)}{\left(\frac{1}{K} + a_0\right) + \left(\frac{1}{K} + a_1\right) + [(1 - \beta)\frac{1}{K} + a_0] \left(\frac{1}{K} + a_1\right) K}. \end{aligned} \quad (\text{A.178})$$

By substituting the expression of ν_1^* given by (A.41) in equation (A.177) we have that

$$\begin{aligned} q_1^* &= \frac{[(1 - \beta)\nu_0^* + a_0](G(b_1) + D) - (b_1 - b_0)}{(\nu_0^* + a_0) + (\nu_1^* + a_1) + [(1 - \beta)\nu_0^* + a_0](\nu_1^* + a_1) K} \\ &= \frac{[(1 - \beta)\nu_0^* + a_0](G(b_1) + D) - (b_1 - b_0)}{(\nu_0^* + a_0) + \left(\frac{(1 - \beta)\nu_0^* + a_0}{1 + [(1 - \beta)\nu_0^* + a_0] K} + a_1\right) + [(1 - \beta)\nu_0^* + a_0] \left(\frac{(1 - \beta)\nu_0^* + a_0}{1 + [(1 - \beta)\nu_0^* + a_0] K} + a_1\right) K} \\ &= \frac{(1 + [(1 - \beta)\nu_0^* + a_0] K) \{[(1 - \beta)\nu_0^* + a_0](G(b_1) + D) - (b_1 - b_0)\}}{(1 + [(1 - \beta)\nu_0^* + a_0] K) \{(\nu_0^* + a_0) + a_1 + (1 + a_1 K) [(1 - \beta)\nu_0^* + a_0]\}} \\ &= \frac{[(1 - \beta)\nu_0^* + a_0](G(b_1) + D) - (b_1 - b_0)}{(\nu_0^* + a_0 + a_1) + (1 + a_1 K) [(1 - \beta)\nu_0^* + a_0]}. \end{aligned} \quad (\text{A.179})$$

By equation (A.41),

$$\nu_1^* = \frac{(1 - \beta)\nu_0^* + a_0}{1 + [(1 - \beta)\nu_0^* + a_0] K},$$

therefore,

$$\begin{aligned}
 \nu_1^* + \frac{1}{2}a_1 &= \frac{(1-\beta)\nu_0^* + a_0}{1 + [(1-\beta)\nu_0^* + a_0]K} + \frac{1}{2}a_1 \\
 &= \frac{(1-\beta)\nu_0^* + a_0 + \frac{1}{2}a_1(1 + [(1-\beta)\nu_0^* + a_0]K)}{1 + [(1-\beta)\nu_0^* + a_0]K} \\
 &= \frac{\frac{1}{2}a_1 + (1 + \frac{1}{2}a_1K)[(1-\beta)\nu_0^* + a_0]}{1 + [(1-\beta)\nu_0^* + a_0]K}.
 \end{aligned} \tag{A.180}$$

On the other hand, from the expression for q_1^c obtained from (A.178) we have that

$$\begin{aligned}
 q_1^c &= \frac{[(1-\beta)\frac{1}{K} + a_0](G(b_1) + D) - (b_1 - b_0)}{(\frac{1}{K} + a_0) + (\frac{1}{K} + a_1) + [(1-\beta)\frac{1}{K} + a_0](\frac{1}{K} + a_1)K} \\
 &= \frac{[(1-\beta)\frac{1}{K} + a_0](G(b_1) + D) - (b_1 - b_0)}{\frac{1}{K}(1 + a_0K) + \frac{1}{K}(1 + a_1K) + \frac{1}{K}[(1-\beta) + a_0K](1 + a_1K)} \\
 &= \frac{\{[(1-\beta)\frac{1}{K} + a_0](G(b_1) + D) - (b_1 - b_0)\}K}{(1 + a_0K) + (1 + a_1K)[(2-\beta) + a_0K]}.
 \end{aligned} \tag{A.181}$$

Plugging equations (A.179), (A.180) and (A.181) in equation (A.176) we deduce

$$\begin{aligned}
 \pi_1^c - \pi_1^* &= (\frac{1}{K} + \frac{1}{2}a_1)q_1^{c2} - (\nu_1^* + \frac{1}{2}a_1)q_1^{*2} \\
 &= (\frac{1}{K} + \frac{1}{2}a_1) \left(\frac{\{[(1-\beta)\frac{1}{K} + a_0](G(b_1) + D) - (b_1 - b_0)\}K}{(1 + a_0K) + (1 + a_1K)[(2-\beta) + a_0K]} \right)^2 \\
 &\quad - \left(\frac{\frac{1}{2}a_1 + (1 + \frac{1}{2}a_1K)[(1-\beta)\nu_0^* + a_0]}{1 + [(1-\beta)\nu_0^* + a_0]K} \right) \left(\frac{[(1-\beta)\nu_0^* + a_0](G(b_1) + D) - (b_1 - b_0)}{(\nu_0^* + a_0 + a_1) + (1 + a_1K)[(1-\beta)\nu_0^* + a_0]} \right)^2 \\
 &= \frac{1}{2}K(2 + a_1K) \left(\frac{[(1-\beta)\frac{1}{K} + a_0](G(b_1) + D) - (b_1 - b_0)}{(1 + a_0K) + (1 + a_1K)[(2-\beta) + a_0K]} \right)^2 \\
 &\quad - \frac{1}{2} \left(\frac{a_1 + (2 + a_1K)[(1-\beta)\nu_0^* + a_0]}{1 + [(1-\beta)\nu_0^* + a_0]K} \right) \left(\frac{[(1-\beta)\nu_0^* + a_0](G(b_1) + D) - (b_1 - b_0)}{(\nu_0^* + a_0 + a_1) + (1 + a_1K)[(1-\beta)\nu_0^* + a_0]} \right)^2.
 \end{aligned} \tag{A.182}$$

Then, to prove the inequalities (A.4) and (A.5) the following conditions has to be met:

$$\pi_1^c(1) - \pi_1^*(1) = (\pi_1^c - \pi_1^*)(1) < 0 \tag{A.183}$$

and

$$\lim_{\beta \rightarrow 0} \pi_1^c(\beta) - \lim_{\beta \rightarrow 0} \pi_1^*(\beta) = \lim_{\beta \rightarrow 0} (\pi_1^c - \pi_1^*)(\beta) > 0. \tag{A.184}$$

Evaluating the expression of ν_0^* , given by (A.40), for $\beta = 1$ and using the notation $\overline{\nu_0^*} = \nu_0^*(1)$, one has

$$\begin{aligned}\overline{\nu_0^*} &= \nu_0^*(1) = \frac{2(a_0 + a_1 + a_0 a_1 K)}{(1 + 2a_0 K + a_1 K + a_0 a_1 K^2) + \sqrt{(1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2}} \\ &= \frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2}.\end{aligned}\quad (\text{A.185})$$

Now, we evaluate (A.182) for $\beta = 1$ to obtain

$$\begin{aligned}(\pi_1^c - \pi_1^*)(1) &= \frac{1}{2} K (2 + a_1 K) \left(\frac{[a_0] (G(b_1) + D) - (b_1 - b_0)}{(1 + a_0 K) + (1 + a_1 K) [1 + a_0 K]} \right)^2 \\ &\quad - \frac{1}{2} \left(\frac{a_1 + (2 + a_1 K) [a_0]}{1 + [a_0] K} \right) \left(\frac{[a_0] (G(b_1) + D) - (b_1 - b_0)}{(\overline{\nu_0^*} + a_0 + a_1) + (1 + a_1 K) [a_0]} \right)^2 \\ &= \frac{1}{2} K (2 + a_1 K) \left(\frac{a_0 (G(b_1) + D) - (b_1 - b_0)}{(1 + a_0 K) (2 + a_1 K)} \right)^2 \\ &\quad - \frac{1}{2} \left(\frac{a_1 + a_0 (2 + a_1 K)}{1 + a_0 K} \right) \left(\frac{a_0 (G(b_1) + D) - (b_1 - b_0)}{(\overline{\nu_0^*} + a_0 + a_1) + a_0 (1 + a_1 K)} \right)^2 \\ &= \frac{1}{2} \frac{[a_0 (G(b_1) + D) - (b_1 - b_0)]^2}{1 + a_0 K} \frac{K}{(1 + a_0 K) (2 + a_1 K)} \\ &\quad - \frac{1}{2} \frac{[a_0 (G(b_1) + D) - (b_1 - b_0)]^2}{1 + a_0 K} \frac{a_1 + a_0 (2 + a_1 K)}{[(\overline{\nu_0^*} + a_0 + a_1) + a_0 (1 + a_1 K)]^2} \\ &= \frac{1}{2} \frac{[a_0 (G(b_1) + D) - (b_1 - b_0)]^2}{1 + a_0 K} \frac{K}{2 + 2a_0 K + a_1 K + a_0 a_1 K^2} \\ &\quad - \frac{1}{2} \frac{[a_0 (G(b_1) + D) - (b_1 - b_0)]^2}{1 + a_0 K} \frac{2a_0 + a_1 + a_0 a_1 K}{(\overline{\nu_0^*} + 2a_0 + a_1 + a_0 a_1 K)^2} \\ &= U_1 \frac{V_1}{W_1},\end{aligned}\quad (\text{A.186})$$

where

$$U_1 = \frac{1}{2} \frac{[a_0 (G(b_1) + D) - (b_1 - b_0)]^2}{1 + a_0 K}, \quad (\text{A.187})$$

$$V_1 = K (\overline{\nu_0^*} + 2a_0 + a_1 + a_0 a_1 K)^2 - (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \quad (\text{A.188})$$

and

$$W_1 = (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) (\overline{\nu_0^*} + 2a_0 + a_1 + a_0 a_1 K)^2. \quad (\text{A.189})$$

Given the values of a_0 , a_1 , $\overline{\nu_0^*}$ and K , it isn't difficult to see that $U_1 > 0$ and $W_1 > 0$. Hence, to prove (A.183) it is enough to show that $V_1 > 0$. Indeed, plugging the expression of $\overline{\nu_0^*}$

given by (A.185) in (A.188), we have that

$$\begin{aligned}
V_1 &= K \left(\bar{\nu}_0^* + 2a_0 + a_1 + a_0 a_1 K \right)^2 - (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\
&= K \left(\frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} + 2a_0 + a_1 + a_0 a_1 K \right)^2 \\
&\quad - (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\
&= K \left[a_0 + a_1 + a_0 a_1 K + (2a_0 + a_1 + a_0 a_1 K) (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) \right]^2 \\
&\quad - (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) (1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2 \\
&< K \left[(2a_0 + a_1 + a_0 a_1 K) + (2a_0 + a_1 + a_0 a_1 K) (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) \right]^2 \\
&\quad - (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) (1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2 \\
&= K \left[(2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \right]^2 \\
&\quad - (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) (1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2 \\
&= \left[K (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \right. \\
&\quad \left. - (1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2 \right] (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\
&= \left[(2a_0 K + a_1 K + a_0 a_1 K^2) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \right. \\
&\quad \left. - (1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2 \right] (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\
&= \left[(1 + 2a_0 K + a_1 K + a_0 a_1 K^2 - 1) (1 + 2a_0 K + a_1 K + a_0 a_1 K^2 + 1) \right. \\
&\quad \left. - (1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2 \right] (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\
&= \left[(1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2 - 1 \right. \\
&\quad \left. - (1 + 2a_0 K + a_1 K + a_0 a_1 K^2)^2 \right] (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\
&= - (2a_0 + a_1 + a_0 a_1 K) (2 + 2a_0 K + a_1 K + a_0 a_1 K^2) < 0.
\end{aligned} \tag{A.190}$$

Therefore $V_1 < 0$. Then, since $U_1 > 0$ and $W_1 > 0$, we have that

$$(\pi_1^c - \pi_1^*)(1) = U_1 \frac{V_1}{W_1} < 0, \tag{A.191}$$

which proves (A.183).

Now, we need only to prove (A.184). Using the notation $\widehat{\nu}_0^* = \lim_{\beta \rightarrow 0} \nu_0^*(\beta)$ given by (A.135), from (A.182) we have that

$$\begin{aligned}
& \lim_{\beta \rightarrow 0} (\pi_1^c - \pi_1^*)(\beta) = \\
&= \frac{1}{2} K (2 + a_1 K) \left(\frac{\left[\frac{1}{K} + a_0 \right] (G(b_1) + D) - (b_1 - b_0)}{(1 + a_0 K) + (1 + a_1 K) [2 + a_0 K]} \right)^2 \\
&\quad - \frac{1}{2} \left(\frac{a_1 + (2 + a_1 K) [\widehat{\nu}_0^* + a_0]}{1 + [\widehat{\nu}_0^* + a_0] K} \right) \left(\frac{[\widehat{\nu}_0^* + a_0] (G(b_1) + D) - (b_1 - b_0)}{(\widehat{\nu}_0^* + a_0 + a_1) + (1 + a_1 K) [\widehat{\nu}_0^* + a_0]} \right)^2 \\
&= \frac{1}{2} K (2 + a_1 K) \left(\frac{\left(\frac{1}{K} + a_0 \right) (G(b_1) + D) - (b_1 - b_0)}{(1 + a_0 K) + (1 + a_1 K) + (1 + a_0 K) (1 + a_1 K)} \right)^2 \\
&\quad - \frac{1}{2} \left(\frac{a_1 + (2 + a_1 K) (\widehat{\nu}_0^* + a_0)}{1 + (\widehat{\nu}_0^* + a_0) K} \right) \left(\frac{(\widehat{\nu}_0^* + a_0) (G(b_1) + D) - (b_1 - b_0)}{a_1 + (2 + a_1 K) (\widehat{\nu}_0^* + a_0)} \right)^2 \tag{A.192} \\
&= \frac{1}{2} K (2 + a_1 K) \left(\frac{\left(\frac{1}{K} + a_0 \right) (G(b_1) + D) - (b_1 - b_0)}{(1 + a_1 K) + (1 + a_0 K) (2 + a_1 K)} \right)^2 \\
&\quad - \frac{1}{2} \frac{1}{1 + (\widehat{\nu}_0^* + a_0) K} \frac{[(\widehat{\nu}_0^* + a_0) (G(b_1) + D) - (b_1 - b_0)]^2}{a_1 + (2 + a_1 K) (\widehat{\nu}_0^* + a_0)} \\
&= \frac{1}{2} \frac{V_2}{W_2},
\end{aligned}$$

where

$$\begin{aligned}
V_2 = & K (2 + a_1 K) \left[\left(\frac{1}{K} + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right]^2 \times \\
& \left[1 + (\widehat{\nu}_0^* + a_0) K \right] \left[a_1 + (2 + a_1 K) (\widehat{\nu}_0^* + a_0) \right] \\
& - [(1 + a_1 K) + (1 + a_0 K) (2 + a_1 K)]^2 \times \\
& \left[(\widehat{\nu}_0^* + a_0) (G(b_1) + D) - (b_1 - b_0) \right]^2 \tag{A.193}
\end{aligned}$$

and

$$\begin{aligned}
W_2 = & [(1 + a_1 K) + (1 + a_0 K) (2 + a_1 K)]^2 \times \\
& \left[1 + (\widehat{\nu}_0^* + a_0) K \right] \left[a_1 + (2 + a_1 K) (\widehat{\nu}_0^* + a_0) \right]. \tag{A.194}
\end{aligned}$$

For arbitrary fixed values of a_0 , a_1 , $\widehat{\nu}_0^*$ and K , it is evident that $W_2 > 0$. Hence, to prove (A.184) it lacks only to show that $V_2 > 0$. Indeed,

$$\begin{aligned}
 V_2 &= \left(\left[(1 + a_1 K) + (2 + a_1 K) (\widehat{\nu}_0^* + a_0) K \right]^2 - 1 \right) \left[\left(\frac{1}{K} + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right]^2 \\
 &\quad - \left[(1 + a_1 K) + (1 + a_0 K) (2 + a_1 K) \right]^2 \left[(\widehat{\nu}_0^* + a_0) (G(b_1) + D) - (b_1 - b_0) \right]^2 \\
 &= \left[(1 + a_1 K) + (\widehat{\nu}_0^* + a_0) (2 + a_1 K) K \right]^2 \left[\left(\frac{1}{K} + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right]^2 \\
 &\quad - \left[(1 + a_1 K) + \left(\frac{1}{K} + a_0 \right) (2 + a_1 K) K \right]^2 \left[(\widehat{\nu}_0^* + a_0) (G(b_1) + D) - (b_1 - b_0) \right]^2 \\
 &\quad - \left[\left(\frac{1}{K} + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right]^2.
 \end{aligned} \tag{A.195}$$

Now introduce the following notation:

$$\eta = 1 + a_1 K > 0, \tag{A.196}$$

$$\xi = K (1 + \eta) = K (2 + a_1 K) > 0, \tag{A.197}$$

$$\mathcal{Z} = \eta + a_0 \xi = (1 + a_1 K) + a_0 K (2 + a_1 K) > 0, \tag{A.198}$$

$$G1 = G(b_1) + D > 0 \tag{A.199}$$

and

$$G3 = a_0 G1 - (b_1 - b_0) = a_0 (G(b_1) + D) - (b_1 - b_0) > 0. \tag{A.200}$$

Based on that, we can rewrite (A.195) as follows:

$$\begin{aligned}
V_2 &= \left[(1 + a_1 K) + \left(\widehat{\nu}_0^* + a_0 \right) (2 + a_1 K) K \right]^2 \left[\left(\frac{1}{K} + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right]^2 \\
&\quad - \left[(1 + a_1 K) + \left(\frac{1}{K} + a_0 \right) (2 + a_1 K) K \right]^2 \left[\left(\widehat{\nu}_0^* + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right]^2 \\
&\quad - \left[\left(\frac{1}{K} + a_0 \right) (G(b_1) + D) - (b_1 - b_0) \right]^2 \\
&= \left(\widehat{\nu}_0^* \xi + \mathcal{Z} \right)^2 \left(\frac{1}{K} G1 + G3 \right)^2 - \left(\frac{1}{K} \xi + \mathcal{Z} \right)^2 \left(\widehat{\nu}_0^* G1 + G3 \right)^2 - \left(\frac{1}{K} G1 + G3 \right)^2 \\
&= \left(\frac{1}{K} \widehat{\nu}_0^* \xi G1 + \frac{1}{K} \mathcal{Z} G1 + \widehat{\nu}_0^* \xi G3 + \mathcal{Z} G3 \right)^2 - \left(\frac{1}{K} \widehat{\nu}_0^* \xi G1 + \widehat{\nu}_0^* \mathcal{Z} G1 + \frac{1}{K} \xi G3 + \mathcal{Z} G3 \right)^2 \\
&\quad - \left(\frac{1}{K^2} G1^2 + 2 \frac{1}{K} G1 G3 + G3^2 \right) \\
&= \left[2 \left(\frac{1}{K} \widehat{\nu}_0^* \xi G1 + \mathcal{Z} G3 \right) + \left(\frac{1}{K} + \widehat{\nu}_0^* \right) (\mathcal{Z} G1 + \xi G3) \right] \left[\left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3) \right] \\
&\quad - \left(\frac{1}{K^2} G1^2 + 2 \frac{1}{K} G1 G3 \right) - G3^2 \\
&= 2 \left(\frac{1}{K} \widehat{\nu}_0^* \xi G1 + \mathcal{Z} G3 \right) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3) + \left(\frac{1}{K^2} - \widehat{\nu}_0^{*2} \right) (\mathcal{Z}^2 G1^2 - \xi^2 G3^2) \\
&\quad - \frac{1}{K} G1 \left(\frac{1}{K} G1 + 2 G3 \right) - G3^2 \\
&= \left(\frac{1}{K^2} - \widehat{\nu}_0^{*2} \right) (\mathcal{Z}^2 G1^2 - \xi^2 G3^2) - \frac{1}{K} G1 \left(\frac{1}{K} G1 + 2 G3 \right) \\
&\quad + 2 \mathcal{Z} G3 \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3) - G3^2 \\
&\quad + 2 \frac{1}{K} \widehat{\nu}_0^* \xi G1 \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3) \\
&= \mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1,
\end{aligned} \tag{A.201}$$

where

$$\mathcal{P}_1 = \left(\frac{1}{K^2} - \widehat{\nu}_0^{*2} \right) (\mathcal{Z}^2 G1^2 - \xi^2 G3^2) - \frac{1}{K} G1 \left(\frac{1}{K} G1 + 2 G3 \right), \tag{A.202}$$

$$\mathcal{Q}_1 = 2 \mathcal{Z} G3 \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3) - G3^2 \tag{A.203}$$

and

$$\mathcal{R}_1 = 2 \frac{1}{K} \widehat{\nu}_0^* \xi G1 \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3). \tag{A.204}$$

Now, we are going to show that

$$\mathcal{Z} G1 - \xi G3 > 0. \tag{A.205}$$

Using (A.198) and (A.200) we have that

$$\begin{aligned}
\mathcal{Z} G1 - \xi G3 &= (\eta + a_0 \xi) G1 - \xi (a_0 G1 - (b_1 - b_0)) \\
&= \eta G1 + \xi (b_1 - b_0) \geq \eta G1 > 0,
\end{aligned} \tag{A.206}$$

which proves (A.205).

Thus, given the values of $\widehat{\nu}_0^*$, K , equations (A.197), (A.199), (A.144) and (A.205), we can conclude that $\mathcal{R}_1 > 0$.

Now,

$$\begin{aligned} \mathcal{Q}_1 &= 2\mathcal{Z}G3 \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z}G1 - \xi G3) - G3^2 \\ &= \left[2\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z}G1 - \xi G3) - G3 \right] G3, \end{aligned} \quad (\text{A.207})$$

and using (A.200) we can rewrite (A.207) as follows:

$$\begin{aligned} \mathcal{Q}_1 &= \left[2\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z}G1 - \xi G3) - G3 \right] G3 \\ &= \left\{ 2\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z}G1 - \xi [a_0G1 - (b_1 - b_0)]) - [a_0G1 - (b_1 - b_0)] \right\} G3 \\ &= \left\{ 2\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) [\mathcal{Z}G1 - a_0\xi G1 + \xi(b_1 - b_0)] - a_0G1 + (b_1 - b_0) \right\} G3 \\ &= \left\{ 2\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) [(Z - a_0\xi)G1 + \xi(b_1 - b_0)] - a_0G1 + (b_1 - b_0) \right\} G3 \\ &= \left\{ 2\mathcal{Z} (Z - a_0\xi) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) G1 + 2\xi\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (b_1 - b_0) - a_0G1 + (b_1 - b_0) \right\} G3 \\ &= \left\{ \left[2\mathcal{Z} (Z - a_0\xi) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - a_0 \right] G1 + \left[2\xi\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) + 1 \right] (b_1 - b_0) \right\} G3. \end{aligned} \quad (\text{A.208})$$

Moreover, from (A.198), we have that

$$\eta = Z - a_0\xi = 1 + a_1K. \quad (\text{A.209})$$

Substituting (A.209) in (A.208) we have that

$$\begin{aligned} \mathcal{Q}_1 &= \left\{ \left[2\mathcal{Z} (Z - a_0\xi) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - a_0 \right] G1 + \left[2\xi\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) + 1 \right] (b_1 - b_0) \right\} G3 \\ &= \left\{ \left[2\mathcal{Z} (1 + a_1K) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - a_0 \right] G1 + \left[2\xi\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) + 1 \right] (b_1 - b_0) \right\} G3 \\ &= \left\{ \left[2a_1K\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) + 2\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - a_0 \right] G1 + \left[2\xi\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) + 1 \right] (b_1 - b_0) \right\} G3 \\ &= [(V_3 + W_3)G1 + U_3(b_1 - b_0)]G3, \end{aligned} \quad (\text{A.210})$$

where

$$V_3 = 2a_1K\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right), \quad (\text{A.211})$$

$$W_3 = 2\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - a_0 \quad (\text{A.212})$$

and

$$U_3 = 2\xi\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) + 1. \quad (\text{A.213})$$

For any given values of a_1 , K , ξ and \mathcal{Z} , one easily deduces that $V_3 > 0$ and $U_3 > 0$. Now, we are going to show that $W_3 > 0$. In order to do that, we first substitute (A.198) in (A.212) to

get:

$$\begin{aligned}
W_3 &= 2\mathcal{Z} \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - a_0 \\
&= 2 \left[(1 + a_1 K) + a_0 K (2 + a_1 K) \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - a_0 \\
&> a_0 K (2 + a_1 K) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - a_0 \\
&= a_0 \left[K (2 + a_1 K) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - 1 \right].
\end{aligned} \tag{A.214}$$

Now, making use of the expression of $\widehat{\nu}_0^*$ given by (A.135) we have that

$$\begin{aligned}
\frac{1}{K} - \widehat{\nu}_0^* &= \frac{1}{K} - \frac{2(a_0 + a_1 + a_0 a_1 K)}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} \\
&= \frac{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} - 2K(a_0 + a_1 + a_0 a_1 K)}{K \left[(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]} \\
&= \frac{\sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} - (2a_1 K + a_0 a_1 K^2)}{K \left[(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]} \\
&= \frac{\sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} - a_1 K(2 + a_0 K)}{K \left[(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]}.
\end{aligned} \tag{A.215}$$

Now plugging (A.215) in (A.214) we get:

$$\begin{aligned}
W_3 &> a_0 \left[K(2 + a_1 K) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - 1 \right] \\
&= a_0 \left[K(2 + a_1 K) \frac{\sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} - a_1 K(2 + a_0 K)}{K \left[(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]} - 1 \right] \\
&= a_0 \left[(2 + a_1 K) \frac{\sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} - a_1 K(2 + a_0 K)}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} - 1 \right] \\
&= \frac{a_0}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} \left[\right. \\
&\quad (2 + a_1 K) \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \\
&\quad - a_1 K(2 + a_0 K)(2 + a_1 K) - (2a_0 K + a_0 a_1 K^2) \\
&\quad \left. - \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right] \\
&= \frac{a_0}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} \left[\right. \\
&\quad [(2 + a_1 K) - 1] \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \\
&\quad \left. - a_1 K(2 + a_0 K)(2 + a_1 K) - a_0 K(2 + a_1 K) \right] \tag{A.216} \\
&= \frac{a_0}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} \left\{ \right. \\
&\quad (1 + a_1 K) \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \\
&\quad \left. - K(2 + a_1 K)[a_1(2 + a_0 K) + a_0] \right\} \\
&= \frac{a_0}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} \left\{ \right. \\
&\quad (1 + a_1 K) \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \\
&\quad \left. - K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1] \right\} \\
&= a_0 \frac{V_4}{W_4},
\end{aligned}$$

where

$$\begin{aligned}
V_4 &= (1 + a_1 K) \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \\
&\quad - K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1] \tag{A.217}
\end{aligned}$$

and

$$W_4 = (2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}. \tag{A.218}$$

For any values of a_0 , a_1 and K , we see that $W_4 > 0$, thus, to compute the value of (A.216) we need only to estimate the value of V_4 . Suppose that

$$V_4 = (1 + a_1 K) \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} - K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1] \leq 0. \quad (\text{A.219})$$

Then, we would have

$$(1 + a_1 K) \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \leq K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1]. \quad (\text{A.220})$$

Both sides of (A.220) are positive, then, by squaring them we have that

$$(1 + a_1 K)^2 \left[(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K) \right] \leq K^2 (2 + a_1 K)^2 [a_0(1 + a_1 K) + 2a_1]^2. \quad (\text{A.221})$$

Thus,

$$(1 + a_1 K)^2 \left[a_0^2 K^2 (2 + a_1 K)^2 + 4K(2 + a_1 K)(a_0 + a_1 + a_0 a_1 K) \right] \leq K^2 (2 + a_1 K)^2 [a_0(1 + a_1 K) + 2a_1]^2, \quad (\text{A.222})$$

which leads to

$$K(2 + a_1 K)(1 + a_1 K)^2 [a_0^2 K(2 + a_1 K) + 4(a_0 + a_1 + a_0 a_1 K)] \leq K^2 (2 + a_1 K)^2 [a_0(1 + a_1 K) + 2a_1]^2, \quad (\text{A.223})$$

where, $K(2 + a_1 K) > 0$. Therefore,

$$(1 + a_1 K)^2 [a_0^2 K(2 + a_1 K) + 4(a_0 + a_1 + a_0 a_1 K)] \leq K(2 + a_1 K)[a_0(1 + a_1 K) + 2a_1]^2. \quad (\text{A.224})$$

Now, expanding the squares we find that:

$$(1 + a_1 K)^2 [a_0^2 K(2 + a_1 K) + 4(a_0 + a_1 + a_0 a_1 K)] \leq K(2 + a_1 K) \left[a_0^2 (1 + a_1 K)^2 + 4a_0 a_1 (1 + a_1 K) + 4a_1^2 \right], \quad (\text{A.225})$$

and by distributing some terms:

$$a_0^2 K(2 + a_1 K)(1 + a_1 K)^2 + 4(a_0 + a_1 + a_0 a_1 K)(1 + a_1 K)^2 \leq a_0^2 K(2 + a_1 K)(1 + a_1 K)^2 + 4a_0 a_1 K(2 + a_1 K)(1 + a_1 K) + 4a_1^2 K(2 + a_1 K). \quad (\text{A.226})$$

Thus,

$$\begin{aligned} & 4(a_0 + a_1 + a_0 a_1 K)(1 + a_1 K)^2 \\ & \leq 4a_0 a_1 K(2 + a_1 K)(1 + a_1 K) + 4a_1^2 K(2 + a_1 K), \end{aligned} \quad (\text{A.227})$$

and distributing again:

$$\begin{aligned} & 4(a_0 + a_1)(1 + a_1 K)^2 + 4a_0 a_1 K(1 + a_1 K)^2 \\ & \leq 4a_0 a_1 K(1 + a_1 K) + 4a_0 a_1 K(1 + a_1 K)^2 + 4a_1^2 K(2 + a_1 K), \end{aligned} \quad (\text{A.228})$$

which leads to

$$\begin{aligned} & 4(a_0 + a_1)(1 + a_1 K)^2 \\ & \leq 4a_0 a_1 K(1 + a_1 K) + 4a_1^2 K(2 + a_1 K). \end{aligned} \quad (\text{A.229})$$

Finally,

$$\begin{aligned} & (a_0 + a_1)(1 + a_1 K)^2 \\ & \leq a_0 a_1 K(1 + a_1 K) + a_1^2 K(2 + a_1 K), \end{aligned} \quad (\text{A.230})$$

and by distributing some terms and expanding the squares again we deduce that

$$\begin{aligned} & (a_0 + a_1)(1 + 2a_1 K + a_1^2 K^2) \\ & \leq a_0 a_1 K(1 + a_1 K) + a_1^2 K(1 + a_1 K) + a_1^2 K, \end{aligned} \quad (\text{A.231})$$

thus,

$$\begin{aligned} & (a_0 + a_1)(1 + 2a_1 K + a_1^2 K^2) \\ & \leq (a_0 + a_1)a_1 K(1 + a_1 K) + a_1^2 K, \end{aligned} \quad (\text{A.232})$$

which leads to

$$\begin{aligned} & (a_0 + a_1)(1 + 2a_1 K + a_1^2 K^2) \\ & \leq (a_0 + a_1)(a_1 K + a_1^2 K^2) + a_1^2 K. \end{aligned} \quad (\text{A.233})$$

Hence,

$$(a_0 + a_1)(1 + a_1 K) \leq a_1^2 K, \quad (\text{A.234})$$

which finally leads to

$$a_0 + a_1 + a_0 a_1 K + a_1^2 K \leq a_1^2 K, \quad (\text{A.235})$$

which is impossible since $a_0 + a_1 + a_0 a_1 K > 0$.

On a base of that, we conclude that

$$V_4 > 0, \quad (\text{A.236})$$

so,

$$W_3 > a_0 \frac{V_4}{W_4} > 0. \quad (\text{A.237})$$

Now, since $V_3 > 0$, $W_3 > 0$ and $U_3 > 0$, and given the values of b_0 , b_1 , $G1$ and $G3$, we have that

$$\mathcal{Q}_1 = [(V_3 + W_3) G1 + U_3(b_1 - b_0)] G3 > 0. \quad (\text{A.238})$$

So we need only to estimate the value of \mathcal{P}_1 .

$$\begin{aligned} \mathcal{P}_1 &= \left(\frac{1}{K^2} - \widehat{\nu}_0^{*2} \right) (\mathcal{Z}^2 G1^2 - \xi^2 G3^2) - \frac{1}{K} G1 \left(\frac{1}{K} G1 + 2G3 \right) \\ &= \left(\frac{1}{K} + \widehat{\nu}_0^* \right) (\mathcal{Z} G1 + \xi G3) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3) - \frac{1}{K} G1 \left(\frac{1}{K} G1 + 2G3 \right) \\ &= \left(\frac{1}{K} \mathcal{Z} G1 + \frac{1}{K} \xi G3 + \widehat{\nu}_0^* \mathcal{Z} G1 + \widehat{\nu}_0^* \xi G3 \right) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3) \\ &\quad - \frac{1}{K} G1 \left(\frac{1}{K} G1 + 2G3 \right). \end{aligned} \quad (\text{A.239})$$

By equations (A.197), (A.200) and (A.209), we have that

$$\xi = K(2 + a_1 K) > 2K, \quad (\text{A.240})$$

$$\mathcal{Z} G1 - \xi G3 = \mathcal{Z} G1 - \xi [a_0 G1 - (b_1 - b_0)] \geq \mathcal{Z} G1 - a_0 \xi G1 = \eta G1, \quad (\text{A.241})$$

$$\mathcal{Z} = \eta + a_0 \xi > \eta \quad (\text{A.242})$$

and

$$\widehat{\nu}_0^* \xi G3 > 0. \quad (\text{A.243})$$

Using inequalities (A.240)-(A.243) in (A.239) we get:

$$\begin{aligned} \mathcal{P}_1 &= \left(\frac{1}{K} \mathcal{Z} G1 + \frac{1}{K} \xi G3 + \widehat{\nu}_0^* \mathcal{Z} G1 + \widehat{\nu}_0^* \xi G3 \right) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (\mathcal{Z} G1 - \xi G3) \\ &\quad - \frac{1}{K} G1 \left(\frac{1}{K} G1 + 2G3 \right) \\ &> \left(\frac{1}{K} \mathcal{Z} G1 + 2G3 + \widehat{\nu}_0^* \eta G1 \right) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \eta G1 \\ &\quad - \frac{1}{K} G1 \left(\frac{1}{K} G1 + 2G3 \right) \\ &= \left[\eta \left(\frac{1}{K} \mathcal{Z} G1 + 2G3 + \widehat{\nu}_0^* \eta G1 \right) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left(\frac{1}{K} G1 + 2G3 \right) \right] G1 \\ &= \left\{ \left[\left(\frac{1}{K} \mathcal{Z} + \widehat{\nu}_0^* \eta \right) \eta G1 + 2\eta G3 \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left(\frac{1}{K} G1 + 2G3 \right) \right\} G1. \end{aligned} \quad (\text{A.244})$$

Now, making use of (A.196), (A.198) and (A.240), we come to

$$\eta = 1 + a_1 K > 1 \quad (\text{A.245})$$

and

$$\mathcal{Z} = \eta + a_0\xi > \eta + 2a_0K. \quad (\text{A.246})$$

Then, applying inequalities (A.245) and (A.246) to (A.244) one gets:

$$\begin{aligned} \mathcal{P}_1 &= \left\{ \left[\left(\frac{1}{K} \mathcal{Z} + \widehat{\nu}_0^* \eta \right) \eta G1 + 2\eta G3 \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left(\frac{1}{K} G1 + 2G3 \right) \right\} G1 \\ &> \left\{ \left[\left(\frac{1}{K} (\eta + 2a_0K) + \widehat{\nu}_0^* \right) G1 + 2\eta G3 \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left(\frac{1}{K} G1 + 2G3 \right) \right\} G1 \\ &= \left\{ \left[\left(\frac{1}{K} \eta + 2a_0 + \widehat{\nu}_0^* \right) G1 + 2\eta G3 \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left(\frac{1}{K} G1 + 2G3 \right) \right\} G1. \end{aligned} \quad (\text{A.247})$$

Substituting the value of $G3$ given by (A.200) in (A.247), we have:

$$\begin{aligned} \mathcal{P}_1 &> \left\{ \left[\left(\frac{1}{K} \eta + 2a_0 + \widehat{\nu}_0^* \right) G1 + 2\eta G3 \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left(\frac{1}{K} G1 + 2G3 \right) \right\} G1. \\ &= \left\{ \left[\left(\frac{1}{K} \eta + 2a_0 + \widehat{\nu}_0^* \right) G1 + 2\eta [a_0 G1 - (b_1 - b_0)] \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left\{ \frac{1}{K} G1 + 2[a_0 G1 - (b_1 - b_0)] \right\} \right\} G1. \\ &= \left\{ \left[\left(\frac{1}{K} \eta + 2a_0 + \widehat{\nu}_0^* \right) G1 + 2a_0 \eta G1 - 2\eta (b_1 - b_0) \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left[\frac{1}{K} G1 + 2a_0 G1 - 2(b_1 - b_0) \right] \right\} G1. \\ &= \left\{ \left[\left(\frac{1}{K} \eta + 2a_0 + 2a_0 \eta + \widehat{\nu}_0^* \right) G1 - 2\eta (b_1 - b_0) \right] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right. \\ &\quad \left. - \frac{1}{K} \left[\left(\frac{1}{K} + 2a_0 \right) G1 - 2(b_1 - b_0) \right] \right\} G1. \\ &= \left\{ \left(\frac{1}{K} \eta + 2a_0 + 2a_0 \eta + \widehat{\nu}_0^* \right) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) G1 - 2\eta \left(\frac{1}{K} - \widehat{\nu}_0^* \right) (b_1 - b_0) \right. \\ &\quad \left. - \frac{1}{K} \left(\frac{1}{K} + 2a_0 \right) G1 + 2\frac{1}{K} (b_1 - b_0) \right\} G1. \\ &= \left\{ \left[\left(\frac{1}{K} \eta + 2a_0 + 2a_0 \eta + \widehat{\nu}_0^* \right) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \frac{1}{K} \left(\frac{1}{K} + 2a_0 \right) \right] G1 \right. \\ &\quad \left. + 2 \left[\frac{1}{K} - \eta \left(\frac{1}{K} - \widehat{\nu}_0^* \right) \right] (b_1 - b_0) \right\} G1. \end{aligned} \quad (\text{A.248})$$

Next, we substitute the value of η given by (A.196) in (A.248) to obtain:

$$\begin{aligned}
\mathcal{P}_1 &> \left\{ \left[\left(\frac{1}{K} \eta + 2a_0 + 2a_0 \eta + \hat{\nu}_0^* \right) \left(\frac{1}{K} - \hat{\nu}_0^* \right) - \frac{1}{K} \left(\frac{1}{K} + 2a_0 \right) \right] G1 \right. \\
&\quad \left. + 2 \left[\frac{1}{K} - \eta \left(\frac{1}{K} - \hat{\nu}_0^* \right) \right] (b_1 - b_0) \right\} G1. \\
&= \left\{ \left[\left(\frac{1}{K} (1 + a_1 K) + 2a_0 + 2a_0 \eta + \hat{\nu}_0^* \right) \left(\frac{1}{K} - \hat{\nu}_0^* \right) - \frac{1}{K} \left(\frac{1}{K} + 2a_0 \right) \right] G1 \right. \\
&\quad \left. + 2 \left[\frac{1}{K} - (1 + a_1 K) \left(\frac{1}{K} - \hat{\nu}_0^* \right) \right] (b_1 - b_0) \right\} G1. \\
&= \left\{ \left[\left(\frac{1}{K} + a_1 + 2a_0 + 2a_0 \eta + \hat{\nu}_0^* \right) \left(\frac{1}{K} - \hat{\nu}_0^* \right) - \frac{1}{K} \left(\frac{1}{K} + 2a_0 \right) \right] G1 \right. \\
&\quad \left. + 2 \left[\frac{1}{K} - \left(\frac{1}{K} - \hat{\nu}_0^* \right) - a_1 K \left(\frac{1}{K} - \hat{\nu}_0^* \right) \right] (b_1 - b_0) \right\} G1. \tag{A.249} \\
&= \left\{ \left[-\hat{\nu}_0^* \left(\frac{1}{K} + 2a_0 \right) + (a_1 + 2a_0 \eta) \left(\frac{1}{K} - \hat{\nu}_0^* \right) + \hat{\nu}_0^* \left(\frac{1}{K} - \hat{\nu}_0^* \right) \right] G1 \right. \\
&\quad \left. + 2 \left[(1 + a_1 K) \hat{\nu}_0^* - a_1 \right] (b_1 - b_0) \right\} G1. \\
&= \left\{ \left[(a_1 + 2a_0 \eta) \left(\frac{1}{K} - \hat{\nu}_0^* \right) - \hat{\nu}_0^* (2a_0 + \hat{\nu}_0^*) \right] G1 \right. \\
&\quad \left. + 2 \left[(1 + a_1 K) \hat{\nu}_0^* - a_1 \right] (b_1 - b_0) \right\} G1. \\
&= [V_5 G1 + 2W_5 (b_1 - b_0)] G1,
\end{aligned}$$

where

$$V_5 = (a_1 + 2a_0 \eta) \left(\frac{1}{K} - \hat{\nu}_0^* \right) - \hat{\nu}_0^* (2a_0 + \hat{\nu}_0^*) \tag{A.250}$$

and

$$W_5 = (1 + a_1 K) \hat{\nu}_0^* - a_1. \tag{A.251}$$

Now, we only need to show that $V_5 > 0$ and $W_5 > 0$.

First, since $\nu_0^*(\beta)$ is strictly decreasing, we have $\hat{\nu}_0^* = \lim_{\beta \rightarrow 0} \nu_0^*(\beta) > \nu_0^*(1) = \overline{\nu}_0^*$, thus,

$$W_5 = (1 + a_1 K) \hat{\nu}_0^* - a_1 > (1 + a_1 K) \overline{\nu}_0^* - a_1. \tag{A.252}$$

Substituting the value of $\overline{\nu}_0^*$ given by (A.185) in (A.252) we have that:

$$\begin{aligned}
W_5 &> (1 + a_1 K) \overline{\nu}_0^* - a_1 \\
&= (1 + a_1 K) \left(\frac{a_0 + a_1 + a_0 a_1 K}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \right) - a_1 \\
&= \frac{(1 + a_1 K) (a_0 + a_1 + a_0 a_1 K) - a_1 (1 + 2a_0 K + a_1 K + a_0 a_1 K^2)}{1 + 2a_0 K + a_1 K + a_0 a_1 K^2} \tag{A.253} \\
&= \frac{V_6}{W_6},
\end{aligned}$$

where

$$V_6 = (1 + a_1 K) (a_0 + a_1 + a_0 a_1 K) - a_1 (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) \quad (\text{A.254})$$

and

$$W_6 = 1 + 2a_0 K + a_1 K + a_0 a_1 K^2. \quad (\text{A.255})$$

Given the values of a_0 , a_1 and K , it's easy to see that $W_6 > 0$. Now, we calculate the value of V_6 :

$$\begin{aligned} V_6 &= (1 + a_1 K) (a_0 + a_1 + a_0 a_1 K) - a_1 (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\ &= (a_0 + a_1 + a_0 a_1 K) + a_1 K (a_0 + a_1 + a_0 a_1 K) - a_1 (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\ &= a_0 + a_1 (1 + a_0 K) + a_1 (a_0 K + a_1 K + a_0 a_1 K^2) - a_1 (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\ &= a_0 + a_1 (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) - a_1 (1 + 2a_0 K + a_1 K + a_0 a_1 K^2) \\ &= a_0 > 0. \end{aligned} \quad (\text{A.256})$$

Therefore $V_6 > 0$ and

$$W_5 > \frac{V_6}{W_6} > 0. \quad (\text{A.257})$$

So we only lack showing that $V_5 > 0$:

$$\begin{aligned} V_5 &= (a_1 + 2a_0 \eta) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* (2a_0 + \widehat{\nu}_0^*) \\ &= (a_1 + a_0 \eta) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) + a_0 \eta \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* (a_0 + \widehat{\nu}_0^*) - a_0 \widehat{\nu}_0^* \\ &= \left[(a_1 + a_0 \eta) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* (a_0 + \widehat{\nu}_0^*) \right] + a_0 \left[\eta \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* \right] \\ &= V_7 + a_0 W_7, \end{aligned} \quad (\text{A.258})$$

where

$$V_7 = (a_1 + a_0 \eta) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* (a_0 + \widehat{\nu}_0^*) \quad (\text{A.259})$$

and

$$W_7 = \eta \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^*. \quad (\text{A.260})$$

Finally, we will demonstrate that $V_7 > 0$ and $W_7 > 0$. Substituting the value of η given by (A.196) in V_7 we get:

$$\begin{aligned}
V_7 &= (a_1 + a_0\eta) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* \left(a_0 + \widehat{\nu}_0^* \right) \\
&= [a_1 + a_0(1 + a_1K)] \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* \left(a_0 + \widehat{\nu}_0^* \right) \\
&= (a_1 + a_0 + a_0a_1K) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* \left(a_0 + \widehat{\nu}_0^* \right) \\
&= \frac{1}{K} (a_1 + a_0 + a_0a_1K) - \widehat{\nu}_0^* (a_1 + a_0 + a_0a_1K) - a_0\widehat{\nu}_0^* - \widehat{\nu}_0^{*2} \\
&= \frac{1}{K} (a_1 + a_0 + a_0a_1K) - \widehat{\nu}_0^* (a_1 + 2a_0 + a_0a_1K) - \widehat{\nu}_0^{*2}.
\end{aligned} \tag{A.261}$$

we get:

$$(1 - \beta) (-2\tau + a_1\tau^2) \nu_0^2 + (\beta - 2a_0\tau - \beta a_1\tau + a_0a_1\tau^2) \nu_0 - (a_0 + a_1 - a_0a_1\tau) = 0,$$

given by (A.15), for $\tau = -K$. Now by applying the limit when $\beta \rightarrow 0$, one obtains the following equality:

$$(2K + a_1K^2) \widehat{\nu}_0^{*2} + (2a_0K + a_0a_1K^2) \widehat{\nu}_0^* - (a_0 + a_1 + a_0a_1K) = 0. \tag{A.262}$$

Therefore,

$$\begin{aligned}
&(2K + a_1K^2) \widehat{\nu}_0^{*2} + (2a_0K + a_0a_1K^2) \widehat{\nu}_0^* - (a_0 + a_1 + a_0a_1K) \\
&= (2 + a_1K) K \widehat{\nu}_0^{*2} + (2a_0 + a_0a_1K) K \widehat{\nu}_0^* - \frac{1}{K} (a_0 + a_1 + a_0a_1K) K \\
&= \left[(2 + a_1K) \widehat{\nu}_0^{*2} + (2a_0 + a_0a_1K) \widehat{\nu}_0^* - \frac{1}{K} (a_0 + a_1 + a_0a_1K) \right] K \\
&= \left[(1 + 1 + a_1K) \widehat{\nu}_0^{*2} + (a_1 + 2a_0 + a_0a_1K - a_1) \widehat{\nu}_0^* - \frac{1}{K} (a_0 + a_1 + a_0a_1K) \right] K \\
&= \left[\widehat{\nu}_0^{*2} + (a_1 + 2a_0 + a_0a_1K) \widehat{\nu}_0^* - \frac{1}{K} (a_0 + a_1 + a_0a_1K) + (1 + a_1K) \widehat{\nu}_0^{*2} - a_1\widehat{\nu}_0^* \right] K \\
&= 0.
\end{aligned} \tag{A.263}$$

Since $K > 0$, we have that

$$\widehat{\nu}_0^{*2} + (a_1 + 2a_0 + a_0a_1K) \widehat{\nu}_0^* - \frac{1}{K} (a_0 + a_1 + a_0a_1K) + (1 + a_1K) \widehat{\nu}_0^{*2} - a_1\widehat{\nu}_0^* = 0, \tag{A.264}$$

thus,

$$(1 + a_1K) \widehat{\nu}_0^{*2} - a_1\widehat{\nu}_0^* = \frac{1}{K} (a_0 + a_1 + a_0a_1K) - (a_1 + 2a_0 + a_0a_1K) \widehat{\nu}_0^* - \widehat{\nu}_0^{*2}. \tag{A.265}$$

Applying equality (A.265) to (A.261) we have that:

$$\begin{aligned}
 V_7 &= \frac{1}{K} (a_0 + a_1 + a_0 a_1 K) - (a_1 + 2a_0 + a_0 a_1 K) \widehat{\nu}_0^* - \widehat{\nu}_0^{*2} \\
 &= (1 + a_1 K) \widehat{\nu}_0^{*2} - a_1 \widehat{\nu}_0^* \\
 &= \left[(1 + a_1 K) \widehat{\nu}_0^* - a_1 \right] \widehat{\nu}_0^* \\
 &= W_5 \widehat{\nu}_0^*.
 \end{aligned} \tag{A.266}$$

Now recalling that $W_5 > 0$ and $\widehat{\nu}_0^* > 0$, we have:

$$V_7 = W_5 \widehat{\nu}_0^* > 0. \tag{A.267}$$

So we only need to show that $W_7 > 0$. To do this, we plug the value of η given by (A.196) in W_7 to get:

$$\begin{aligned}
 W_7 &= \eta \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* \\
 &= (1 + a_1 K) \left(\frac{1}{K} - \widehat{\nu}_0^* \right) - \widehat{\nu}_0^* \\
 &= \frac{1}{K} (1 + a_1 K) - (1 + a_1 K) \widehat{\nu}_0^* - \widehat{\nu}_0^* \\
 &= \frac{1}{K} (1 + a_1 K) - (2 + a_1 K) \widehat{\nu}_0^*.
 \end{aligned} \tag{A.268}$$

Using relationship (A.262) we have that:

$$\begin{aligned}
 &(2K + a_1 K^2) \widehat{\nu}_0^{*2} + (2a_0 K + a_0 a_1 K^2) \widehat{\nu}_0^* - (a_0 + a_1 + a_0 a_1 K) \\
 &= \left[(2 + a_1 K) \widehat{\nu}_0^{*2} + (2a_0 + a_0 a_1 K) \widehat{\nu}_0^* - \frac{1}{K} (a_0 + a_1 + a_0 a_1 K) \right] K \\
 &= \left\{ (2 + a_1 K) \widehat{\nu}_0^{*2} + a_0 (2 + a_1 K) \widehat{\nu}_0^* - \frac{1}{K} [a_1 + a_0 (1 + a_1 K)] \right\} K \\
 &= \left\{ (2 + a_1 K) \widehat{\nu}_0^{*2} + a_0 (2 + a_1 K) \widehat{\nu}_0^* - \frac{1}{K} a_1 - \frac{1}{K} a_0 (1 + a_1 K) \right\} K \\
 &= \left\{ (2 + a_1 K) \widehat{\nu}_0^{*2} - \frac{1}{K} a_1 + a_0 \left[(2 + a_1 K) \widehat{\nu}_0^* - \frac{1}{K} (1 + a_1 K) \right] \right\} K.
 \end{aligned} \tag{A.269}$$

Since $K > 0$, we have that:

$$(2 + a_1 K) \widehat{\nu}_0^{*2} - \frac{1}{K} a_1 + a_0 \left[(2 + a_1 K) \widehat{\nu}_0^* - \frac{1}{K} (1 + a_1 K) \right] = 0, \tag{A.270}$$

which implies

$$(2 + a_1 K) \widehat{\nu}_0^{*2} - \frac{1}{K} a_1 = -a_0 \left[(2 + a_1 K) \widehat{\nu}_0^* - \frac{1}{K} (1 + a_1 K) \right]. \tag{A.271}$$

As $a_0 > 0$, one gets:

$$\frac{1}{a_0} \left[(2 + a_1 K) \widehat{\nu}_0^{*2} - \frac{1}{K} a_1 \right] = \frac{1}{K} (1 + a_1 K) - (2 + a_1 K) \widehat{\nu}_0^*. \tag{A.272}$$

Now, applying equality (A.272) to (A.268), we deduce:

$$\begin{aligned} W_7 &= \frac{1}{K} (1 + a_1 K) - (2 + a_1 K) \widehat{\nu}_0^* \\ &= \frac{1}{a_0} \left[(2 + a_1 K) \widehat{\nu}_0^{*2} - \frac{1}{K} a_1 \right] \\ &= \frac{U_8}{a_0}, \end{aligned} \quad (\text{A.273})$$

where

$$U_8 = (2 + a_1 K) \widehat{\nu}_0^{*2} - \frac{1}{K} a_1. \quad (\text{A.274})$$

Finally, let's suppose that

$$U_8 \leq 0. \quad (\text{A.275})$$

Substituting the value of $\widehat{\nu}_0^*$, given by (A.135), in U_8 we have:

$$\begin{aligned} U_8 &= (2 + a_1 K) \widehat{\nu}_0^{*2} - \frac{1}{K} a_1 \\ &= (2 + a_1 K) \left[\frac{2(a_0 + a_1 + a_0 a_1 K)}{(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)}} \right]^2 - \frac{1}{K} a_1 \\ &= \left\{ (2 + a_1 K) [2(a_0 + a_1 + a_0 a_1 K)]^2 - \frac{1}{K} a_1 \left[(2a_0 K + a_0 a_1 K^2) \right. \right. \\ &\quad \left. \left. + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]^2 \right\} \\ &\quad / \left[(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]^2 \\ &= \frac{V_9}{W_9} \leq 0, \end{aligned} \quad (\text{A.276})$$

where

$$\begin{aligned} V_9 &= (2 + a_1 K) [2(a_0 + a_1 + a_0 a_1 K)]^2 - \frac{1}{K} a_1 \left[(2a_0 K + a_0 a_1 K^2) \right. \\ &\quad \left. + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]^2 \end{aligned} \quad (\text{A.277})$$

and

$$W_9 = \left[(2a_0 K + a_0 a_1 K^2) + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]^2. \quad (\text{A.278})$$

For any given values of a_0 , a_1 and K , one has $W_9 > 0$. Therefore, equations (A.276) implies

$$V_9 \leq 0. \quad (\text{A.279})$$

Hence

$$\begin{aligned}
V_9 &= (2 + a_1 K) [2(a_0 + a_1 + a_0 a_1 K)]^2 - \frac{1}{K} a_1 \left[(2a_0 K + a_0 a_1 K^2) \right. \\
&\quad \left. + \sqrt{(2a_0 K + a_0 a_1 K^2)^2 + 4(2K + a_1 K^2)(a_0 + a_1 + a_0 a_1 K)} \right]^2 \\
&= 4(2 + a_1 K) [a_1 + a_0(1 + a_1 K)]^2 - \frac{1}{K} a_1 \left[a_0 K(2 + a_1 K) \right. \\
&\quad \left. + \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \right]^2 \\
&= 4a_1^2(2 + a_1 K) + 8a_0 a_1(2 + a_1 K)(1 + a_1 K) + 4a_0^2(2 + a_1 K)(1 + a_1 K)^2 \\
&\quad - a_1 a_0^2 K(2 + a_1 K)^2 - a_1(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\} \\
&\quad - 2a_0 a_1(2 + a_1 K) \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \\
&= (2 + a_1 K) \left[4a_1^2 + 8a_0 a_1(1 + a_1 K) + 4a_0^2(1 + a_1 K)^2 \right. \\
&\quad \left. - a_1 a_0^2 K(2 + a_1 K) - a_1 a_0^2 K(2 + a_1 K) - 4a_1[a_1 + a_0(1 + a_1 K)] \right] \\
&\quad - 2a_0 a_1(2 + a_1 K) \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \\
&= (2 + a_1 K) \left[4a_0 a_1(1 + a_1 K) + 4a_0^2(1 + a_1 K)^2 - 2a_1 a_0^2 K(2 + a_1 K) \right] \\
&\quad - 2a_0 a_1(2 + a_1 K) \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \\
&= a_0(2 + a_1 K) \left[4a_1(1 + a_1 K) + 4a_0(1 + 2a_1 K + a_1^2 K^2) - 2a_0 a_1 K(2 + a_1 K) \right] \\
&\quad - 2a_0 a_1(2 + a_1 K) \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \\
&= a_0(2 + a_1 K) (4a_0 + 4a_1 + 4a_0 a_1 K + 4a_1^2 K + 2a_0 a_1^2 K^2) \\
&\quad - 2a_0 a_1(2 + a_1 K) \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \\
&= 2a_0(2 + a_1 K) \left[(2a_0 + 2a_1 + 2a_0 a_1 K + 2a_1^2 K + a_0 a_1^2 K^2) \right. \\
&\quad \left. - a_1 \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \right] \\
&= 2a_0(2 + a_1 K) \left[2a_0 + a_1 K \left(2\frac{1}{K} + 2a_0 + 2a_1 + a_0 a_1 K \right) \right. \\
&\quad \left. - a_1 \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \right] \leq 0.
\end{aligned} \tag{A.280}$$

Since $2a_0(2 + a_1 K) > 0$, then, from (A.280) the following condition must be met:

$$\begin{aligned}
&2a_0 + a_1 K \left(2\frac{1}{K} + 2a_0 + 2a_1 + a_0 a_1 K \right) \\
&- a_1 \sqrt{K(2 + a_1 K) \{a_0^2 K(2 + a_1 K) + 4[a_1 + a_0(1 + a_1 K)]\}} \leq 0.
\end{aligned} \tag{A.281}$$

Therefore,

$$\begin{aligned} & 2a_0 + a_1K \left(2\frac{1}{K} + 2a_0 + 2a_1 + a_0a_1K\right) \\ & \leq a_1 \sqrt{K(2 + a_1K) \{a_0^2K(2 + a_1K) + 4[a_1 + a_0(1 + a_1K)]\}}. \end{aligned} \quad (\text{A.282})$$

Since both sides of inequality (A.282) are positive, then,

$$\begin{aligned} & [2a_0 + a_1K \left(2\frac{1}{K} + 2a_0 + 2a_1 + a_0a_1K\right)]^2 \\ & \leq a_1^2 [K(2 + a_1K) \{a_0^2K(2 + a_1K) + 4[a_1 + a_0(1 + a_1K)]\}], \end{aligned} \quad (\text{A.283})$$

which leads to

$$\begin{aligned} 0 & \leq a_1^2 [K(2 + a_1K) \{a_0^2K(2 + a_1K) + 4[a_1 + a_0(1 + a_1K)]\}] \\ & \quad - [2a_0 + a_1K \left(2\frac{1}{K} + 2a_0 + 2a_1 + a_0a_1K\right)]^2 \\ & = a_1^2 K(2 + a_1K) (2a_0^2K + a_0^2a_1K^2 + 4a_1 + 4a_0 + 4a_0a_1K) \\ & \quad - (2a_0 + 2a_1 + 2a_0a_1K + 2a_1^2K + a_0a_1^2K^2)^2 \\ & = a_1^2 K(2 + a_1K) [4(a_0 + a_1 + a_0a_1K) + a_0^2K(2 + a_1K)] \\ & \quad - [2(a_0 + a_1 + a_0a_1K) + a_1^2K(2 + a_0K)]^2 \\ & = 4a_1^2K(a_0 + a_1 + a_0a_1K) [(2 + a_1K) - (2 + a_0K)] \\ & \quad + a_1^2K^2 [a_0^2(2 + a_1K)^2 - a_1^2(2 + a_0K)^2] \\ & \quad - 4(a_0 + a_1 + a_0a_1K)^2 \\ & = 4a_1^2K^2(a_0 + a_1 + a_0a_1K)(a_1 - a_0) \\ & \quad + a_1^2K^2(2a_0 + a_0a_1K + 2a_1 + a_0a_1K)(2a_0 + a_0a_1K - 2a_1 - a_0a_1K) \\ & \quad - 4(a_0 + a_1 + a_0a_1K)^2 \\ & = 4a_1^2K^2(a_0 + a_1 + a_0a_1K)(a_1 - a_0) \\ & \quad - 4a_1^2K^2(a_0 + a_1 + a_0a_1K)(a_1 - a_0) \\ & \quad - 4(a_0 + a_1 + a_0a_1K)^2 \\ & = -4(a_0 + a_1 + a_0a_1K)^2. \end{aligned} \quad (\text{A.284})$$

Thus,

$$-4(a_0 + a_1 + a_0a_1K)^2 \geq 0,$$

which cannot happen due to $a_0 + a_1 + a_0a_1K > 0$.

Therefore, the assumption was false, so $U_8 > 0$. Thus,

$$W_7 = \frac{U_8}{a_0} > 0, \quad (\text{A.285})$$

then,

$$V_5 = V_7 + a_0 W_7 > 0, \quad (\text{A.286})$$

which proves that

$$\mathcal{P}_1 > [V_5 G_1 + 2W_5(b_1 - b_0)] G_1 \geq V_5 G_1^2 > 0. \quad (\text{A.287})$$

So we have that $\mathcal{P}_1 > 0$, $\mathcal{Q}_1 > 0$ and $\mathcal{R}_1 > 0$, then,

$$V_2 = \mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1 > 0, \quad (\text{A.288})$$

and therefore,

$$\lim_{\beta \rightarrow 0} (\pi_1^c - \pi_1^*)(\beta) = \frac{1}{2} \frac{V_2}{W_2} > 0, \quad (\text{A.289})$$

which proves (A.184). The proof of the theorem is complete ■

APPENDIX B

PROOFS OF CHAPTER 2

Theorem 2.2. *Let the number of oligopoly producers be at least three, i.e., $n \geq 3$, then, under assumptions **A2.1-A2.3**, there exists an interior equilibrium. Moreover, if the number of producers is two, i.e., $n = 2$, in addition to assumptions **A2.1-A2.3**, suppose that there exists an $\varepsilon > 0$ such that $G'(p) \leq -\varepsilon$ for all $p > 0$, then, there exists interior equilibrium.*

Proof. For any given set of influence coefficients $\nu = (\nu_1 \dots \nu_n) \geq 0$, by theorem 2.1, there exists the unique exterior equilibrium $(p(\nu), q_1(\nu), \dots, q_n(\nu))$.

Now, we define the following functions:

$$F_i(\nu) = \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{\nu_j + f_j''(q_j(\nu))} - G'(p(\nu))}, \quad i = 1, \dots, n. \quad (\text{B.1})$$

These functions are well-defined and continuous with respect to $\nu = (\nu_1 \dots \nu_n) \geq 0$, due to assumptions **A2.1** and **A2.2**.

Therefore, the function $H = (F_1 \dots, F_n) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is also continuous.

Next, we define the value $s = \max\{f_i''(q_i) \mid 0 \leq q_i \leq G(p_0), i = 1, \dots, n\} > 0$.

For $n \geq 3$, if $0 \leq \nu_i \leq \frac{s}{n-2}$ for all $i = 1, \dots, n$, we have:

$$\begin{aligned}
 0 \leq F_i(\nu) &= \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{\nu_j + f_j''(q_j(\nu))} - G'(p(\nu))} \leq \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{\nu_j + f_j''(q_j(\nu))}} \\
 &\leq \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{\frac{s}{n-2} + s}} = \frac{1}{\frac{n-1}{\frac{s}{n-2} + s}} = \frac{s}{n-2}, \quad i = 1, \dots, n.
 \end{aligned} \tag{B.2}$$

Thus, the function $H = (F_1, \dots, F_n)$ maps the convex compact subset $\left[0, \frac{s}{n-2}\right]^n$ onto itself. Therefore, by Brouwer's fixed-point theorem, H has a fixed point, i.e., there exists $\nu^* = (\nu_1^* \dots \nu_n^*) \geq 0$ such that $F_i(\nu^*) = \nu_i^*$ for all $i = 1, \dots, n$.

On the other hand, for $n = 2$ and $G'(p) \leq -\varepsilon$ for some $\varepsilon > 0$, if $0 \leq \nu_i \leq \frac{1}{\varepsilon}$ for all $i = 1, \dots, n$, we have:

$$\begin{aligned}
 0 \leq F_1(\nu) &= \frac{1}{\frac{1}{\nu_2 + f_2''(q_2(\nu))} - G'(p(\nu))} \leq \frac{1}{-G'(p(\nu))} \leq \frac{1}{\varepsilon}, \\
 0 \leq F_2(\nu) &= \frac{1}{\frac{1}{\nu_1 + f_1''(q_1(\nu))} - G'(p(\nu))} \leq \frac{1}{-G'(p(\nu))} \leq \frac{1}{\varepsilon}.
 \end{aligned} \tag{B.3}$$

Thus, the function $H = (F_1, F_2)$ maps the convex compact subset $\left[0, \frac{1}{\varepsilon}\right]^2$ onto itself, then, again by Brouwer's fixed-point theorem, H has a fixed point $F_i(\nu^*) = \nu_i^*$, $i = 1, 2$.

By the definition of the functions (B.1), the influence coefficients $\nu^* = (\nu_1^* \dots \nu_n^*) \geq 0$, given by Brouwer's fixed-point theorem, satisfy the consistency criterion 2.2 and, therefore, the vector $(p(\nu^*), q_1(\nu^*), \dots, q_n(\nu^*), \nu_1^*, \dots, \nu_n^*)$ is the interior equilibrium. The proof is complete ■

Theorem 2.6. *Suppose that the stronger assumption **A2.4** is true, together with **A2.1** and **A2.3**, and suppose that the function G is concave. Then, the consistency criterion for the original oligopoly is a necessary and sufficient condition for the collection of influence conjectures $\nu = (\nu_1, \dots, \nu_n)$ to produce Nash equilibrium in the meta-game.*

Proof. Note that the necessity is a particular case of theorem 2.5, thus, to prove theorem 2.6, we just need to establish the sufficiency.

We assume **A2.4**, that is, for all i , the cost functions f_i are quadratic (and strictly convex) with $f_i(0) = 0$, $f'_i(0) > 0$, and $f''_i(0) > 0$, i.e.,

$$f_i(q_i) = \frac{1}{2}a_i q_i^2 + b_i q_i,$$

where $a_i > 0$, $b_i > 0$, $i = 1, \dots, n$. Now we are in a position to demonstrate that in this specific case, each interior equilibrium $(p^*; q_1^*, \dots, q_n^*; \nu_1^*, \dots, \nu_n^*)$ of the original oligopoly provides the Nash equilibrium in the meta-game $\Gamma = (N, V, \Pi, D)$. Namely, the consistent conjectures (influence coefficients) $\nu^* = (\nu_1^*, \dots, \nu_n^*)$ satisfying (2.15) form the Nash equilibrium in the meta-game.

Indeed, first of all, equations (2.15) in this particular case are reduced to the system

$$\nu_i^* = \frac{1}{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k} - G'(p^*)}, \quad i = 1, \dots, n, \quad (\text{B.4})$$

which clearly implies that all components of the vector ν^* are positive: $\nu_i^* > 0$, $i = 1, \dots, n$.

Next, equations

$$\begin{aligned} \frac{\partial \pi_i}{\partial \nu_i} &= \frac{q_i^2}{\nu_i + f'_i(q_i)} \left[\frac{1}{\sum_{k=1}^n \frac{1}{\nu_k + f''_k(q_k)} - G'(p)} - \frac{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k + f''_k(q_k)} - G'(p)}{\sum_{k=1}^n \frac{1}{\nu_k + f''_k(q_k)} - G'(p)} \nu_i \right] \\ &= \frac{q_i^2}{\nu_i + f'_i(q_i)} \frac{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k + f''_k(q_k)} - G'(p)}{\sum_{k=1}^n \frac{1}{\nu_k + f''_k(q_k)} - G'(p)} \left[\frac{1}{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k + f''_k(q_k)} - G'(p)} - \nu_i \right] \\ &= 0, \quad i = 1, \dots, n. \end{aligned} \quad (\text{B.5})$$

guarantee that the first-order optimality conditions for the meta-game payoff functions hold:

$$\frac{\partial \pi_i}{\partial \nu_i}(\nu^*) = 0, \quad i = 1, \dots, n. \quad (\text{B.6})$$

Therefore, the value ν_i^* may be the maximum point of the i -th producer's payoff function

$$\tilde{\pi}_i(\nu_i) \equiv \pi(\nu_i, \nu_{-i}^*), \quad i = 1, \dots, n, \quad (\text{B.7})$$

where $\nu_{-i}^* = (\nu_1^*, \dots, \nu_{i-1}^*, \nu_{i+1}^*, \dots, \nu_n^*)$. In order to establish the maximum point property, we are going to fix an arbitrary i and to show that the function $\tilde{\pi}_i = \tilde{\pi}_i(\nu_i)$:

- (a) doesn't increase along the ray $(\nu_i^*, +\infty)$,
- (b) doesn't decrease in the interval $(0, \nu_i^*)$.

In order to prove (a), taking into account (B.5), it suffices to show that

$$\frac{1}{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k}} - (\nu_i^* + \delta) \leq 0, \quad \forall \delta > 0. \quad (\text{B.8})$$

By inverting both sides of the consistency equation (B.4) one gets

$$\frac{1}{\nu_i^*} = \sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k} - G'(p^*), \quad (\text{B.9})$$

which clearly implies the relationships

$$\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k} = \frac{1}{\nu_i^*} + G'(p^*) = \frac{1 + \nu_i^* G'(p^*)}{\nu_i^*} > 0. \quad (\text{B.10})$$

Making use of (B.10), rewrite the left-hand side of (B.8) in the form

$$\begin{aligned}
& \frac{1}{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k}} - (\nu_i^* + \delta) \\
&= \frac{1}{\frac{1}{\nu_i^*} + G'(p^*) - G'(p(\nu_i^* + \delta, \nu_{-i}^*))} - (\nu_i^* + \delta) \\
&= \frac{\nu_i^*}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* + \delta, \nu_{-i}^*))} - (\nu_i^* + \delta) \\
&= \frac{(\nu_i^*)^2 [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] - \delta + \nu_i^* \delta [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)]}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* + \delta, \nu_{-i}^*))} \\
&= \frac{\nu_i^* [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* + \delta) - \delta}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* + \delta, \nu_{-i}^*))}.
\end{aligned} \tag{B.11}$$

Since $1 + \nu_i^* G'(p^*) > 0$ from (B.10), and $-\nu_i^* G'(p(\nu_i^* + \delta, \nu_{-i}^*)) \geq 0$ by assumption **A2.1**, then the denominator of (B.11) is strictly positive, thus the sign of ratio (B.11) is determined by that of its numerator. Now since the derivative $G'(p)$ is non-increasing by hypothesis, and $\frac{\partial p}{\partial \nu_i} > 0$ by (2.11), it isn't difficult to show that $[G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] \leq 0$, hence the numerator of (B.11) is strictly negative for any $\delta > 0$:

$$\nu_i^* [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* + \delta) - \delta < 0, \quad \forall \delta > 0. \tag{B.12}$$

The latter brings about the desired inequality

$$\frac{d\tilde{\pi}_i}{d\nu_i}(\nu_i, \nu_{-i}^*) < 0, \quad \forall \nu_i > \nu_i^*, \tag{B.13}$$

which finishes the proof of (a).

Now to demonstrate that (b) is also true, again taking into account (B.5), it is enough to check that

$$\frac{1}{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k}} - (\nu_i^* - \delta) \geq 0, \quad \forall \delta \text{ such that } 0 < \delta < \nu_i^*. \tag{B.14}$$

Again employing (B.10) yields the following transformation of the left-hand side of (B.14):

$$\begin{aligned}
& \frac{1}{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k}} - (\nu_i^* - \delta) \\
&= \frac{1}{\frac{1}{\nu_i^*} + G'(p^*) - G'(p(\nu_i^* - \delta, \nu_{-i}^*))} - (\nu_i^* - \delta) \\
&= \frac{\nu_i^*}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* - \delta, \nu_{-i}^*))} - (\nu_i^* - \delta) \\
&= \frac{(\nu_i^*)^2 [G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)] + \delta - \nu_i^* \delta [G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)]}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* - \delta, \nu_{-i}^*))} \\
&= \frac{\nu_i^* [G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* - \delta) + \delta}{1 + \nu_i^* G'(p^*) - \nu_i^* G'(p(\nu_i^* - \delta, \nu_{-i}^*))}.
\end{aligned} \tag{B.15}$$

Similar to the proof of case (a), the denominator of the fraction (B.15) is strictly positive, hence, the fraction's sign coincides with that of its numerator. Again, since the derivative $G'(p)$ is non-increasing by hypothesis, and $\frac{\partial p}{\partial \nu_i} > 0$ by (2.11), it is evident that $[G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)] \geq 0$, hence, the numerator of (B.15) is strictly positive for any $0 < \delta < \nu_i^*$:

$$\nu_i^* [G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* - \delta) + \delta > 0, \quad \forall \delta \text{ that } 0 < \delta < \nu_i^*, \tag{B.16}$$

which deduces the desired inequality:

$$\frac{d\tilde{\pi}_i}{d\nu_i}(\nu_i, \nu_{-i}^*) > 0, \quad \forall \nu_i < \nu_i^*. \tag{B.17}$$

Therefore, the proof of (b) is also completed.

Now we can conclude that the Nash equilibrium condition has been established:

$$\pi_i(\nu^*) = \max_{\nu_i > 0} \pi_i(\nu_i, \nu_{-i}^*), \text{ for any } i \in \{1 \dots, n\}, \tag{B.18}$$

which finishes the proof of theorem 2.6 ■

Theorem 2.7. *Suppose that apart from assumptions **A2.1**, **A2.3** and **A2.4**, the regular demand function's derivative is Lipschitz continuous. In more detail, for $n \geq 3$ assume*

that for any $p_1 > 0$ and $p_2 > 0$ the following inequality holds:

$$|G'(p_1) - G'(p_2)| \leq \frac{1}{2s^2 G(p_0)} |p_1 - p_2|, \quad (2.19)$$

where $s = \max\{a_1, \dots, a_n\}$, and the price p_0 is the one defined in the assumption **A2.3**. Next, if $n = 2$ (duopoly), we again suppose that there exists $\varepsilon > 0$ such that $G'(p) \leq -\varepsilon$ for all $p > 0$, and the Lipschitz continuity of the demand function is described in the form:

$$|G'(p_1) - G'(p_2)| \leq \frac{2}{\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\}\right)^2 G(p_0)} |p_1 - p_2|, \quad \forall p_1, p_2 > 0. \quad (2.20)$$

Then, the consistency criterion for the original oligopoly is a necessary and sufficient condition for the collection of influence conjectures $\nu = (\nu_1 \dots \nu_n)$ to be the Nash equilibrium in the meta-game.

Proof. Again, the necessity is just a particular case of theorem 2.5, then, we proceed to show the sufficiency.

Just like in the proof of theorem 2.6, we need to establish that the i -th producer's payoff function

$$\tilde{\pi}_i(\nu_i) \equiv \pi(\nu_i, \nu_{-i}^*), \quad i = 1, \dots, n, \quad (B.7)$$

has a maximum point at $\nu_i = \nu_i^*$ for a fixed value of i , for which, again, it will suffice to show that:

$$(a) \quad \nu_i^* [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* + \delta) - \delta < 0, \quad \forall 0 < \delta < s, \quad (B.19)$$

$$(b) \quad \nu_i^* [G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* - \delta) + \delta > 0, \quad \forall 0 < \delta < \nu_i^*, \quad (B.20)$$

where $s = \max\{a_1, \dots, a_n\} > 0$. From the proof of theorem 2.2 (Bulavsky (1997)) we have:

$$0 \leq \nu_i^* \leq \frac{s}{n-2}, \quad i = 1, \dots, n. \quad (B.21)$$

Now, from assumption (2.19) and the fact that $\frac{\partial p}{\partial \nu_i} > 0$, it follows that

$$\begin{aligned}
& \nu_i^* [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* + \delta) - \delta \\
& \leq \nu_i^* |G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)| (\nu_i^* + \delta) - \delta \\
& \leq \nu_i^* (\nu_i^* + \delta) \frac{1}{2s^2 G(p_0)} |p(\nu_i^* + \delta, \nu_{-i}^*) - p^*| - \delta \\
& \leq \nu_i^* (\nu_i^* + \delta) \frac{1}{2s^2 G(p_0)} (p(\nu_i^* + \delta, \nu_{-i}^*) - p^*) - \delta.
\end{aligned} \tag{B.22}$$

By the mean value theorem here exists a value $\hat{\nu}_i$ such that $\nu_i < \hat{\nu}_i < \nu_i + \delta$ and

$$p(\nu_i^* + \delta, \nu_{-i}^*) - p^* = \delta \frac{\partial p}{\partial \nu_i}(\hat{\nu}_i, \nu_{-i}^*). \tag{B.23}$$

Using (2.11) we get

$$\begin{aligned}
\frac{\partial p}{\partial \nu_i}(\hat{\nu}_i, \nu_{-i}^*) &= \frac{\frac{q_i(p(\hat{\nu}_i, \nu_{-i}^*), (\hat{\nu}_i, \nu_{-i}^*))}{\hat{\nu}_i + a_i}}{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k} + \frac{1}{\hat{\nu}_i + a_i} - G'(p(\hat{\nu}_i, \nu_{-i}^*))} \leq \frac{\frac{q_i(p(\hat{\nu}_i, \nu_{-i}^*), (\hat{\nu}_i, \nu_{-i}^*))}{\hat{\nu}_i + a_i}}{\sum_{\substack{k=1 \\ k \neq i}}^n \frac{1}{\nu_k^* + a_k} + \frac{1}{\hat{\nu}_i + a_i}} \\
&\leq \frac{\frac{q_i(p(\hat{\nu}_i, \nu_{-i}^*), (\hat{\nu}_i, \nu_{-i}^*))}{\hat{\nu}_i + a_i}}{\frac{1}{\hat{\nu}_i + a_i}} = q_i(p(\hat{\nu}_i, \nu_{-i}^*), (\hat{\nu}_i, \nu_{-i}^*)) \leq G(p_0).
\end{aligned} \tag{B.24}$$

Applying (B.23) and (B.24) to (B.22) we find:

$$\begin{aligned}
\nu_i^* [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* + \delta) - \delta &\leq \nu_i^* (\nu_i^* + \delta) \frac{1}{2s^2} \delta - \delta \\
&= \left[\nu_i^* (\nu_i^* + \delta) \frac{1}{2s^2} - 1 \right] \delta,
\end{aligned} \tag{B.25}$$

moreover, since $0 < \nu_i^* \leq s$ and $0 < \delta < s$, it follows that

$$\nu_i^* [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* + \delta) - \delta \leq \left[\nu_i^* (\nu_i^* + \delta) \frac{1}{2s^2} - 1 \right] \delta < 0, \tag{B.26}$$

which proves (a).

Analogous to the previous case, we can find that

$$\nu_i^* [G'(p^*) - G'(p(\nu_i^* - \delta, \nu_{-i}^*))] (\nu_i^* - \delta) - \delta \leq \left[\nu_i^* (\nu_i^* - \delta) \frac{1}{2s^2} - 1 \right] \delta, \quad (\text{B.27})$$

and, since $0 < \delta < \nu_i^* \leq s$, we have

$$\nu_i^* [G'(p^*) - G'(p(\nu_i^* - \delta, \nu_{-i}^*))] (\nu_i^* - \delta) - \delta \leq \left[\nu_i^* (\nu_i^* - \delta) \frac{1}{2s^2} - 1 \right] \delta < 0, \quad (\text{B.28})$$

then

$$\nu_i^* [G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* - \delta) + \delta > 0. \quad (\text{B.29})$$

Therefore, the vector ν^* is Nash equilibrium for $n \geq 3$.

Finally, let $n = 2$. We can repeat the steps for the case $n \geq 3$ to get the following inequality:

$$\begin{aligned} & \nu_i^* [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* + \delta) - \delta \\ & \leq \left[\nu_i^* (\nu_i^* + \delta) \frac{2}{\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\} \right)^2} - 1 \right] \delta. \end{aligned} \quad (\text{B.30})$$

From

$$\begin{aligned} \nu_i & \leq \frac{1}{2} \left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + \max\{a_1, a_2\} \right) + \max\{a_1, a_2\} \\ & = \frac{1}{2} \left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\} \right), \quad i = 1, 2. \end{aligned} \quad (\text{B.31})$$

we have

$$0 < \nu_i^* \leq \frac{1}{2} \left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\} \right) \quad (\text{B.32})$$

and

$$0 < \delta < \max\{a_1, a_2\} < \frac{1}{2} \left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\} \right), \quad (\text{B.33})$$

thus

$$\begin{aligned} & \nu_i^* [G'(p(\nu_i^* + \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* + \delta) - \delta \\ & \leq \left[\nu_i^* (\nu_i^* + \delta) \frac{2}{\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\} \right)^2} - 1 \right] \delta < 0. \end{aligned} \quad (\text{B.34})$$

which finally proves (a).

Analogously, to prove (b), it is easy to show that

$$\begin{aligned} & \nu_i^* [G'(p(\nu_i^* - \delta, \nu_{-i}^*)) - G'(p^*)] (\nu_i^* - \delta) + \delta \\ & \geq \left[1 - \nu_i^* (\nu_i^* - \delta) \frac{2}{\left(\frac{a_1 + a_2}{\varepsilon \min\{a_1, a_2\}} + 3 \max\{a_1, a_2\} \right)^2} \right] \delta > 0. \end{aligned} \quad (\text{B.35})$$

The theorem has been proved ■

APPENDIX C

PROOFS OF CHAPTER 3

Theorem 3.1. *The quadratic programming problem (3.11)-(3.15) is convex and any of its solutions provides the Nash equilibrium for the non-cooperative game (3.6)-(3.10).*

Proof. First, in order to prove that the quadratic programming problem (3.11)-(3.15) is convex, we only need to prove that the symmetric matrix associated with the quadratic objective function (3.12) is positive semidefinite, i.e., we need to prove that

$$\sum_{k \in K} \sum_{\ell \in K \setminus \{k\}} \sum_{a \in A} \frac{1}{2} d_a x_a^k x_a^\ell + \sum_{k \in K} \sum_{a \in A} d_a (x_a^k)^2 \geq 0, \quad \forall x \in \mathbb{R}^{M\kappa}. \quad (\text{C.1})$$

Indeed, let $x \in \mathbb{R}^{M\kappa}$, then, we have that

$$\begin{aligned} & \sum_{k \in K} \sum_{\ell \in K \setminus \{k\}} \sum_{a \in A} \frac{1}{2} d_a x_a^k x_a^\ell + \sum_{k \in K} \sum_{a \in A} d_a (x_a^k)^2 \\ &= \sum_{k \in K} \sum_{a \in A} \frac{1}{2} d_a (x_a^k)^2 + \sum_{k \in K} \sum_{\ell \in K} \sum_{a \in A} \frac{1}{2} d_a x_a^k x_a^\ell \\ &= \sum_{k \in K} \sum_{a \in A} \frac{1}{2} d_a (x_a^k)^2 + \sum_{a \in A} \frac{1}{2} d_a \left(\sum_{k \in K} \sum_{\ell \in K} x_a^k x_a^\ell \right) \\ &= \sum_{k \in K} \sum_{a \in A} \frac{1}{2} d_a (x_a^k)^2 + \sum_{a \in A} \frac{1}{2} d_a \left(\sum_{k \in K} x_a^k \right)^2. \end{aligned} \quad (\text{C.2})$$

Since all of the congestion coefficients d_a are nonnegative, we can easily see that (C.2) is also nonnegative. Moreover, if all congestion coefficients d_a are strictly positive, then, (C.2) is also strictly positive and the quadratic programming problem (3.11)-(3.15) is strictly convex so it has a unique solution.

Now, we will prove that any solution of (3.11)-(3.15) provides the Nash equilibrium for (3.6)-(3.10). In order to do this, we rewrite both problems in their matrix forms.

Let $\{t_a \mid a \in A_1\}$ satisfy (3.4) and (3.5), then, we can consider the vector $z \in R^M$ whose a -th component is given by c_a if $a \in A_2$ and by $t_a + c_a$ if $a \in A_1$. Thus, the Nash equilibrium problem (3.6)-(3.10) is given as follows:

$$x^k \in \Psi_k(t, x^{-k}), \quad \forall k \in K, \quad (\text{C.3})$$

where

$$\Psi_k(t, x^{-k}) = \underset{x^k}{\text{Argmin}} \quad f_k(x^k) = z^T x^k + \sum_{\ell \in K \setminus \{k\}} x^{k^T} H x^\ell + x^{k^T} H x^k, \quad (\text{C.4})$$

$$\text{subject to} \quad B x^k = b^k, \quad (\text{C.5})$$

$$x^k \leq q - \sum_{\ell \in K \setminus \{k\}} x^\ell, \quad (\text{C.6})$$

$$x^k \geq 0. \quad (\text{C.7})$$

Here, the matrix H is the diagonal matrix $M \times M$ matrix corresponding to the congestion factors, i.e., the a -th diagonal element of H is d_a . The matrix $B \in \mathbb{R}^{\eta \times M}$ and the vector $b^k \in \mathbb{R}^\eta$ corresponds to the equality constraints (3.8) (the matrix B depends solely upon the network so it is the same for any commodity k), and the vector $q \in \mathbb{R}^M$ has the capacity upper bounds q_a , $a \in A$, as its components. Using the above notation, the quadratic programming problem (3.11)-(3.15) is given by:

$$x \in \Psi(t), \quad (\text{C.8})$$

where

$$\Psi(t) = \underset{x}{\text{Argmin}} \quad f(x) = \sum_{k \in K} z^T x^k + \sum_{k \in K} \sum_{\ell \in K \setminus \{k\}} \frac{1}{2} x^{k^T} H x^\ell + \sum_{k \in K} x^{k^T} H x^k, \quad (\text{C.9})$$

$$\text{subject to} \quad B x^k = b^k, \quad \forall k \in K, \quad (\text{C.10})$$

$$\sum_{\ell \in K} x^\ell \leq q, \quad (\text{C.11})$$

$$x \geq 0. \quad (\text{C.12})$$

The matrix D is a $M\kappa \times M\kappa$ symmetric block matrix whose $\kappa \times \kappa$ block components are the matrix H in its diagonal blocks and the matrix $\frac{1}{2}H$ in its non-diagonal blocks. Since all the values d_a are nonnegative, both, the matrix H and the matrix D are symmetric and positive semi-definite, and if all the values d_a are strictly positive, the matrices H and D will be positive definite.

The programs appearing in (C.3)-(C.7) and program (C.8)-(C.12) are differentiable and convex (with linear constraints) quadratic programming problems, so these problems can be equivalently transformed into a nonlinear system of equations and inequalities using the Lagrange multipliers and Karush-Kuhn-Tucker conditions. Therefore, in order to show that a solution of (C.8)-(C.12) generates Nash-equilibrium for (C.3)-(C.7), it will suffice to demonstrate that a solution for the Lagrange multipliers and Karush-Kuhn-Tucker conditions of (C.3)-(C.7) lead to a solution for the Lagrange multipliers and Karush-Kuhn-Tucker conditions of (C.3)-(C.7). The Lagrange multipliers and Karush-Kuhn-Tucker condition for problem (C.3)-(C.7) are as follows:

$$\frac{df_k}{dx^k} + \mu^k + B^T \lambda^k = z + \sum_{\ell \in K \setminus \{k\}} Hx^\ell + 2Hx^k + \mu^k + B^T \lambda^k \geq 0, \quad (\text{C.13})$$

$$x^k \left(z + \sum_{\ell \in K \setminus \{k\}} Hx^\ell + 2Hx^k + \mu^k + B^T \lambda^k \right) = 0, \quad (\text{C.14})$$

$$Bx^k = b^k, \quad (\text{C.15})$$

$$x^k \leq q - \sum_{\ell \in K \setminus \{k\}} x^\ell, \quad (\text{C.16})$$

$$\mu^k \left(\sum_{\ell \in K} x^\ell - q \right) = 0, \quad (\text{C.17})$$

$$x^k, \mu^k \geq 0, \quad (\text{C.18})$$

where $\mu^k \in \mathbb{R}^M$ and $\lambda^k \in \mathbb{R}^\eta$, for all $k \in K$. And the Lagrange multipliers and Karush-Kuhn-Tucker conditions for problem (C.8)-(C.12) are:

$$\frac{\partial f}{\partial x^k} + \mu + B^T \lambda^k = z + \sum_{\ell \in K \setminus \{k\}} Hx^\ell + 2Hx^k + \mu + B^T \lambda^k \geq 0, \quad \forall k \in K, \quad (\text{C.19})$$

$$x^k \left(z + \sum_{\ell \in K \setminus \{k\}} Hx^\ell + 2Hx^k + \mu + B^T \lambda^k \right) = 0, \quad \forall k \in K, \quad (\text{C.20})$$

$$Bx^k = b^k, \quad \forall k \in K, \quad (\text{C.21})$$

$$\sum_{\ell \in K} x^\ell \leq q, \quad (\text{C.22})$$

$$\mu \left(\sum_{\ell \in K} x^\ell - q \right) = 0, \quad (\text{C.23})$$

$$x, \mu \geq 0, \quad (\text{C.24})$$

where $\mu \in \mathbb{R}^M$ and $\lambda^k \in \mathbb{R}^\eta$, $k \in K$. Now let $x^k, \mu \in \mathbb{R}^M$ and $\lambda^k \in \mathbb{R}^\eta$, $k \in K$, satisfy (C.19)-(C.24). Then, for a fixed $k \in K$, we have that:

$$z + \sum_{\ell \in K \setminus \{k\}} Hx^\ell + 2Hx^k + \mu + B^T \lambda^k \geq 0, \quad (\text{C.25})$$

$$x^k \left(z + \sum_{\ell \in K \setminus \{k\}} Hx^\ell + 2Hx^k + \mu + B^T \lambda^k \right) = 0, \quad (\text{C.26})$$

$$Bx^k = b^k, \quad (\text{C.27})$$

$$x^k \leq q - \sum_{\ell \in K \setminus \{k\}} x^\ell, \quad (\text{C.28})$$

$$\mu \left(\sum_{\ell \in K} x^\ell - q \right) = 0, \quad (\text{C.29})$$

$$x^k, \mu \geq 0. \quad (\text{C.30})$$

Therefore, the vectors $x^k, \mu \in \mathbb{R}^M$ and $\lambda^k \in \mathbb{R}^\eta$ satisfy (C.13)-(C.18), for all $k \in K$, which proves the theorem.

Finally, if we remove the capacity constraints, we can easily see that the KKT conditions (C.13)-(C.18) imply the KKT conditions (C.19)-(C.24) taking $\mu := \max \{ \mu^k \mid k \in K \}$, which proves corollary 3.1 ■

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