UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS

TESIS

DISEÑO DE CONTROLADORES CONTINUOS CONVERGENTES POR UNTIEMPO FIJO PARA SISTEMAS DINÁMICOS CON INCERTIDUMBRES

POR

DANY FERNANDO GUERRA AVELLANEDA

EN OPCIÓN AL GRADO DE DOCTOR EN CIENCIAS CON ORIENTACIÓN EN MATEMÁTICAS

DICIEMBRE, 2019

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Universidad Autónoma de Nuevo León
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Doctorado en Ciencias con Orientación en Matemáticas

Los miembros del Comité de Tesis recomendamos que la Tesis "Diseño de controladores continuos convergentes por un tiempo fijo para sistemas dinámicos con incertidumbres", realizada por el alumno Dany Fernando Guerra Avellaneda, con número de matrícula 1769837, sea aceptada para su defensa como opción al grado de Doctor en Ciencias con orientación en Matemáticas.

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Doctorado en Ciencias con Orientación en Matemáticas

San Nicolás de los Garza, Nuevo León, Diciembre de 2019

Dedicatoria

A la memoria de mi mami. Sigues dando luz a mi camino.

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Resumen

Este documento presenta controladores no lineales que proveen convergencia en tiempo fijo al origen (o a una vecindad del origen) para sistemas dinámicos de alto orden sujetos a incertidumbres (disturbios deterministicos no desvanescentes y disturbios estocásticos desvanescentes dependientes de los estados y el tiempo). Dos de los tres controladores diseñados incluyen un diferenciador convergente en tiempo fijo, un observador de disturbios convergente en tiempo fijo, y un regulador convergente en tiempo fijo. El diferenciador se da en el caso que el único estado medible del sistema dinámico es el de mayor grado relativo. El observador de disturbios convergente en tiempo fijo se emplea para estimar variaciones de disturbios no desvanecentes y no acotados. En caso que las cotas para los disturbios sean desconocidas se incluye un observador adaptable convergente en tiempo fijo caracterizado por no incrementar de manera excesiva las ganancias del controlador. En cuanto a la presencia simultanea de disturbios deterministicos no desvanescentes y disturbios estocásticos desvanescentes dependientes de los estados y el tiempo, se presenta un algoritmo Super-twisting estocástico convergente en tiempo fijo.

El problema de estimación del tiempo de convergencia de los controladores se resuelve calculando una cota superior uniforme del tiempo fijo de convergencia. Finalmente, los algoritmos diseñados se verifican en dos casos de estudio: Un motor DC con armadura y un problema de gestión de stocks. Resultados de las simulaciones confirman convergencia en tiempo fijo y robustez de los controladores diseñados.

Abstract

This document presents non-linear controllers that provides fixed-time convergence to the origin (a vicinity of the origin) of higher-order dynamic systems subjects to uncertainties (unbounded non-vanishing deterministic disturbances and stochastic vanishing state dependent disturbances). Two of three designed controllers include a fixed-time convergent differentiator, a fixed-time convergent disturbance observer, and a fixed-time convergent regulator. The differentiator is in case that the only measurable state of the system is the highest relative degree one. The fixed-time convergent disturbance observer is used to estimate unbounded non-vanishing deterministic disturbances. In case of uncertainties bounds are unknown is included using an adaptive fixed-time convergent disturbance observer without excessively increasing the controller gains. While, unbounded non-vanishing deterministic disturbances and stochastic vanishing state dependent disturbances are both considered, a fixed-time convergent Super-twisting Algorithm is proposed.

The settling time estimation problem is solved computing an uniform upper bound of the fixed convergence time of the designed controller. Finally, the designed algorithms are verified in a case study of an industrial armature-controlled DC motor, and a stock Management problem. Simulation results confirm fixed-time convergence and robustness of the designed controller.

Contents

Chapter 1

Introduction

"Natura non facit saltus"

Dynamic systems problems are usually unstructured and under-defined. Their dynamics can be seen affected by different factors, for instance, very small errors in setting coefficients, parameters, initial and boundary conditions give rise to finite and even large solution errors, or relative solution errors are much greater than relative coefficient errors. These factors show the importance to take into account the specific features about the problem statement, the measurement accuracy provided by particular measuring devices when build the mathematical model, and the requirements to the precision in obtaining the solutions bearing in mind the solution techniques [1].

Mathematical models of dynamic systems are built to understand patterns, and designing artificial patterns. The model is good if the desired behaviour is considered, and if the model may feasibly be implemented. In that sense a model could be enough simple to make analysis and numerical computing feasible while retains enough of the complexity to give an acceptable approximation of the events one might observe and the same time left out certain dynamics (unknown external uncertainties, parameters, etc). So, always there are mismatches between dynamic system and their mathematical model [2, 3].

In most cases those mismatches and factors related to discretization, delay or noise cannot be expressed in explicit form, instead we see like a state and time dependent disturbance. Disturbances are pervasive in dynamic systems and affect the performance of controllers, up to bringing instability. Depending on the situation the modeller can know some information about the disturbance, for instance, an upper bound, or whether disturbance vanishes on equilibrium points of the system.

That way, designing accurate control algorithms for disturbed dynamical systems is one of the most important challenges in robust control systems area. This document is about fixed-time convergent controllers. Robust and accurate control algorithms for dynamical systems developed under the Sliding Mode Control (SMC) approach.

Fixed-time convergence occurs when the solutions of a dynamical system converge in finite-time to the equilibrium, and the settling time function is independent of initial conditions. The main property of fixed-time convergent systems is rejection to unbounded non-evanescent Lipschitz disturbances.

Content of this chapter is organized as follows.

Antecedents of finite-time and fixed-time deterministic/stochastic controllers are showed in section 1.1. Motivation and justification of this thesis are given in section 1.2. Finally, Section 1.3 consists of contributions, and organization of the thesis.

1.1 Antecedents

The sliding mode control (SMC) technique consists of two phase motions. The reaching phase when the system trajectory moves towards the sliding surface, and the sliding phase when the system trajectory moves along the sliding surface. So, the SMC design involves two tasks. First, a sliding surface design in accordance to some performance criterion to which the controlled system trajectories must belong. Then, a control (discontinuous or not) strategy design to force the system state to reach the sliding surface such that a sliding mode occurs on this manifold. Once the system reaches the sliding surface, the system exhibits robustness properties with respect to parameter perturbations and external disturbances.

1.1.1 Finite and fixed-time Controllers for deterministic systems subject to uncertainties

SMC is an effective technique to ensure finite time convergence of uncertain non-linear systems. Finite-time convergence appears for the first time when discontinuous controllers (relay or unit) are designed to ensure sliding modes. This kind of controllers guaranteed finite-time convergence to the sliding surface, but the state variables only converge asymptotically. Once in sliding mode the system is robust against a wide class of uncertainties and disturbances.

Extending finite-time convergence from scalar variable to a higher order vector variable becomes very challenging. This topic has been intensively studied in the nineties with the introduction of higher order sliding modes [4]. These challenges such that chattering, noise sensitivity and rejection of matched/unmatched disturbances, related to SMC and finite time convergence are the topics to covering hereunder.

Forcing the trajectory onto the sliding manifold in the presence of disturbances, requires the relay to switch at theoretically infinite frequency. However, in practical applications due to small imperfections in switching devices the switching frequency is limited by e.g., delays, hysteresis in the switching elements and discretization of the controller. Also, the state is not confined to the switching line but oscillates within its small vicinity. The limitation of the switching frequency leads to the chattering phenomenon, high frequency oscillations of the state trajectories around the sliding surface that may lead to lower control accuracy, to degradation of the system efficiency, or to damage of physical parts in certain systems [5, 6]. So, a first drawback for the implementation of SMC algorithms is chattering.

It has been shown that chattering phenomenon is mainly caused by sources grouped into two categories: the discretization chattering, which comprises effects related to the discrete-time realization of the controller, and the control chattering which embodies the effects designated to model imperfections: unmodelled dynamics along with high feedback gain or discontinuous control which increase the systems relative degree and perturb the ideal sliding mode [5, 6]. i.e. in order to adjust the chattering it is necessary that not only the sliding variable tends to zero, but also its derivative.

The discretization chattering effects in SMC systems are mainly studied in the context of the unperturbed Euler discretized closed loop system [7]. For the analysis of the control chattering effects the frequency domain technique such as the describing function (DF) method is the preferred tool [6].

At the beginning, sliding surface design is restricted to have relative degree one with respect to the control. Latter, second order sliding modes are proposed as solution of chattering phenomenon, but, in order to adjust the chattering for a relative degree one sliding variable, a continuous control signal should be generated without requiring information on the derivative of the sliding variable, i.e. on the perturbations.

In 1993 Levant in [4] introduces the super-twisting algorithm (STA) for Lipschitz systems allows substituting a discontinuous control by means of an continuous one. Their use offers: Chattering attenuation and robustness against disturbances. However, there are some disadvantages: For systems with relative degree $r = 2$, the design of a sliding surface is still needed. Hence, there is finite-time convergence to the surface, but the convergence of the states to the origin is again asymptotic. Also, the first order sliding mode controllers with constant gains could compensate Lebesgue but bounded perturbations. The STA is insensible to perturbations whose time derivative is bounded. However, these perturbations could grow no faster than linear function of time, i.e., they do not need to be bounded.

Higher order sliding modes with discontinuous injections for chain of integrators are proposed in [8]. The advantages are to ensure the r-th order accuracy for the sliding output with respect to the discretization step, and the sliding surface design is no longer needed. However, discontinuity of HOSM algorithms for relative degree r systems cannot reduce the chattering substantially [9].

Finite, fixed-time convergence and Continuous higher order sliding modes

The finite-time convergence property was started within ordinary differential framework in [10]. As shown in [11], certain classes of non-linear systems can be transformed into a chain of integrators form by using an appropriate change of variables (for instance feedback linearisation). For this reason, research on SMC and finite time convergence usually focuses on integrator chains.

The study of finite-time convergence and stability for equilibrium points of continuous but non-Lipschitzian autonomous systems, as well as properties of the finite-time convergence time functions and their robustness to perturbations was initiated [12] and [13]. The first result regarding finite-time regulators was obtained in [11], which consists in establishing existence of a continuous regulator driving all states of a chain of integrators at the origin in finite time. The authors demonstrated that all states of a controllable linear system can be driven at the origin in finite time by a continuous state feedback. A more in-depth study of finite-time stabilization of a chain of integrators can be found in [14]. Finite-time convergent algorithms have recently been applied to various classes of dynamic systems, such as finite-time state-bounded systems [15], automotive suspension systems [16], rigid spacecrafts [17], Takagi–Sugeno fuzzy systems [18], Markovian jump systems [19, 20, 21], and many others.

In most practical cases, it is also desirable to ensure that the convergence time function is uniformly bounded regardless of the initial condition. The problem of designing continuous control laws convergent for a fixed-time was considered in [22] and [23]. The paper [24] presents comprehensive studies of fixed-time stability and control design for non-autonomous systems, including discontinuous ones. Various approaches to estimating finite-convergence times for different control algorithms can be found in [4, 3]. A fixed-time convergent multivariable super-twisting-like control law was proposed [25], where an upper estimate of its convergence time was calculated. In [26], an estimate of the convergence time was obtained for a fixed-time convergent control algorithm driving at the origin all states of a finite-dimensional integrator chain.

In the presence of unknown disturbances a linear feedback observer is unable to force the estimation error to zero and get the estimate to converge to the system states only asymptotically. For this reason, a finite-time convergent observer is necessary. This observer feeds the output estimation error via a nonlinear term that makes a switch providing a solution to the drawbacks that arise with the Luenberger observer [27],[28]. One of these drawbacks is: If a bound on the magnitude of the disturbances is known, the finite time convergent observer can force the output estimation error to converge to zero in finite time, whereas a linear feedback observer converges asymptotically to states of the system. In addition, disturbances of the system can be reconstructed.

As with the observers for derivatives (differentiators), the studies have focused mostly on observers in sliding modes in low dimensions ($n = 2$) for a finite/fixed time, both discontinuous and continuous.

From super-twisting algorithm we highlight the first-order super-twisting differentiator proposed by Levant [29] and characterized by combining the exactness (exactness of some input occurs if the output coincides with its derivative) of the differentiator with the robustness respect to measurement errors or input noise. Particularly, in [30] and [31], Levant extends the obtained result to high order differentiation, and establishes the accuracy of the proposed algorithms, while Perruquetti et al. [32], propose the first continuous finite-time observer for systems of an arbitrary dimension with non-linear feedback.

Another relevant problem consists in estimating the convergence (settling) time for the finite-time convergent control laws. Some results providing settling time estimates for supertwisting algorithm, can be found in [33, 34]. Settling time estimates for a super-twisting algorithm, based on an explicit Lyapunov function or a geometric approach are obtained in [35, 36, 3].

In [37], a fixed-time convergence differentiator is proposed for a chain of integrators of an arbitrary dimension and the uniform upper bound of the settling time is computed. A comprehensive survey summarizing existing results on finite and fixed-time convergence can be found in [24].

A recent result concerning the observer and observer-based controller design problems for systems with unknown disturbances are presented in Lopez et al. [38]. The authors design a state feedback controller and a dynamic observer in order to guarantee both fixed-time estimation and fixed-time control. The Robustness with respect to exogenous disturbances and measurement noises is established by means of an optimization algorithm to tune the parameters involved in the feedback controller. In the perturbed case, fixed-time stabilization to a vicinity of the origin whose radius depends on the size of the perturbation is obtained.

1.1.2 Fixed time Adaptive Controllers for deterministic systems subject to uncertainties

State and time-dependent disturbances are pervasive in dynamic systems and affect the performance of control, up to bringing instability. Higher-Order Sliding Modes (HOSM) are robust and accurate for controlling disturbed systems of arbitrary relative degree with bounded disturbances; however, in practice, disturbance bounds are not available. To deal with this problem, some HOSM algorithms use conservative upper bounds to guarantee that sliding mode takes place. However, this approach has a drawback: the bound of disturbance can be overestimated when tuning the control gains, yielding to chattering [4, 39, 40, 41].

Particularly, A. Levant in [4], introduces a new class of bounded continuous time-dependent control sliding algorithm with discontinuities only in the control derivative. The notion of sliding mode order is presented. Accuracy and conditions for convergence for twisting and super-twisting algorithms are demonstrated. While in [40] presents a second-order sliding modes controllers with finite-time convergence using homogeneity approach for SISO systems with relative degree 2 in presence of uncertainties. A procedure to attenuate chattering with first order sliding modes based on their replacement by 2-sliding modes is developed. In the presence of measurement noise the tracking accuracy is proportional to the unknown noise magnitude.

G. Bartolini et al. in [39] give a procedure for chattering elimination based on the extension to multi-input uncertain non-linear systems from a control approach able of enforcing a second-order sliding mode in case of single-input non-linear systems with different types of uncertainties. The generalization to multi-input systems is obtained by introducing an auxiliary system, and a mixed second order and first order sliding mode control with a hierarchy in the reaching phase.

F. Plestan, A. Glumineau, and S. Laghrouche, in [41], design a robust high-order sliding mode controller for a class of uncertain minimum phase single-input single-output (SISO) non-linear systems. The sliding manifold is designed in order to ensure finite-time convergence of sliding variable and its high-order time derivatives. The control law can be applied for every value of sliding mode order, can be arbitrarily adjusted, and allows the establishment of an rth-order sliding mode. Results are extended to multi-input multi-output (MIMO) systems.

The drawback of continuous HOSM algorithms; the overestimation of bound of disturbance when tuning the control gains yielding to chattering is counteracted with an adaptive finite-time convergent continuous controller. The idea is to develop adaptive laws for increasing the control gains so that a sliding mode is obtained. Once it is obtained, the adaptive gains start reducing. If the sliding mode is not sustained in view of uncertainties destroying the real sliding mode, the adaptation resumes and the gains increase again. Adaptive control algorithms attenuate chattering and provide robustness against bounded disturbances with unknown bounds, without their overestimation.

For instance, in [42], the authors propose two methodologies for adaptive sliding mode controller design whose gain adaptation is realized without a priori knowing uncertainties/perturbations bounds while the adaptive gains values are not over-estimated. The first algorithm is based on evaluation of uncertainties/perturbation by using equivalent control concept that requires employment of low-pass filter. The second adaptive control law does not estimate the boundary of perturbations and yields establishing a real sliding mode. The efficacy is verified for an electropneumatic actuator.

Shtessel et al. in [43], a novel super-twisting adaptive sliding mode control law is proposed for the control of an electropneumatic actuator. The authors consider that the bounds of uncertainties and perturbations are not known. The control consists in using dynamically adapted control gains that ensure finite time convergence of a real second order sliding mode. The main characteristic of the adaptation algorithm is non-overestimating the values of the control gains. The proof of the finite time convergence of the closed-loop system is derived using the Lyapunov function technique.

In [44], G. Bartolini et al. develop adaptive sliding mode strategies of first and second sliding orders for single-input single output systems of the first and second relative degrees respectively. Since the only concrete known uncertainty bounds are assumed to be the upper bounds of logarithmic derivatives, the gain is to be increased until the sliding mode is reached, then it is decreased until the sliding mode is lost. The accuracy of strategies is proportional to the sampling period to the power of the relative degree r. the robustness of the system is obtained by increase the sampling period.

M. Taleb, A. Levant, and F. Plestan, [45] propose an adaptive twisting algorithm, which actually presents a new second-order sliding-mode algorithm. Due to the dynamic adaptation of the gains the controller design does not require information on the bounds of uncertainties and perturbations. It automatically decreases the gains and respectively also the dangerous oscillations due to a too large discontinuous-control magnitude. The performance and the accuracy of the closed-loop system are demonstrated. The algorithm is applied to control the position of a pneumatic actuator.

Recently, Edwards and Shtessel [46] design a Continuous Adaptive HOSM control algorithm with two parameters adapt allows better self-tuning. The adaptive algorithm does not requires knowledge of the bounds on the matched uncertainty and its derivatives, and the gains are not overestimated by the adaptation scheme mitigating chattering.

In addition to the chattering problem, the settling (convergence) time depending on initial conditions, which is common for HOSM control algorithms, is undesirable in certain situations as well, since the settling time may go to infinity as the magnitude of initial conditions grows. This problem is solved if a fixed-time convergence controller, whose convergence time is bounded for any initial condition, is used. It is demonstrated [47] that adaptive uniform finite-time convergence increases the controller robustness against disturbances with unknown constraints.

Controllers above mentioned ensure a finite/fixed-time convergence of the sliding output to the $(r+1)$ th-order sliding set using information on the sliding output and its derivatives up to the order $(r-1)$.

1.1.3 Fixed time Controllers for stochastic Systems

A first approximation to finite-time stability, called stochastically finite-time attractiveness is introduced in [48, 49]. The authors verify the finite-time attractiveness for a class of Ito stochastic non linear autonomous systems using Lyapunov functions, and characterize the settling time function of solutions starting from a vicinity of the origin as a stochastic variable whose expectation is finite. Yin et al., [50] introduce a new definition that involves finite-time stability in probability in addition of finite time attractiveness in probability. Also a Lyapunov theorem on finite-time stability is established for stochastic non-linear non-autonomous systems. The Lyapunov second method has been developed to deal with stochastic stability in [51, 52, 53].

Criteria on almost sure exponential stability and p-th-moment stability for stochastic systems are obtained in [54], and for stochastic systems with hysteresis in [55]. Meanwhile, [56], addresses the problems of finite-time stochastic stabilization, optimal finite-time stochastic stabilization, as well as partial state stabilization and optimal partial-state stochastic stabilization, for stochastic continuous systems. In [57], the authors extend concepts of p-th moment exponential stability and almost sure exponential stability to stochastic switched non-linear systems. Recently, the p-th moment exponential stability of stochastic switched systems has been studied in [58].

In many applications, it is desirable the trajectories of dynamical systems converge to a stable equilibrium state in finite time, rather than asymptotically or exponentially. Based in finite-time stability theorem given in [50], Khoo et al. [59] investigate the problem of almost sure finite-time stabilization of a class of stochastic non-linear autonomous systems. The authors prove that almost sure global finite-time stability of stochastic non-linear systems in strict-feedback form can be achieved by a continuous control law. Using the results of [59], Zhai [60] designs an output feedback controller for a stochastic high order non-linear system such that the closed-loop equilibrium is global finite-time stable in probability. In [61] Wang et al. design a backstepping control law for finite-time stabilization of stochastic higher-order nonlinear systems in strict-feedback form.

Works about homogeneous autonomous non-linear stochastic systems rises with [62] and follows with [63]. [62] shown that homogeneous stochastic systems can exhibit exponential stability or finite-time stability according to the degree of homogeneity of coefficients. Yin and Khoo [63] present conditions for finite time stability of Lyapunov functions non differentiable at the origin.

As for non-autonomous systems, works [50, 64, 65] prove direct Lyapunov theorems and converse Lyapunov theorems on finite time stability in probability for stochastic non-linear continuous non-autonomous systems. Some regular properties (continuity, positive definiteness and boundedness of the expected stochastic settling time) of their stochastic settling time function are established under appropriate conditions.

The problem of design a sliding mode controller for Markovian jump systems (MJSs) is treated starting from [66]. The authors prove that stochastic stability of a class of linear continuous-time systems with stochastic jumps is achieved, if a set of coupled Linear matrix inequalities (LMIs) is solvable. Wu et al. [61] design an observer to estimate system states in finite time and a sliding-mode control based on the state estimates for singular MJSs. Results related to stabilization on a finite time interval for different MJSs can be consulted in [67, 68].

A survey on modeling, analysis, and design of control and filtering algorithms for MJSs is addressed in Shi and Li [69]. The paper [70] studies a robust adaptive sliding mode control problem for a class of non-linear uncertain neutral MJSs with unmeasurable states. A sufficient condition of stochastic stability of the overall closed-loop system can be achieved in terms of LMIs based on the SMC strategy. Zhao in [71] investigates finite-time stability in probability and finite time stability in $\mathcal{L}_1$ for networked semi-Markovian jump non-linear systems with parametric and dynamic uncertainties. Other results about sliding modes for MJSs can be found in [72, 73, 74].

In [75] a sliding mode control is proposed for continuous time switched system with time varying delay in the state equation. Zhao et al. [76] study finite-time globally asymptotical stability in probability (FGSP) and finite-time stochastic input-to-state stability (FSISS) for a class of switching stochastic non-autonomous non-linear systems, while finite-time stabilization problem for uncertain (MJSs) is considerated by Yin and Khoo [77]. The authors present conditions for almost surely finite time stability and discuss existence and uniqueness of strong solutions. Weak solutions and uncertain transition probabilities are studied in [78], and stability of switched systems in which all the subsystems may be unstable is treated in [79].

Introduction of sliding modes in the problem of design a control strategy to guarantee stability in probability (not finite time stability) for second order and high-order stochastic systems taking into account nonvanishing diffusion terms is addressed in [80, 81].

Finally, to the best of our knowledge, the only note on fixed-time stability of stochastic non-linear systems described by the Ito differential equations is recently published in [82]. Yu et al., give a definition of fixed-time stable in probability and present the correspondent conditions using Lyapunov functions for non-autonomous systems.

1.2 Motivation and Justification

HOSM control techniques allow driving to zero the sliding variable and its consecutive derivatives in the presence of the uncertainties increasing the accuracy of the sliding variable stabilization and has still been successfully applied. The previous review suggests main drawbacks of HOSM controllers to be remarked:

1. Continuous injections in HOSM yield to decrease chattering. However, in some cases the sliding trajectories do not converge to the sliding surface but to its vicinity losing the robustness to the disturbances.
2. The settling time dependence on initial conditions of finite time controllers results in the fact that if the norm of an initial condition grows in an unbounded manner, the corresponding convergence time tends to infinity.
3. In practice, disturbance bounds are not available. To deal with this problem, some HOSM algorithms use conservative upper bounds to guarantee that sliding mode takes place. However, the bound of disturbance can be overestimated when tuning the control gains, yielding to chattering.
4. There is no continuous fixed-time convergent control law driving the states of a stochastic super-twisting system at the origin for a fixed time in presence of stochastic white noise and unbounded deterministic disturbance satisfying a Lipschitz condition.

HOSM controllers use high order time derivatives of the sliding variable. The main advantage of the second order sliding mode super-twisting algorithm is it only requires measurement of the sliding variable. Also, fixed-time controllers overcome the drawback given in item 3, providing uniform finite-time convergence to the origin for a certain time, which is no greater than the calculated upper bound.

Being well established the drawbacks, it is needed to:

Design a fixed time convergent CHOSM controller that increases robustness against Lipschitz disturbances, when a fixed-time Super-twisting disturbance observer (a compensation mechanism) is added.

Design an adaptive fixed-time convergent continuous controller to decreasing chattering phenomenon increasing the controller robustness against disturbances with unknown constraints.

Design a continuous fixed-time convergent control law driving the states of a stochastic super-twisting system at the origin for a fixed time in presence of stochastic white noise and unbounded deterministic disturbance satisfying a Lipschitz condition.

1.3 Contributions

Chapter 2 presents a non-linear continuous observer-based controller that provides fixed-time convergence to the origin (a vicinity of the origin) of a higher-order dynamic system subject to unbounded disturbances satisfying a Lipschitz condition, whose only measurable state is the highest relative degree one.

Designing a fixed-time convergent observer-based controller is important and practically useful, taking into account that the obtained finite-time convergence is uniform and the calculated upper bound for the convergence time is independent of initial conditions. Algorithm is verified in a case study of an industrial armature-controlled DC motor.

This chapter is published in [83, 84]

Michael Basin, Fernando Guerra-Avellaneda, Continuous fixed-time controller Design for dynamic systems with unmeasurable states, IFAC-PapersOnLine, Volume 51, no. 13, pp. 597-602, 2018. 2nd IFAC Conference on Modelling, Identification and Control of Nonlinear Systems MICNON 2018.

M. Basin and F. G. Avellaneda, Continuous fixed-time Controller Design for Dynamic Systems with Unmeasurable States Subject to Unbounded Disturbances, Asian Journal of Control, vol. 21, no. 1, pp. 194-207, 2019.

Chapter 3 presents an adaptive fixed-time convergent continuous HOSM controller designed to solve a Stock Management Problem with the objective to drive stock and supply chain levels at the reference values, subject to loss rate disturbances whose bounds are unknown. The only measurable state of the supply chain is the inventory retailer stock level, whereas the supply line inventory level should be estimated. The designed controller includes a fixed-time convergent differentiator, an adaptive fixed-time convergent disturbance observer, and a fixed-time convergent regulator. The adaptive fixed-time convergent observer is used to estimate a disturbance without excessively increasing the controller gains. The calculated upper estimate for the total settling (convergence) time and obtained simulation results confirm fixed-time convergence and robustness of the designed controller.

This chapter is included in [85]

M. V. Basin, F. Guerra-Avellaneda, and Y. B. Shtessel, Stock management problem: Adaptive fixed-time convergent continuous controller design, IEEE Transactions on Systems, Man, and Cybernetics: Systems, 2019, DOI 10.1109/TSMC.2019.2930563.

Chapter 4 presents a continuous fixed-time Super-twisting stochastic algorithm driving the states at the origin for a finite pre-established (fixed) time using a scalar input. The algorithm involves both unbounded non-vanishing deterministic disturbances and stochastic vanishing state dependent disturbances. Performance of the developed algorithm is verified with numerical simulations.

The content of this chapter was submitted to Journal of the Franklin Institute. [86].

F. Guerra-Avellaneda, M. Basin, Continuous Fixed-Time Convergent Control Design for Stochastic Super-Twisting System, submitted to JFI, 2020.

Document is organized as follows.

An introduction that contains antecedents, motivation and justification, and contributions is given in chapter 1. Conceptual framework about Higher sliding modes, finite and fixed-time convergence, and settling time estimations are presented in Chapter 2. A Continuous Fixed-Time Controller based-observer Design for Deterministic systems with Lipschitz uncertainties and unmeasurable states is obtained in Chapter 3. Results are confirmed in case study of controlling a DC motor. Chapter 4 presents an adaptive continuous higher order sliding mode controller for dynamical systems with unmeasurable states and non-vanishing Lipschitz disturbances whose bounds are unknown. An upper estimate of the convergence (settling) time for the control algorithm is calculated. Results are tested in a stock management problem. Chapter 5 gives a continuous fixed-time convergent Controller for Ito-Stochastic Super-Twisting System. Conclusions and future work are given in Chapter 6.

Chapter 2

Higher sliding modes and Fixed time convergence

"The popular view that scientists proceed inexorably from well-established fact to well-established fact, never being influenced by any unproved conjecture, is quite mistaken. Provided it is made clear which are proved facts and which are conjectures, no harm can result. Conjectures are of great importance since they suggest useful lines of research."

Sir Alan Turing (1912-1954)

2.1 Introduction

Some definitions and theorems needed for contents understanding of next chapters are given here. The first section presents Filippov and Utkin solutions for piecewise continuous systems. In section 2 Higher sliding modes, Lie derivative and Relative degree are introduced. Finite and fixed-time convergence of piecewise dynamical systems are defined in Section 3. Finally, in section 4 Settling time estimations for a class of dynamics systems are showed. Contents of this chapter are taken from [87], [88], [89], [3], and [90], mainly.

2.2 Discontinuous and Piecewise continuous dynamical systems

Consider the dynamical system

$$\dot{x} = f(t, x) \tag{2.1}$$

where the vector field $f(t, x)$ is a continuous function. It is known that a classical solution for (2.1) consists of a differentiable function $x(t)$ satisfying (2.1) everywhere on a given interval.

However in case that vector field is not continuous the solution concept can be modified as can be seen from following examples.

Example 1. *[88] Consider the dynamical system*

$$\dot{x} = 1 - 2sign(x) \tag{2.2}$$

here

$$sign(x) = \begin{cases} 1, & x > 0 \\ 0 & x = 0 \\ -1, & x < 0 \end{cases}$$

whose vector field is discontinuous at zero. If $x(t) < 0$, $\dot{x}(t) = 3$, we have the solution $x(t) = 3t+c_1$. If $x(t) > 0$, $\dot{x}(t) = -1$, we have the solution $x(t) = -t + c_2$. As t increases each solution reaches $x = 0$, but, in case that $x(t) = 0$, $\dot{x}(t) = 1-2sign(0) = 1 \neq 0$, in consequence a classical solution does not exist starting from every initial condition.

Definition 2.2.1. *[87, 24] A function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is called piecewise continuous function if $\mathbb{R}^{n+1}$ consists of a finite number of open connected sets (domains) $G_j \subseteq \mathbb{R}^{n+1}$, $j = 1, 2, \cdots, N$; $G_i \bigcap G_j = \emptyset$, for $i \neq j$ and the boundary set $\mathcal{S} = \bigcup_{i=1}^{n} \partial G_j$ of measure zero such that $f(t, x)$ is continuous in each G_j and for each $(t^*, x^*) \in \partial G_j$ there exists a vector $f^j(t^*, x^*)$, possibly dependent on j, such that for any sequence $(t^k, x^k) \in G_j : (t^k, x^k) \to (t^*, x^*)$ we have $f(t^k, x^k) \to f^j(t^*, x^*)$. Let functions $f^j : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be defined on ∂G according to this limiting process i.e.*

$$\lim_{(t^k,x^k)\to(t,x)} f(t^k, x^k) = f^j(t, x)$$

Differential equations with a piece-wise continuous right-hand side are accepted as a basic mathematical model of a discontinuous system.

Definition 2.2.2. *[87] will said the dynamical system*

$$\dot{x} = f(t, x) \tag{2.3}$$

is piecewise continuous, if the vector field $f(t, x)$ is a piecewise continuous function.

Example 2. *[88] Consider the dynamical system*

$$\dot{x} = sign(t) \tag{2.4}$$

This is a piecewise dynamical system consist of two domains $G_1 = \{t \in \mathbb{R} : t > 0\}$, $G_2 = \{t \in \mathbb{R} : t < 0\}$, the vector field is continuous inside each one domain G_j. The boundary set is $\mathcal{S} = \{t \in \mathbb{R} : t = 0\}$. For $t^* = 0 \in (S)$ there exists two vectors $f^1(t^*) = -1$, $f^2(t^*) = 1$, such that for any sequence $t^k \in G_j$: $t^k \to t^* = 0$ we have $f(t^k) \to f^j(t^*)$. Let functions $f^j : \mathbb{R} \to \mathbb{R}$ be defined on ∂G according to this limiting process i.e. $\lim_{t^k\to t} f(t^k) = f^1(t) = 1$, and $\lim_{t^k\to t} f(t^k) = f^2(t) = -1$, then we have a piecewise continuous vector field.

Example 3. *[89] Consider the dynamical system*

$$\dot{x} = \begin{cases} -1, & x > 0 \\ 1, & x \leq 0 \end{cases} \tag{2.5}$$

This is a piecewise dynamical system consist of two domains $G_1 = \{x \in \mathbb{R} : x > 0\}$, $G_2 = \{x \in \mathbb{R} : x < 0\}$, the vector field is continuous inside each one domain G_j. The boundary set is $\mathcal{S} = \{x \in \mathbb{R} : x = 0\}$. For $x^* = 0 \in (S)$ there exists two vectors $f^1(x^*) = -1$, $f^2(x^*) = 1$, such that for *any sequence* $x^k \in G_j : x^k \to x^* = 0$ *we have* $f(x^k) \to f^j(x^*)$. *Let functions* $f^j : \mathbb{R} \to \mathbb{R}$ *be defined on* ∂G *according to this limiting process i.e.* $\lim_{x^k \to x} f(x^k) = f^1(x) = -1$, *and* $\lim_{x^k \to x} f(x^k) = f^2(x) = 1$, *then we have a piecewise continuous vector field.*

2.2.1 Existence of solutions

Now, solutions of piecewise continuous vector fields for points inside of regions G_j *satisfy the classical solution concept, but for points in boundaries of domains or intersection of boundaries of these domains, solutions do not exist. It is the case with vector field in example* (3) *which is discontinuous in zero. Suppose that exists a continuous differentiable function* $x : [0, t] \to \mathbb{R}$ *such that* $\dot{x}(t) = f(x(t))$, *and* $x(0) = 0$. *Then* $\dot{x}(0) = f(x(0)) = f(0) = 1$ *which implies for all* t *sufficiently small* $x(t) > 0$ *and hence* $\dot{x}(t) = f(x(t)) = -1$, *which contradicts the fact that* $t \to \dot{x}(t)$ *is continuous.*

In the case of system in example 2, for $t > 0$ *we have* $\dot{x} = 1$, $x(t) = t + c_1$; *for* $t < 0$ *we have* $\dot{x} = -1$, $x(t) = -t + c_2$. *From the requirement of solution continuity in* $t = 0$, $x(0) = \lim_{t \to 0^+} x(t) = \lim_{t \to 0^-} x(t)$ *we obtain* $x(0) = c_1 = c_2$, *so the solution is* $x(t) = |t| + c$, *and for* $t = 0$ *the derivative* $\dot{x}(t)$ *does not exist.*

Example 4. *[89] Let change the value at zero in vector field given in example 3. The following example shows that discontinuity of vector fields does not always implies the non-existence of classical solutions.*

Consider the dynamical system

$$\dot{x} = -sign(x) = \begin{cases} -1, & x > 0 \\ 0 & x = 0 \\ 1, & x < 0 \end{cases} \tag{2.6}$$

which is discontinuous at zero. If $x(0) > 0$, *then the solution* $x : [0, x(0)) \to \mathbb{R}$ *is* $x(t) = x(0) - t$, *whereas, if* $x(0) < 0$ *then the solution* $x : [0, -x(0)) \to \mathbb{R}$ *is* $x(t) = x(0) + t$. *In case that* $x(0) = 0$ *then the solution* $x : [0, \infty) \to \mathbb{R}$ *is* $x(t) = 0$, *in consequence has a classical solution starting from every initial condition.*

The examples suggest the question to ask about the conditions: does a piecewise continuous vector field have a unique solution starting from each initial condition? The answer to this question relies on the solution concept itself. The case of example 2 where the vector field is discontinuous in t *and continuous in* x *requires a new consideration about of the solution concept (consider continuity of solutions instead differentiability). In examples 1, 3 and 4, when the discontinuity is in state* x, *requires a novel solution concept taking into account physical issues of the problem addressed.*

2.2.2 Filippov and Utkin solutions

What happen if, instead of focusing on the value of vector field in points, we consider vicinities of points? This is the core of Filippov solutions.

Definition 2.2.3. *[87] Given the dynamical system* (2.3) *let us introduce for each point* $(t, x) \in \mathbb{R}^{n+1}$ *the smallest convex closed set* $F(t, x)$ *which contains all the limit points of* $f(t, x^*)$ *as* $x^* \to x$, $t = const$, $(t, x) \in \mathbb{R}^{n+1}$ *and* $\bigcup_{j=1}^{N} \partial G_j$. *An absolutely continuous function* $x(\cdot)$, *defined on an interval* I, *is said to be a Filippov solution of dynamical system* (2.3) *if it satisfies the differential inclusion*

$$\dot{x}(t) \in F(t,x) \tag{2.7}$$

almost everywhere on I. The set $F(t,x)$ is referred as the Filippov set.

2.2.3 Existence and uniqueness of Filippov solutions

Let us assume now that the discontinuities of function $f(t,x)$ live on a smooth surface $\mathcal{S}$ only, such that this surface is governed by the equation $s(x) = 0$. Then we have two subspaces $G^- = \{x \in \mathbb{R}^n : s(x) < 0\}$ and $G^+ = \{x \in \mathbb{R}^n : s(x) > 0\}$. given t, the Filippov set is a convex combination joining the endpoints of the vectors

$$f^-(t,x) = \lim_{(t,\hat{x}) \in G^-, \hat{x} \to x} f(t,\hat{x}),$$

$$f^+(t,x) = \lim_{(t,\hat{x} \in G^+, \hat{x} \to x} f(t,\hat{x}))$$

assuming that these vectors have the same initial point x.

We are interested in situation that the vectors $f^-(t,x)$ and $f^+(t,x)$ appoint to opposite directions, and the Filippov set $F(t,x)$ intersects but not lie in the plane normal T to the vector gradient of surface $s(x) = 0$. e.d.,

$$\nabla s(x) f^-(t,x) > 0$$

and

$$\nabla s(x) f^+(t,x) < 0$$

for all $t \in [t_0, t_1]$, and we say that a sliding mode takes place on the discontinuity surface $\mathcal{S}$ for these t.

According to definition of Filippov set, this mode is governed by the equation

$$\dot{x} = f^0(t,x) = \mu(t,x) f^+(t,x) + (1 - \mu(t,x)) f^-(t,x) \tag{2.8}$$

where

$$\mu(t,x) = \frac{\nabla s(x) f^-(t,x)}{\nabla s(x)[f^-(t,x) - f^+(t,x)]} \tag{2.9}$$

$\mu(t,x) \in [0,1]$ is obtained from the condition that the velocity vector is in the plane T, tangential to $\mathcal{S}$. i.e.;

$$\nabla s(x) f^0(t,x) = \nabla s(x)[\mu(t,x) f^+(t,x) + (1 - \mu(t,x)) f^-(t,x)] = 0.$$

Remark 2.2.4. *$\nabla s(x)[f^-(t,x) - f^-(t,x) = 0$ occurs, in case that the velocity vector is completely lie in the tangential plane T, therefore Filippov solution are not unique.*

Remark 2.2.5. *If $s(t,x)$ (s depends of t), the conditions are*

$$\frac{\partial s(x,t)}{\partial t} + \nabla s(x) f^-(t,x) > 0$$

and

$$\frac{\partial s(x,t)}{\partial t} + \nabla s(x) f^+(t,x) < 0$$

for all $t \in [t_0, t_1]$, *and we said that a sliding mode takes place on the discontinuity surface* $\mathcal{S}$ *for these* t, *whereas the velocity vector* (2.20) *is determined by*

$$\mu(t,x) = \frac{\partial s(t,x)/\partial t + \nabla s(t,x) f^-(t,x)}{\nabla s(t,x)[f^-(t,x) - f^+(t,x)]} \tag{2.10}$$

$\mu(t,x) \in [0,1]$ *and obtained from the condition*

$$\frac{\partial s(t,x)}{\partial t} + \nabla s(t,x)[\mu(t,x) f^+(t,x) + (1-\mu(t,x)) f^-(t,x)] = 0$$

ensuring $\frac{\partial s(t,x)}{\partial t} = 0$ *for the sliding mode* $x(t)$ *evolving along discontinuity surface* $\mathcal{S}$.

Example 5. *Consider the vector fields in examples 3 and 4. They differs only at zero value (a set of measure zero). hence, they have the same Filippov set*

$$F(x) = \begin{cases} \{-1\}, & x > 0 \\ [-1,\ 1], & x = 0 \\ \{1\}, & x < 0 \end{cases} \tag{2.11}$$

The Filippov set $F(x)$ *for the vector fields of examples 3 and 4 is a set-valued map, and its multiple-valued only at discontinuity points of these vector fields.*

Now, we replace the dynamical system $\dot{x} = f(x)$ *by the differential inclusion* $\dot{x} \in F(x)$. *A Filippov solution of dynamical systems given in examples 3 and 4 on* $[0, t_1] \subseteq \mathbb{R}$ *is an absolutely continuous map* $x : [0, t_1] \to \mathbb{R}$ *that satisfies* $\dot{x} \in F(x)$ *for almost all* $t \in [0, t_1]$. *So, if* $x(0) > 0$ *the solution* $x : [0, \infty] \to \mathbb{R}$ *is*

$$x(t) = \begin{cases} x(0) - t, & t \leq x(0) \\ 0, & t \geq x(0) \end{cases} \tag{2.12}$$

If $x(0) < 0$ *the solution* $x : [0, \infty] \to \mathbb{R}$ *is*

$$x(t) = \begin{cases} x(0) + t, & t \leq -x(0) \\ 0, & t \geq -x(0) \end{cases} \tag{2.13}$$

Finally, if $x(0) = 0$, *then the solution* $x : [0, \infty] \to \mathbb{R}$ *is* $x(t) = 0$.

Remark 2.2.6. *The Filippov set* $F(x)$ *for the vector fields of examples 3 and 4 given in* (2.11) *is denoted as* $-sgn(x)$. *Therefore, differential inclusion* $\dot{x} \in F(x)$ *can be rewritten* $\dot{x} \in -sgn(x)$.

2.2.4 Utkin solutions

Utkin solutions can be viewed as a modification of Filippov solutions in terms of dynamical control systems. The control law $u(t,x)$ *may be interpreted as a physical parameter responsible of the switching.*

Consider the dynamical control system

$$\dot{x}(t) = f(t, x, u(t,x)) \tag{2.14}$$

with $f = (f_1, f_2, \ldots, f_n)^T$ *a continuous vector field, where* $x(t) \in \mathbb{R}^n$*, and* $t \in \mathbb{R}$ *are the same as before, the control input* $u(t,x) \in \mathbb{R}^m$ *is a piecewise continuous function of the states and time variables. The components of control signal presents discontinuities on possibly intersecting smooth surfaces* $\mathcal{S}_i = \{(t,x) \in \mathbb{R}^{n+1} : s_i(t,x) = 0\}$ *and to vary during the segments* $U_i(t,x) = [u_i^-(t,x), u_i^+(t,x)]$ *where*

$$u_i^-(t,x) = \lim_{(t,\hat{x}) \in \mathcal{S}_i^-, \hat{x} \to x} u(t,\hat{x}) \quad \mathcal{S}_i^- = \{(t,x) \in \mathbb{R}^{n+1} : s(t,x) < 0\} \tag{2.15}$$

$$u_i^+(t,x) = \lim_{(t,\hat{x}) \in \mathcal{S}_i^+, \hat{x} \to x} u(t,\hat{x}) \quad \mathcal{S}_i^+ = \{(t,x) \in \mathbb{R}^{n+1} : s(t,x) > 0\}' \tag{2.16}$$

At the continuity points (t,x) *of* $u_i(t,x)$*, the set* $U_i(t,x)$ *contains one unique point.*

The sliding modes at an intersection of some sets S_{jk} $k = 1, 2, \ldots, r$ *are governed by*

$$\dot{x}(t) = f(t, x, u^{eq}(t,x)) \tag{2.17}$$

where the components $u^{eq}(t,x) \in [u_i^-(t,x), u_i^+(t,x)]$ $i = 1, 2, \ldots m$ *of the equivalent control input* $u^{eq}(t,x)$ *are such that the velocity vector* f *in* (2.17) *is tangent to the sets* S_{jk} $k = 1, 2, \ldots, r$*. i.e., for* $(t,x) \in \bigcap_{k=1}^r S_{jk}$ *the equivalent control vector on components* $u^{eq}(t,x) \in [u_i^-(t,x), u_i^+(t,x)]$ *satisfies*

$$\frac{\partial s_{jk}(t,x)}{\partial t} + \nabla s_{jk}(t,x) f(t, x, u^{eq}) = 0 \quad k = 1, 2, \ldots r. \tag{2.18}$$

Definition 2.2.7. *[87] An absolutely continuous function* $x(\cdot)$*, defined on an interval* I*, is said to be an Utkin solution of* (2.14) *if it satisfies* (2.18) *beyond the surfaces* $\mathcal{S}_i$*,* $i = 1, 2, \ldots, m$ *and it satisfies equations of the form* (2.17) *on these surfaces and their intersections.*

Utkin solutions constitute a class of generalized solutions of (2.14) *under a particular regularization into* (2.14) *for the control input* u*.*

2.2.5 Filippov and Utkin solutions of affine systems

Unless stated otherwise, all dynamical systems studied here admit an affine representation, i.e., systems nonlinear with respect to a state vector and linear with respect to a control input. It is worth noting that this class of systems are the most common in practical applications [91].

Example 6. *Suppose that control system given in equation* (2.14) *admits the affine representation*

$$\dot{x} = a(t,x) + b(t,x)u, \quad (t,x) \in \mathbb{R}^{n+1}, \quad u \in \mathbb{R}^m \tag{2.19}$$

Filippov solutions

$$\dot{x} = f^0(t,x) = \mu(t,x) f^+(t,x) + (1 - \mu(t,x)) f^-(t,x) \tag{2.20}$$

yields

$$\begin{aligned}
\dot{x} &= [\mu(t,x)f^{+}(t,x) + (1-\mu(t,x))f^{-}(t,x)] \\
\dot{x} &= [\mu(t,x)(a(t,x)+b(t,x)u^{+}) + (1-\mu(t,x))(a(t,x)+b(t,x)u^{-})] \\
\dot{x} &= [\mu(t,x)(a(t,x)+b(t,x)u^{+}) + (a(t,x)+b(t,x)u^{-}) - \mu(t,x))(a(t,x)+b(t,x)u^{-})] \\
\dot{x} &= [\mu(t,x)(b(t,x)u^{+}) + (a(t,x)+b(t,x)u^{-}) - \mu(t,x))(b(t,x)u^{-})] \\
\dot{x} &= [\mu(t,x)(b(t,x)u^{+} - b(t,x)u^{-}) + (a(t,x)+b(t,x)u^{-})]
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
\mu(t,x) &= \frac{\partial s(t,x)/\partial t + \nabla s(t,x)(a(t,x)+b(t,x)u^{-})}{\nabla s(t,x)[(a(t,x)+b(t,x)u^{-}) - (a(t,x)+b(t,x)u^{+})]} \\
\mu(t,x) &= \frac{\partial s(t,x)/\partial t + \nabla s(t,x)(a(t,x)+b(t,x)u^{-})}{\nabla s(t,x)[b(t,x)u^{-} - b(t,x)u^{+})]}
\end{aligned} \tag{2.22}$$

replacing $\mu(t,x)$

$$\begin{aligned}
\dot{x} &= [\frac{\partial s(t,x)/\partial t + \nabla s(t,x)(a(t,x)+b(t,x)u^{-})}{\nabla s(t,x)[b(t,x)u^{-} - b(t,x)u^{+})]}(b(t,x)u^{+} - b(t,x)u^{-}) + (a(t,x)+b(t,x)u^{-})] \\
\dot{x} &= [\frac{-\partial s(t,x)/\partial t - \nabla s(t,x)(a(t,x)+b(t,x)u^{-})}{\nabla s(t,x)} + (a(t,x)+b(t,x)u^{-})] \\
\dot{x} &= [\frac{-\partial s(t,x)/\partial t - \nabla s(t,x)(a(t,x)+b(t,x)u^{-})}{\nabla s(t,x)} + (a(t,x)+b(t,x)u^{-})] \\
\dot{x} &= [\frac{-\partial s(t,x)/\partial t}{\nabla s(t,x)}] \\
0 &= \frac{\partial s(t,x)}{\partial t} + \nabla s(t,x)\dot{x}.
\end{aligned} \tag{2.23}$$

The last equation coincides with Utkin conditions given in (2.18).

In fact u_{eq} *satisfies*

$$\begin{aligned}
0 &= \frac{\partial s(t,x)}{\partial t} + \nabla s(t,x)[a(t,x)+b(t,x)u^{eq}] \\
u^{eq} &= -(\nabla s(t,x)b(t,x))^{-1}[\frac{\partial s(t,x)}{\partial t} + \nabla s(t,x)a(t,x)] \\
\dot{x} &= a(t,x) - b(t,x)(\nabla s(t,x)b(t,x))^{-1}[\frac{\partial s(t,x)}{\partial t} + \nabla s(t,x)a(t,x)]
\end{aligned} \tag{2.24}$$

provided that the matrix function $(\nabla s(t,x)b(t,x))^{-1}$ *is not singular for all* $(t,x) \in \mathbb{R}^{n+1}$*, the last equation describes Utkin solutions of the affine system* (2.19) *on the discontinuity manifold* $s(t,x) = 0$.

Hence, for affine systems Utkin and Filippov solutions coincide whenever $(\nabla s(t,x)b(t,x))^{-1}$ *is not singular. This coincidence of solutions is an advantage proper to affine systems in theory of sliding mode control.*

2.3 Higher sliding modes

Definition 2.3.1. *[3] Consider a piecewise continuous differential equation $\dot{x} = f(x)$ (a Filippov differential inclusion $\dot{x} = F(x)$) with a smooth output function $\sigma = \sigma(x)$, and let it be understood in the Filippov sense. Then, if*

1. *The total time derivatives $\sigma, \dot{\sigma}, \cdots, \sigma^{r-1}$ are continuous functions of x*
2. *The set*

$$\mathcal{S} = \{x : \sigma(x) = \dot{\sigma}(x) = \cdots = \sigma^{(r-1)}(x) = 0\}, \tag{2.25}$$

   is a non-empty integral set (i.e. consists of Filippov trajectories)

3. *The Filippov set of admissible velocities at the r-sliding points in equation (5.8) contains more than one vector.*

the motion on the set $\mathcal{S}$ with respect to sliding variable σ in (5.8) is said to exist in an r-sliding (rth-order sliding) mode. The set in Equation (5.8) is called the r-sliding set. The non-autonomous case is reduced to the one considered above by introducing the fictitious equation $\dot{t} = 1$.

2.3.1 Lie derivative and Relative degree.

Definition 2.3.2. *[3] Let $\sigma(x)$ a differentiable function, $x \in \mathbb{R}^n$, $f(x)$ a vector field (i.e., an n-dimensional vector function), then, the Lie derivative of σ with respect to f at the point x_0 is defined as*

$$L_f\sigma(x_0) = \nabla(\sigma(x_0)) \cdot f(x_0) = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\sigma(x_0) f_i(x_0) \tag{2.26}$$

Example 7. *Consider the autonomous dynamical system*

$$\begin{aligned} \dot{x} &= a(x) + b(x)u, \\ y &= \sigma(x) \end{aligned} \tag{2.27}$$

where u a scalar control input, $\sigma(x)$ a scalar output. $a : D \to \mathbb{R}^n$, $b : D \to \mathbb{R}^n$ and $\sigma : D \to \mathbb{R}^n$ are sufficiently smooth in a domain $D \subseteq \mathbb{R}^n$. The total derivative of the output σ is the time derivative of σ along the trajectory of the system

$$\begin{aligned} \dot{\sigma} &= \frac{d}{dt}\sigma(x(t)) \\ &= \nabla(\sigma(x(t))) \cdot f(x(t)) \\ &= \nabla(\sigma(x(t)))(a(x) + b(x)u) \\ &= L_a(\sigma(x)) + L_b(\sigma(x))u \end{aligned} \tag{2.28}$$

the following notation is used

$$L_bL_a(\sigma(x)) = \frac{\partial L_a\sigma}{\partial x} b(x) = \nabla(L_a\sigma(x(t)))(b(x)) \tag{2.29}$$

$$
\begin{aligned}
L_a^2\sigma(x) &= L_a L_a \sigma(x) = \frac{\partial L_a \sigma}{\partial x} a(x) = \nabla(L_a\sigma(x(t)))(a(x)) \\
L_a^i\sigma(x) &= L_a L_a^{i-1} \sigma(x) = \frac{\partial L_a^{i-1} \sigma}{\partial x} a(x) = \nabla(L_a^{i-1}\sigma(x(t)))(a(x)) \\
L_a^0\sigma(x) &= \sigma(x)
\end{aligned}
$$

Definition 2.3.3. *[90] Consider the smooth dynamic system* (2.27)*, u a scalar control input and $\sigma(x)$ a scalar output. The number r is called the relative degree of the output σ of the system* (2.27) *with respect to the input u at the point $x_0 \in D$, if the conditions*

$$
L_b(\sigma(x)) = L_b L_a(\sigma(x)) = \cdots = L_b L_a^{r-2}(\sigma(x)) = 0, \quad L_b L_a^{r-1}(\sigma(x)) \neq 0 \tag{2.30}
$$

hold in some vicinity of point x_0.

Remark 2.3.4. *The case when system* (2.27) *is non autonomous*

$$
\begin{aligned}
\dot{x} &= a(t,x) + b(t,x)u, \\
y &= \sigma(t,x)
\end{aligned} \tag{2.31}
$$

can be handled introducing the fictitious equation $\dot{t} = 1$, and definition above can be applied for the vector field $\hat{a}(t,x) + \hat{b}(t,x)u$ in some vicinity of (t_0, x_0), where $\hat{a}(t,x) = [a(t,x), 1]^T$, $\hat{b}(t,x) = [b(t,x), 0]^T$.

Example 8. *In case that the output of the system of equation* (2.31) *satisfies $\sigma^{(i)} = L_{\hat{a}}^i \sigma, i = 1, 2, \cdots r-1$, and $\sigma^{(r)} = L_{\hat{a}}^r \sigma + L_{\hat{b}} L_{\hat{a}}^{r-1}(\sigma)u$ the relative degree of the system is r.*

The equation

$$
\sigma^{(r)} = L_{\hat{a}}^r \sigma + L_{\hat{b}} L_{\hat{a}}^{r-1}(\sigma)u
$$

shows that the system (2.31) *is input-output linearisable since the control law*

$$
u = \frac{1}{L_{\hat{b}} L_{\hat{a}}^{r-1}(\sigma)}[v]
$$

reduces the input output map to a chain of r integrators

$$
\begin{aligned}
\sigma^{(r)} &= v + L_{\hat{a}}^r \sigma \\
&= v + h(t,x)
\end{aligned} \tag{2.32}
$$

where the term $L_{\hat{a}}^r \sigma = h(t,x)$ possibly contains vanishing and nonvanishing disturbances.

It is worth nothing that σ^i, $i = 1, 2, \cdots r-1$ are continuous functions of t, x, and keeping $\sigma = 0$ by means of switching control is only possible in the r-sliding mode.

2.3.2 Single-Input Single-Output control problem

Consider the uncertain non-linear system (2.31) *where $x \in \mathbb{R}^n$ is the state variable, u a scalar control input, and $\sigma(t,x) \in \mathbb{R}$ a scalar output available on real time. $a(t,x)$, $b(t,x)$ and $\sigma(t,x) : D \to \mathbb{R}^n$ are uncertain smooth functions in a domain $D \subseteq \mathbb{R}^{n+1}$.*

Note that dimension n might be uncertain.

We assume that:

- *The relative degree r of the system with respect to σ is constant and known.*
- *The solutions are understood in the Filippov sense, and system trajectories are infinitely extendible in time for any Lebesgue-measurable bounded control u.*

Following the arguments given in subsection above, the output σ satisfies

$$\sigma^r = h(t,x) + g(t,x)u \tag{2.33}$$

where $h(t,x) = L_{\hat{a}}^r \sigma$, and $g(t,x) = L_{\hat{b}} L_{\hat{a}}^{r-1}(\sigma(x))$.

It is assume that $g(t,x) > 0$, and for some constants K_m, $K_M, C > 0$

the inequalities

$$0 < k_m \leq g(t,x) \leq K_M, \quad |h(t,x)| \leq C \tag{2.34}$$

hold (always true at least locally).

The aim is to design a control law $u(t,x)$ that produces fixed-time convergence of the states $\sigma, \dot{\sigma}, \ldots \sigma^{(n-1)}$ of the uncertain system (2.33) *under the conditions in* (2.34)*. Note that exact knowledge of the parameters K_m, K_M, and C is not necessary in practice, because they only influence the magnitude of the control to be designed.*

Then, the r-sliding mode control u of (2.33) *with respect to sliding variable σ is equivalent to fixed-time stabilization of a n-dimensional chain of integrators ($r = n$)*

$$\begin{array}{rcll} \dot{x}_1(t) &=& x_2(t), & x_1(t_0) = x_{10}, \\ \dot{x}_2(t) &=& x_3(t), & x_2(t_0) = x_{20}, \\ \vdots & \vdots & \vdots & \vdots \\ \dot{x}_n(t) &=& h(t,x) + g(t,x)u(t,x), & x_n(t_0) = x_{n0}, \\ y(t) &=& x_1(t), & \end{array} \tag{2.35}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T = [\sigma, \dot{\sigma}, \ldots \sigma^{(n-1)}] \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}$ is the state measurement (observation), $u(t) \in \mathbb{R}$ is the control input.

It is worth nothing that implementing the HOSM controller given by Eq. (2.33) *requires observation of the derivatives or real-time differentiation of the output.*

Remark 2.3.5. *The latter result is important, because under some assumptions, it reduces research in control-affine dynamics to research in control of chain of integrators.*

The following section introduce the finite and fixed time convergence notions.

2.4 Finite and fixed time convergence

Finite time convergence of asymptotically convergent homogeneous dynamical systems whose homogeneity degree is negative has been well-studied for continuous vector fields [11, 12, 13, 14]. Extending this result to piecewise discontinuous systems has required non-smooth Lyapunov functions and follows two strands. [92, 87] introduces the equistability concept and shows that finite time convergence can be demonstrated for piecewise continuous homogeneous vector fields in presence of inhomogeneous perturbations. On the other hand [23, 24], the convergence of a discontinuous system (2.3) *with possibly non-uniquely defined trajectories is introduced by means of a combination of Bhat and Bernstein definitions [11, 12, 13] and the corresponding differential inclusion. The latter is the strand followed hereafter.*

Suppose $x = 0$ *is an equilibrium point of the differential inclusion* (2.7) *and* $\phi^{x_0}(t)$ *denotes a solution of* (2.7) *under the initial conditions* $x(t_0) = x_0$.

2.4.1 Finite time convergence

Definition 2.4.1. *[24] The origin of the* (2.7) *is said to be finite time convergent at the origin, if there exists an open vicinity* $N \subseteq D$ *at the origin and a function* $T : N - \{0\} \to \mathbb{R}_+$, *called the settling time function, such that for all* $x_0 \in N - \{0\}$, $\phi^{x_0}(t)$ *is defined and unique on* $[0, T(x_0))$, $\phi^{x_0}(t) \in N - \{0\}$ *for all* $t \in [0, T(x_0))$, $\lim_{t\to T(x_0)} \phi^{x_0}(t) = 0$ *and* $\phi^{x_0}(t) = 0$, *for all* $t \geq T(x_0)$.

The origin is said to be globally finite time convergent equilibrium if it is finite time convergent with $D = N = \mathbb{R}^n$

Example 9. *Consider the vector fields in examples 3 and 4. Any solution of these systems reach the origin in a finite time* $T(x_0) = |x_0|$ *and remain there for all future time.*

Example 10. *For* $q > 0$ *consider the scalar system*

$$\frac{dx}{dt} = -k|x|^q sign(x) \quad , x(t_0) = x_0 \tag{2.36}$$

equation (5.9) *can be readily integrated to obtain*

$$x(\tau) = \begin{cases} sign(x(\tau))[|x(\tau_0)|^{1-q} + k(1-q)(\tau_0 - \tau)]^{1/(1-q)}, & if \quad 0 \leq \tau(x) < \tau_0 + \frac{|x(\tau_0)|^{1-q}}{k(1-q)}, \quad x \neq 0 \\ 0, & if \quad \tau(x) \geq \tau_0 + \frac{|x(\tau_0)|^{1-q}}{k(1-q)}, \quad x \neq 0 \\ 0, & if \quad \tau \geq 0, \quad x = 0 \end{cases} \tag{2.37}$$

The origin of dynamical system (5.9) *is asymptotically convergent if and only if* $k > 0$ *and finite time convergent if and only if* $k > 0$ *and* $q < 1$. *In these cases the vector field is everywhere continuous, locally Lipschitz except at the origin, in such way that any initial condition in* $\mathbb{R} - \{0\}$ *has a unique solution in all future time on a sufficiently small time interval. This system converges in finite time at the origin for* $k > 0$,

The dynamical system is said to be globally convergent in finite time if $k > 0$ *and* $q < 1$ *for any initial condition.*

Sometimes, in addition to finite-time convergence, uniform boundedness of the settling time function is assumed on a set of initial conditions (attraction domain). This extension is called Fixed-time convergence.

2.4.2 Fixed time convergence

Definition 2.4.2. *[24] The origin of the differential inclusion* (2.7) *is called equilibrium convergent in fixed time if there exists an open vicinity $N \subseteq D$ at the origin and a function $T : N - \{0\} \to \mathbb{R}_+$, called settling time function such that*

1. *the system is convergent in finite time*
2. *The settling time function T is independent of initial conditions $x_0 \in N - \{0\}$. e.d; function T is bounded on N, such that exists a number $T_{max} \in \mathbb{R}^{\geq 0}$ with $T(x_0) \leq T_{max}$ for all $x_0 \in \mathbb{R}^n$.*

the origin of the system (2.7) *is said to be globally convergent in fixed time if it is convergent in fixed time with $D = N = \mathbb{R}^n$.*

Example 11. *Let consider*

$$\frac{dx}{dt} = -k_1|x|^q sign(x) - k_2|x|^p sign(x), \quad x(t_0) = x_0 \tag{2.38}$$

This system converges in fixed time at the origin for $k > 0$, $0 < q < 1$, $1 < p$, and $q + p = 2$.

equation (2.38) *can be readily integrated to obtain $x(\tau) =$*

$$\begin{cases} sign(x(\tau_0))[(\frac{k_1}{k_2})^{\frac{1}{2}} \tan[arctan((\frac{k_2}{k_1})^{\frac{1}{2}}|x(\tau_0)|^{1-q}) - k_1(1-q)(\frac{k_2}{k_1})^{\frac{q+(q-p)^2}{q-p}}(\tau-\tau_0)]]^{\frac{1}{1-q}}, & if \quad 0 \leq \tau(x) < \tau_0 + \tau_{max}, \quad x \neq 0 \\ 0, & if \quad \tau(x) \geq \tau_0 + \tau_{max}, \quad x \neq 0 \\ 0, & if \quad \tau \geq 0, \quad x = 0 \end{cases} \tag{2.39}$$

here

$$\tau_{max} = \frac{arctan((\frac{k_2}{k_1})^{\frac{1}{2}}|x(\tau_0)|^{1-q})}{k_1(1-q)(\frac{k_2}{k_1})^{\frac{q+(q-p)^2}{q-p}}}$$

.

If $x \to \infty$, then $\arctan(x) \to \frac{\pi}{2}$, any solution of this system converges to the origin in finite time for all

$$\tau(x) \leq \tau_0 + \frac{\pi}{2k_1(1-q)(\frac{k_2}{k_1})^{\frac{q+(q-p)^2}{q-p}}}$$

i.e the system is fixed time convergent with $T_{max} = \frac{\pi}{2k_1(1-q)(\frac{k_2}{k_1})^{\frac{q+(q-p)^2}{q-p}}}$ (the convergence is independent of initial conditions).

Note that if $q = 0.5$, $p = 1.5$, we obtain Therefore $x(\tau) =$

$$\begin{cases} sign(x(\tau_0))[(\frac{k_1}{k_2})^{\frac{1}{2}} \tan[arctan((\frac{k_2}{k_1})^{\frac{1}{2}}|x(\tau_0)|^{0.5}) - k_1(0.5)(\frac{k_1}{k_2})^{\frac{3}{2}}(\tau-\tau_0)]]^2, & if \quad 0 \leq \tau(x) < \tau_0 + \tau_{max}, \quad x \neq 0 \\ 0, & if \quad \tau(x) \geq \tau_0 + \tau_{max}, \quad x \neq 0 \\ 0, & if \quad t \geq 0, \quad x = 0 \end{cases} \tag{2.40}$$

$$\tau_{max} = \frac{\pi}{k_1(\frac{k_1}{k_2})^{\frac{3}{2}}}$$

Any solution of this system converges to the origin in finite time for all $\tau(x) \leq \tau_0 + \tau_{max}$ *i.e the system is fixed time convergent with convergence time* τ_{max} *(the convergence time is independent of initial conditions).*

The example below shows that fixed time convergence and finite time convergence concepts are very similar from a local perspective

Example 12. *[24] Consider*

$$\dot{x} = -|x|^{1/2} sign(x)[1 - |x|] \tag{2.41}$$

$$\dot{x} = \begin{cases} -|x|^{1/2} sign(x) & if \quad x < 1 \\ 0 & if \quad x \geq 1 \end{cases} \tag{2.42}$$

Take the open ball centered at 0 *and ratio* 1, $N = B(0,1)$. *The settling time functions of these systems*

$$T_1(x_0) = ln(\frac{1 + |x_0|^{1/2}}{1 - |x_0|^{1/2}})$$

$$T_2(x_0) = 2|x_0|^{1/2}$$

are continuous on N, *so for any* $y \in N$ *fix the ball* $\{y\} + B(0,\epsilon) \subseteq N$ *(open ball centered at* y *and ratio* ϵ*) where* $\epsilon = (1 - |y|)/2$, *such that* $sup_{x_0 \in \{y\}+B(0,\epsilon)} T_1(x_0) < \infty$ *and* $sup_{x_0 \in \{y\}+B(0,\epsilon)} T_2(x_0) < \infty$.

Now, $T_1(x_0) \to \infty$ *if* $x_0 \to 1$, *whereas* $T_2(x_0) \to 2$ *if* $x_0 \to 1$. *It is said that the systems are fixed time convergent on* N. *however, the first system loses the property in set* $N \cup \{1\}$.

2.4.3 Fixed time convergent control systems

Definition 2.4.3. *[47]. A dynamic control system*

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0, \tag{2.43}$$

where $x(t) \in R^n$ *is the vector state,* $u(t) \in \mathbb{R}^m$ *is the control law, is said to be fixed-time convergent at the origin, if there exists a time moment* T *(independent of initial conditions) such that the system state* $x(t)$ *is equal to zero,* $x(t) = 0$, *for all* $t \geq T$, *starting from any initial condition* $x_0 \in \mathbb{R}^n$.

Definition 2.4.4. *[47]. A dynamic control system*

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0, \tag{2.44}$$

where $x(t) \in R^n$ *is the vector state,* $u(t) \in \mathbb{R}^m$ *is the control law, is said to be fixed-time convergent at a certain vicinity* $N \subset R^n$ *of the origin, if there exists a time moment* T *(independent of initial conditions) such that if the system state* $\psi(t)$ *enters* N, $\psi(t) \in N$, *it remains there for all* $t \geq T$, *starting from any initial condition* $\psi_0 \in R^n$.

Remark 2.4.5. *Proof of finite/fixed-time convergence from definition is not easy task. Hence in various occasions we use Lyapunov Functions, homogeneity theory (see appendix B), and settling time estimations.*

Hereafter, a result is included about settling time estimation that will be useful in succeeding chapters.

2.4.4 Settling time estimation of Fixed- time convergent Super-twisting Algorithm

In this section and upper bound of admissible disturbances and lower bound of the convergence time are calculated for the Super-twisting algorithm [25, 93]. The desired accuracy of these quantities can be guaranteed follows the Utkin approach [94].

Theorem 2.4.6. *[25, 93] Consider the dynamical system*

$$\begin{aligned} \dot{s}(t) &= -\lambda_1 \mid s(t) \mid^{1/2} sign(s(t)) \\ &- \lambda_2 \mid s(t) \mid^p sign(s(t)) - \alpha \int_{t_0}^{t} sign(s(\tau))d\tau. \end{aligned} \tag{2.45}$$

Here, the control gains $\lambda_1, \lambda_2, \alpha > 0$ are greater than zero, $s(t)$ is the state, and $p > 1$.

In the presence of a disturbance $\xi(t)$ satisfying Lipschitz condition with constant L, and an initial condition $y(t_0) = \xi(t_0)$ bounded by a constant K, the system above can be rewritten,

$$\begin{aligned} \dot{r}(t) &= -\lambda_1 \lfloor r(t) \rceil^{1/2} - \lambda_2 \lfloor r(t) \rceil^p + y(t), \quad r(t_0) = r_0, \\ \dot{y}(t) &= -\frac{\alpha}{2} sign(r(t)) + \dot{\xi}(t), \quad y(t_0) = y_0, \end{aligned} \tag{2.46}$$

where $\dot{y}(t)$ exists almost everywhere and bounded by the constant L, $\mid \dot{y}(t) \mid \leq L$. Then, both states $r(t)$ and $y(t)$ converge to the origin uniformly in fixed time

$$T_{SW} \leq \left(\frac{1}{\lambda_2(p-1)\epsilon^{p-1}} + \frac{K}{B} + \frac{2\epsilon^{1/2}}{\lambda_1}\right) \times \left(1 + \frac{1}{b\left(\frac{1}{B} - \frac{h(\lambda_1)}{\lambda_1}\right)}\right) + \frac{K}{B} \tag{2.47}$$

Here, $\epsilon > 0$, $B = \alpha + L$, $b = \alpha - L$, $h(\lambda_1) = 1/\lambda_1 + (2e/b\lambda_1)^{1/3}$, and e is the base of natural logarithms, provided that the following conditions hold for control gains: $\alpha > L$, $\lambda_1 h^{-1}(\lambda_1) > B$. The minimum value of $T_f(\epsilon)$ is reached for $\epsilon = (\lambda_1/\lambda_2)^{\frac{1}{p+1/2}}$.

Proof of theorem. A. *Consider $|r_0| > \epsilon$, where $\epsilon > 0$, is a constant, and $sign(y_0)sign(r_0) \leq 0$. The first equation in system (2.46) yields*

$$\begin{aligned} \frac{d|r(t)|}{dt} &= sign(r(t))\frac{dr(t)}{dt} \\ &= -\lambda_1 |r(t)|^{1/2} - \lambda_2 |r(t)|^p + y(t) sign(r(t)) \\ &\leq -\lambda_2 |r(t)|^p. \end{aligned} \tag{2.48}$$

taking into account that sign(y(t)) remains opposite to sign(r(t)) for $t > t_0$*, while* $r(t)$ *does not cross* $r(t) = 0$*, and* $\alpha > L$*. Solving (2.48) with an initial condition* $r(t_0) = r_0$ *yields*

$$\frac{|r(t)|^{1-p}}{1-p} \leq -\lambda_2(t-t_0) + \frac{|r(0)|^{1-p}}{1-p} \leq -\lambda_2(t-t_0)$$

and

$$|r(t)|^{p-1} \leq \frac{1}{\lambda_2(p-1)(t-t_0)}.$$

Therefore, $|r(t)|$ *decreases and reaches the value* $|r(t)| = \epsilon$ *for a time* $T_1 \leq \frac{1}{\lambda_2(p-1)\epsilon^{p-1}}$*, which corresponds to the first term in (2.47) and is independent of an unknown initial condition* $r(t_0) = r_0$*. Step **A** ends with* $|r(T_1)| = \epsilon > 0$*. If* $|r_0| \leq \epsilon$*, then step **A** is not executed; therefore the term* $\frac{1}{\lambda_2(p-1)\epsilon^{p-1}}$ *would be absent in (2.47).*

B*. For* $t > T_1$*,* $|r(t)|$ *decreases to zero in time* T_2*. Taking into account that sign(y(t)) remains opposite to sign(r(t)) on interval* $[T_1, T_2]$*, the first equation in 2.46 yields*

$$\frac{d|r(t)|}{dt} \leq -\lambda_1|r(t)|^{1/2}. \tag{2.49}$$

solving 2.49 between T_1 *and* t *implies*

$$\begin{aligned} |r(t)|^{-1/2}d|r(t)| &\leq -\lambda_1 dt \\ 2[|r(t)|^{1/2} - |r(T_1)|^{1/2}] &\leq -\lambda_1(t-T_1) \\ |r(t)|^{1/2} &\leq |r(T_1)|^{1/2} - \frac{1}{2}\lambda_1(t-T_1) \\ |r(t)|^{1/2} &\leq \epsilon^{1/2} - \frac{1}{2}\lambda_1(t-T_1) \end{aligned} \tag{2.50}$$

so $|r(t)|$ *decreases and reaches zero for a time* $T_2 \leq \frac{2\epsilon^{1/2}}{\lambda_1}$ *which corresponds to the second term in* (2.47).

*Step **B** ends with* $|r(T_2)| = 0$*,* $y(t)$ *increases in* $[t_0, T_2]$ *because the sign(y(t)) remains opposite to sign(r(t)) and* $\alpha > L$*. The value* $|y(T_2)|$ *is bounded by* $|y(T_2)| < K + B(T_2 - t_0) = K + B(\frac{1}{\lambda_2(p-1)\epsilon^{p-1}} + \frac{2\epsilon^{1/2}}{\lambda_1})$ *by the second equation in* (2.46).

C*. Consider now that* $sign(y_0) = sign(r_0)$*. Then,* $r(t)$ *cannot reach zero until sign(y(t)) becomes opposite to sign(r(t)), since* $y(t)$ *has to reach zero first. In view of the second equation in (2.46), the convergence time of* $y(t)$ *to zero can be estimated from above as* $T_3 = \frac{K}{B}$ *and added to the time* T_2 *calculated in step **B**.*

D*.The trajectory of the system* (2.46) *starting at (0, y(T2)) is dominated by the trajectory of the system* (2.46) *starting at* $(0, y_2 = K + B(\frac{1}{\lambda_2(p-1)\epsilon^{p-1}} + \frac{2\epsilon^{1/2}}{\lambda_1}))$ *and converges to the origin faster. In according to Theorem 4.5 in [3], the finite convergence time for the latter trajectory is bounded by the expression can be estimated by the formula* $T_{STW} \leq \sum \dot{r}(t_i)/b$*, where* t_i *are subsequent time moments such that* $r(t_i) = 0$*,* $\dot{r}(t_i) = y(t_i)$ *if* $q_c < 1$ *for* $q_c = \frac{\dot{r}(t_2)}{\dot{r}(t_1)}$*. As step **D** begins in* $T_2 + T_3$*, then* T_{STW} *could be calculated* $T_{STW} = y_2/(1-q_c)b$*, where* $q_c \leq Bh(\lambda_1)/\lambda_1$ *a formula derived in [94]. The condition* $q_c < Bh(\lambda_1)/\lambda_1$ *here corresponds to gains in second condition of this theorem. Following [94], the first* *condition* $\alpha > L$ *is required for convergence of the super-twisting algorithm. Substituting the obtained estimates for* $y_2 = |y(T_2)|$ *and* q_c *yields to* $T_{STW} \leq (\frac{1}{\lambda_2(p-1)\epsilon^{p-1}} + \frac{K}{B} + \frac{2\epsilon^{1/2}}{\lambda_1})/b(1 - B\frac{h(\lambda_1)}{\lambda_1})$ *which, after dividing both parts of the fraction by B, corresponds to the former terms in* (2.47)*. The optimal value of* ϵ *is determined in [25] minimizing the first two terms in* (2.47) *respect to* ϵ*.* ■

A review of settling time estimations explicit and easily computable for the corresponding convergence times of finite and fixed-time convergent algorithms for different classes of dynamic systems is presented in [95].

Chapter 3

Continuous Fixed-Time Controller Design

"Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object."

Charles Sanders Peirce (1839–1914)

3.1 Introduction

This chapter presents a non-linear continuous observer-based controller that provides fixed-time convergence to the origin (or its vicinity) of all states of a higher order dynamic system subject to matched and unmatched Lipschitz disturbances, whose only measurable state is the highest relative degree one. The problem concerning the settling time estimation is studied computing an uniform upper bound of the fixed convergence time of the designed controller. Content of this chapter is published in [83, 84].

The importance of having fixed-time convergent observer-based controller for finite-dimensional non-linear systems is that the convergence is uniform and therefore the calculation of the upper bounds is independent of the initial conditions of the system. The main drawback with finite-time convergent observer-based controllers is that although the error converges accurately, their dependence on the initial conditions result in the fact that if the norm of an initial condition in an unbounded manner, its convergence time tends to infinity, preventing the pre-establishment of the time for which errors converges to zero even for a set of initial conditions. Finally, it should be noted that in most of the revised literature, fixed-time convergence is obtained by discontinuous actions [96, 97]. Our paper works with continuous injections, and the considered disturbances may be unbounded.

Examples of various technical systems, where the designed controller would be applicable, can be found in [98, 99].

The contribution of this chapter is in

- *Presenting a continuous fixed-time convergent observer-based controller algorithm driving the output (the highest relative degree state) of an n-dimensional integrator to a vicinity of zero for a fixed time using a scalar input in the equation for the lowest relative degree state against unbounded disturbances, which do not assume knowledge of all system states: only the output should be measured.*

- *The controller involves continuous injections, the considered disturbances may be unbounded and a calculation of uniform upper bound (independent of the initial conditions of the system) for the convergence time of the convergent controller is obtained.*

- *The performance of the controller is verified through the design and computational implementation in a case study of a DC motor. The accuracy of the controller is examined and found consistent with the results obtained in [31].*

Sections of chapter are organized as follows: The problem statement is given in section 3.2. A continuous fixed time observer-based controller driving the output to a vicinity of zero of a dynamic system subject to matched and unmatched disturbances is designed in section 3.3. The main theorem providing a settling time estimate is given in section 3.4. A case study: an industrial armature-controlled DC motor system affected with unbounded disturbances and conclusions are presented in Sections 3.5 and 3.6 respectively.

3.2 Problem Statement

Consider an n-dimensional chain of integrators

$$\begin{array}{rcll} \dot{x}_1(t) & = & x_2(t), & x_1(t_0) = x_{10}, \\ \dot{x}_2(t) & = & x_3(t), & x_2(t_0) = x_{20}, \\ \vdots & \vdots & \vdots & \quad\vdots \\ \dot{x}_n(t) & = & u(t) + \xi(t), & x_n(t_0) = x_{n0}, \\ y(t) & = & x_1(t), & \end{array} \tag{3.1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^n$ is the vector state, $y(t) \in R$ is the state measurement (observation), $u(t) \in R$ is the control input, and $\xi(t)$ is an unbounded external disturbance satisfying the Lipschitz condition with a constant L.

Remark 3.2.1. *In [32], the following dynamical system is considered:*

$$\dot{x} = \eta(x, u), \tag{3.2}$$

$$y = h(x), \tag{3.3}$$

where $\xi \in \mathbb{R}^d$ is the state vector, $u \in \mathbb{R}^m$ is a control law, $y(t) \in \mathbb{R}$ is the output depending on the state of the highest relative degree, $\eta : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is a continuous vector field. It is assumed that the nonlinear system (3.2),(3.3) is locally observable [100] and there exists a transformation of this nonlinear system into the canonical form (3.1) [101], [102]. This naturally extends the class of considered systems to nonlinear systems (3.2),(3.3).

The control objective is to design continuous control law that drives all state variables of the system (3.1) at the origin (a vicinity of the origin) for a finite pre-established (fixed) time. Since only the scalar output $y(t) = x_1(t)$ is measured, a fixed-time convergent differentiator should first be employed to estimate values of all state components for a finite pre-established (fixed) time. Then, based on the obtained estimates, a continuous fixed-time convergent controller with a compensation term should be designed to *drive all the states at the origin (a vicinity of the origin) for a finite pre-established (fixed) time using a scalar control input.*

The next section presents continuous fixed-time observer-based controller design for the system (3.1) and an upper estimate for the corresponding fixed convergence time.

3.3 Observer-based Controller Design

3.3.1 Fixed-Time Convergent Differentiator

Consider the n-dimensional chain of integrators given in (3.1). Since only the scalar output $y(t) = x_1(t)$ *can be measured, a smooth fixed-time convergent differentiator is designed to reconstruct the states of the system (3.1):*

$$\begin{aligned}
\dot{z}_1(t) &= z_2(t) - k_1 \mid y(t) - z_1(t) \mid^{\alpha_1} sign(y(t) - z_1(t)) \\
&\quad -\kappa_1 \mid y(t) - z_1(t) \mid^{\beta_1} sign(y(t) - z_1(t)), \\
&\vdots \\
\dot{z}_i(t) &= z_{i+1}(t) - k_i \mid y(t) - z_1(t) \mid^{\alpha_i} sign(y(t) - z_1(t)) \\
&\quad -\kappa_i \mid y(t) - z_1(t) \mid^{\beta_i} sign(y(t) - z_1(t)), \\
&\quad i = 2, \ldots, n-1, \\
&\vdots \\
\dot{z}_n(t) &= -k_n \mid y(t) - z_1(t) \mid^{\alpha_n} sign(y(t) - z_1(t)) \\
&\quad -\kappa_n \mid y(t) - z_1(t) \mid^{\beta_n} sign(y(t) - z_1(t)).
\end{aligned} \tag{3.4}$$

Here, the exponents α_i, $i = 1, \ldots, n$, *and* β_i, $i = 1, \ldots, n$, *are selected as follows:* $\alpha_i \in (0, 1)$, $i = 1, \ldots, n$ *satisfy the recurrent relations* $\alpha_i = i\alpha - (i-1)$, $i = 2, \ldots, n$, *and* $\alpha_1 = \alpha$, *where* α *belongs to an interval* $(1 - \epsilon_1, 1)$ *for a sufficiently small* $\epsilon_1 > 0$; $\beta_i > 1$, $i = 1, \ldots, n$ *satisfy the recurrent relations* $\beta_i = i\beta - (i-1)$, $i = 2, \ldots, n$, *and* $\beta_1 = \beta$, *where* β *belongs to an interval* $(1, 1 + \epsilon_2)$ *for a sufficiently small* $\epsilon_2 > 0$. *Observer gains* k_i, $i = 1, \ldots, n$, *and* κ_i, $i = 1, \ldots, n$, *are assigned such that the matrices* A_1 *and* A_1' *defined as*

$$A_1 = \begin{pmatrix} -k_1 & 1 & 0 & \ldots & 0 \\ -k_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{n-1} & 0 & 0 & \ldots & 1 \\ -k_n & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$A_1' = \begin{pmatrix} -\kappa_1 & 1 & 0 & \ldots & 0 \\ -\kappa_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\kappa_{n-1} & 0 & 0 & \ldots & 1 \\ -\kappa_n & 0 & 0 & \ldots & 0 \end{pmatrix},$$

are Hurwitz. The observer variable $z_i(t)$ *is the estimate for* $y^{(i-1)}(t)$, $i = 1, \ldots, n-1$. *Note that* $z_n(t)$ *is a smooth function, so* $z_i(t)$ *has continuous derivatives up to order* $n - i + 1$.

The symmetric positive definite matrices P_1 and P_1' are introduced to satisfy Lyapunov equations

$$P_1A_1 + A_1^TP_1 = -Q_1, \qquad P_1'A_1' + A_1'^TP_1' = -Q_1',$$

respectively, where $Q_1, Q_1' \in R^{n\times n}$ are symmetric positive definite matrices.

The resulting observer estimation error system for the error $e(t) = e_1(t) = y(t) - z_1(t)$ and its derivatives $e_i(t) = e^{(i-1)}(t) = y^{(i-1)}(t) - z_i(t)$, $i = 2,\dots,n-1$ takes the form

$$\begin{aligned}
\dot{e}_1(t) &= e_2(t) - k_1 \mid e(t) \mid^{\alpha_1} sign(e(t)) \\
&\quad -\kappa_1 \mid e(t) \mid^{\beta_1} sign(e(t)), \\
&\vdots \\
\dot{e}_i(t) &= e_{i+1}(t) - k_i \mid e(t) \mid^{\alpha_i} sign(e(t)) \\
&\quad -\kappa_i \mid e(t) \mid^{\beta_i} sign(e(t)), \\
&\quad i = 2,\dots,n-1, \\
&\vdots \\
\dot{e}_n(t) &= -k_n \mid e(t) \mid^{\alpha_n} sign(e(t)) \\
&\quad -\kappa_n \mid e(t) \mid^{\beta_n} sign(e(t)).
\end{aligned} \tag{3.5}$$

Remark 3.3.1. *The convergence of differentiator to zero occurs in absence of disturbances. In the presence of disturbances the convergence is to a tube around zero.*

3.3.2 Fixed-Time Convergent Regulator

Consider the following control input for the system (3.1)

$$u(t) = u_1(t) + u_2(t), \tag{3.6}$$

where the control law $u_1(t)$, is given by

$$\begin{aligned}
u_1(t) = v_1(t) + v_2(t) + \dots + v_n(t) + \\
+w_1(t) + w_2(t) + \dots + w_n(t),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
v_i(t) &= -m_i \mid x_i(t) \mid^{\gamma_i} sign(x_i(t)), \\
w_i(t) &= -M_i \mid x_i(t) \mid^{\delta_i} sign(x_i(t)), \\
&\quad i = 1,\dots,n.
\end{aligned}$$

Here, the exponents γ_i and δ_i, $i = 1,\dots,n$, are selected as follows: $\gamma_i \in (0,1)$, $i = 1,\dots,n$, satisfy the recurrent relations $\gamma_{i-1} = \gamma_i\gamma_{i+1}/(2\gamma_{i+1} - \gamma_i)$, $i = 2,\dots,n$, $\gamma_{n+1} = 1$ and $\gamma_n = \gamma$; $\delta_i > 1$, $i = 1,\dots,n$ satisfy the recurrent relations $\delta_{i-1} = \delta_i\delta_{i+1}/(2\delta_{i+1} - \delta_i)$, $i = 2,\dots,n$, $\delta_{n+1} = 1$ and $\delta_n = \delta$, where γ belongs to an interval $(1-\epsilon, 1)$ and δ belongs to an interval $(1, 1+\epsilon_1)$, for sufficiently small $\epsilon > 0$ and $\epsilon_1 > 0$. Control gains m_i and M_i, $i = 1,\dots,n$, are assigned such that $s^n + m_ns^{n-1} + \dots + m_1$ and $s^n + M_ns^{n-1} + \dots + M_1$ are Hurwitz polynomials.

The disturbance compensator control law $u_2(t)$ is assigned as

$$\begin{aligned} u_2(t) &= -\lambda_1 \mid s(t) \mid^{1/2} sign(s(t)) \\ &- \lambda_2 \mid s(t) \mid^{p} sign(s(t)) - \alpha \int_{t_0}^{t} sign(s(\tau))d\tau. \end{aligned} \tag{3.8}$$

Here, the control gains $\lambda_1, \lambda_2, \alpha > 0$ are greater than zero, the sliding surface $s(t)$ is defined as $s(t) = x_n(t) - \vartheta(t)$, $\dot{\vartheta}(t) = u(t) - u_2(t)$, and $p > 1$ (see appendix A).

The symmetric positive definite matrices P and P' are introduced to satisfy Lyapunov equations

$$PA + A^T P = -Q, \qquad P'A' + A'^T P' = -Q',$$

respectively, where $Q, Q' \in R^{n \times n}$ are symmetric positive definite matrices, the matrices A and A' are defined as

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -m_1 & -m_2 & -m_3 & \dots & -m_n \end{pmatrix}$$

and

$$A' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -M_1 & -M_2 & -M_3 & \dots & -M_n \end{pmatrix},$$

$\lambda_{\min}(Q)$ and $\lambda_{\min}(Q')$ are the minimum eigenvalues of the matrices Q and Q', respectively, and $\lambda_{\max}(P)$ and $\lambda_{\max}(P')$ are the maximum eigenvalues of the matrices P and P', respectively.

Remark 3.3.2. *: Note that the disturbance compensator control law u_2 in (3.8) actually defines a super-twisting observer for the disturbance $\xi(t)$, so that u_2 converges to $\xi(t)$ for a fixed time.*

3.3.3 Fixed-Time Convergent Controller

Combining the fixed-time convergent differentiator and regulator from the preceding subsections, the fixed-time convergent controller for the system (3.1) is designed employing control law (3.6) where the control input $u_1(t)$ is given by

$$\begin{aligned} u_1(t) = v_1(t) + v_2(t) + \dots + v_n(t) + \\ + w_1(t) + w_2(t) + \dots + w_n(t), \end{aligned} \tag{3.9}$$

$$\begin{aligned} v_i(t) &= -m_i \mid z_i(t) \mid^{\gamma_i} sign(z_i(t)), \\ w_i(t) &= -M_i \mid z_i(t) \mid^{\delta_i} sign(z_i(t)), \\ i &= 1, \dots, n. \end{aligned}$$

Here, z_i $i = 1, \dots, n$ are the differentiator variables, the exponents γ_i and δ_i, $i = 1, \dots, n$, are selected as in (3.7), the control law $u_2(t)$ is assigned as in (4.4), and the sliding surface $s(t)$ is defined as $s(t) = z_n(t) - \vartheta(t)$.

The resulting closed-loop system (3.1) with a control input (3.6) takes the form

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \qquad x_1(t_0) = x_{10}, \\
\dot{x}_2(t) &= x_3(t), \qquad x_2(t_0) = x_{20}, \\
&\dots \qquad\qquad \dots \\
\dot{x}_{n-1}(t) &= x_n(t), \qquad x_{n-1}(t_0) = x_{(n-1)0}, \\
\dot{x}_n(t) &= \sum_{i=1}^{n} v_i(t) + \sum_{i=1}^{n} w_i(t) + \\
-\lambda_1 \mid s(t) \mid^{1/2} sign(s(t)) &- \lambda_2 \mid s(t) \mid^{p} sign(s(t)) + y(t), \\
x_n(t_0) &= x_{n0}, \\
\dot{y}(t) &= -\alpha sign(s(t)) + \xi(t), \quad y(t_0) = 0,
\end{aligned} \tag{3.10}$$

where the disturbance $\xi(t) = \dot{\zeta}(t)$ exists almost everywhere and is bounded by the constant L.

3.4 Settling Time Estimate

Theorem 3.4.1. *Consider the dynamical system (3.1) in the presence of a disturbance $\xi(t)$ satisfying the Lipschitz condition with a constant L, the smooth fixed-time convergent differentiator (3.4), and the continuous control law (3.6-3.9).*

Case 1. *If $\xi(t) = 0$ and only the highest relative degree state $x_1(t)$ is measurable, then the vector of estimates $z(t) = [z_1(t), z_2(t), \dots, z_n(t)] \in \mathbb{R}^n$ of (3.4) converges to the vector of the states $x(t) = [x_1(t), x_2(t), \dots, x_n(t)] \in \mathbb{R}^n$ of the system (3.1) for a fixed (pre-established) time no greater than T_{FD} given by*

$$T_{FD} \;\leq\; \frac{\lambda^{\rho}_{\max}(P_1)}{r_0 \rho} + \frac{1}{r_1 \sigma \Upsilon^{\sigma}} \tag{3.11}$$

and in absence of disturbances, the continuous fixed-time convergent control law (3.9) drives all states $[x_1(t), x_2(t), \dots, x_n(t)]$ of the system (3.1) at the origin uniformly for a fixed (pre-established) time no greater than T_{F_1} given by

$$\begin{aligned}
T_{F_1} \;\leq\; & \frac{\lambda^{\rho}_{\max}(P_1)}{r_0 \rho} + \frac{1}{r_1 \sigma \Upsilon^{\sigma}} \\
+ \; & \frac{\lambda^{\hat{\rho}}_{\max}(P)}{\hat{r}_0 \hat{\rho}} + \frac{1}{\hat{r}_1 \hat{\sigma} \hat{\Upsilon}^{\hat{\sigma}}}
\end{aligned} \tag{3.12}$$

Case 2. *If $\xi(t) \neq 0$ and all states of the system (3.1) are measurable, then the continuous fixed-time convergent control law (3.6) drives all states $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]$ of the system (3.1) at* *the origin uniformly for a fixed (pre-established) time no greater than T_{F_2} given by*

$$
\begin{aligned}
T_{F_2} \;\leq\; & \frac{\lambda_{\max}^{\hat{\rho}}(P)}{\hat{r}_0\hat{\rho}} + \frac{1}{\hat{r}_1\hat{\sigma}\hat{\Upsilon}^{\hat{\sigma}}} \\
+ & \left(\frac{1}{\lambda_2(p-1)\epsilon^{p-1}} + \frac{2\epsilon^{1/2}}{\lambda_1}\right) \\
\times & \left(1 + \frac{1}{b\left(\frac{1}{B} - \frac{h(\lambda_1)}{\lambda_1}\right)}\right),
\end{aligned} \tag{3.13}
$$

Case 3. *If $\xi \neq 0$ and only the highest relative degree state $x_1(t)$ is measurable, then the continuous fixed-time convergent control law (3.6-3.9) drives all states $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]$ of the system (3.1) to a vicinity of the origin uniformly for a fixed (pre-established) time no greater than T_F given by*

$$
\begin{aligned}
T_F \;\leq\; & \frac{\lambda_{\max}^{\rho}(P_1)}{r_0\rho} + \frac{1}{r_1\sigma\Upsilon^{\sigma}} \\
+ & \frac{\lambda_{\max}^{\hat{\rho}}(P)}{\hat{r}_0\hat{\rho}} + \frac{1}{\hat{r}_1\hat{\sigma}\hat{\Upsilon}^{\hat{\sigma}}} \\
+ & \left(\frac{1}{\lambda_2(p-1)\epsilon^{p-1}} + \frac{2\epsilon^{1/2}}{\lambda_1}\right) \\
\times & \left(1 + \frac{1}{b\left(\frac{1}{B} - \frac{h(\lambda_1)}{\lambda_1}\right)}\right)
\end{aligned} \tag{3.14}
$$

Here, $\rho = \frac{1-\alpha}{\alpha}$, $\hat{\rho} = \frac{1-\gamma}{\gamma}$,$\sigma = \frac{\beta-1}{\beta}$, $\hat{\sigma} = \frac{\delta-1}{\delta}$, $r_0 = \frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P_1)}$, $\hat{r_0} = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$, $r_1 = \frac{\lambda_{\min}(Q_1')}{\lambda_{\max}(P_1')}$, $\hat{r_1} = \frac{\lambda_{\min}(Q')}{\lambda_{\max}(P')}$, $\Upsilon \leq \lambda_{\min}(P')$ and $\Upsilon' \leq \lambda_{\min}(P)$ are positive numbers, $\epsilon > 0$, $B = \alpha + L$, $b = \alpha - L$, $h(\lambda_1) = 1/\lambda_1 + (2e/b\lambda_1)^{1/3}$, and e is the base of natural logarithms, provided that the following conditions hold for control gains: $\alpha > L$, $\lambda_1 h^{-1}(\lambda_1) > B$. The minimum value of $T_f(\epsilon)$ is reached for $\epsilon = (\lambda_1/\lambda_2)^{\frac{1}{p+1/2}}$.

Proof of Case 1. *The result in case 1 is proved in [103]. Let consider the dynamical system* (3.1) *in absence of disturbances ($\xi(t) = 0$), the smooth fixed time differentiator* (3.4)*, and the corresponding estimation error system* (3.5)*. According to theorem 1 in [37], the error $[e_1, e_2, \dots e_n(t)] \in \mathbb{R}^n$ converges to the origin and the state $[z_1(t), z_2(t), \dots, z_n(t)] \in \mathbb{R}^n$ of the continuous fixed-time differentiator (3.4) converges to the derivatives vector of the output $y(t)$ $[y_1(t), \dot{y}(t), \dots, y^{(n-1)}(t)]$ for the system (3.1) in a fixed time not exceeding T_{FD} given by expression in* (3.11)*. Therefore, starting from $t = T_{FD}$, the equalities $z_i = x_i$, $i = 1, \dots, n$, holds for the differentiator variables and system states.*

From Theorem 2 in [26], the continuous fixed-time convergent control law ((3.10)*) activated at $t = T_{FD}$ using variables z_i instead of states x_i, drives all states $[x_1(t), x_2(t), \dots, x_n(t)]$ of the system (3.1) in absence of disturbances to the origin uniformly for a fixed (pre-established) time not exceeding T_{F1} given by* (3.12)*, and independent of any initial conditions of the system (3.1) and differentiator (3.4), i.e., the fixed-time convergence is provided.* ***Case 1*** *of theorem is proven.*

Proof of Case 2. *Consider the dynamics of the sliding variable $s(t)$ defined in (4.4) as $s(t) = x_n(t) - \vartheta(t)$, $\dot{\vartheta}(t) = u(t) - u_2(t)$, where $u(t)$ is obtained as (3.6) and $x_n(t)$ satisfies the last equation in (3.1). The derivative $\dot{s}(t)$ satisfies the relation*

$$
\begin{aligned}
\dot{s}(t) &= \dot{x}_n(t) - \dot{\vartheta}(t) \\
&= u(t) + \xi(t) - u(t) + u_2(t) \\
&= u_2(t) + \xi(t) \\
&= -\lambda_1 \mid s(t) \mid^{1/2} sign(s(t)) \\
&- \lambda_2 \mid s(t) \mid^{p} sign(s(t)) \\
&- \alpha \int_{t_0}^{t} sign(s(\tau))d\tau + \xi(t).
\end{aligned} \tag{3.15}
$$

In accordance with [3, 25], the last equation provides fixed-time convergence of both, $s(t)$ and $\dot{s}(t)$, to the origin for a time no greater than

$$
\begin{aligned}
T_{FC} &= \left(\frac{1}{\lambda_2(p-1)\epsilon^{p-1}} + \frac{2\epsilon^{1/2}}{\lambda_1}\right) \\
&\times \left(1 + \frac{1}{b\left(\frac{1}{B} - \frac{h(\lambda_1)}{\lambda_1}\right)}\right)
\end{aligned} \tag{3.16}
$$

independently of initial conditions for $s(t)$. This expression corresponds to the product terms in formula (5.13).

In addition, since $\dot{s}(t) = 0$ is reached in finite time, it yields $0 = u_2(t) + \xi(t)$, $t \geq T_1$, i.e., $u_2(t) = -\xi(t)$ holds for all $t \geq T_1$. Next, substituting $u(t) = u_1(t) + u_2(t)$ defined by (3.6) into (3.1) compensates for the disturbance $\xi(t)$ and results in the equations

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t), \qquad x_1(t_0) = x_{10}, \\
\dot{x}_2(t) &= x_3(t), \qquad x_2(t_0) = x_{20}, \\
\vdots \; &\vdots \; \vdots \qquad\qquad \vdots \\
\dot{x}_{n-1}(t) &= x_n(t), \qquad x_{n-1}(t_0) = x_{(n-1)0}, \\
\dot{x}_n(t) &= u_1(t) = \sum_{i=1}^{n} v_i(t) + \sum_{i=1}^{n} w_i(t) \\
& \quad x_n(t_0) = x_{n0}.
\end{aligned} \tag{3.17}
$$

Fixed-time convergence of the system (3.1) for a time no greater than $T_{FR} = \frac{\lambda^{\rho}_{\max}(P)}{r_0\rho} + \frac{1}{r_1\sigma\Upsilon^{\sigma}}$ follows from Theorem 2 in [26]. Finally, summing up the convergence time upper estimates T_{FC} and T_{FR} yields the complete formula (5.13), $T_{F_2} = T_{FC} + T_{FR}$.

Proof of Case 3. *The formula (3.14) of Case 3 is obtained summing up the relations (3.11) of Case 1 and (5.13) of Case 2, $T_F = T_{FD} + T_{FC} + T_{FR}$. The estimate T_F presents an upper bound for the fixed convergence time of all states of the system (3.1) to the origin in the absence of disturbance $\xi(t)$. If the disturbance is present, T_F serves as an upper bound for the fixed convergence time of all states of the system (3.1) to a vicinity of the origin, so that the magnitudes of the estimation errors depend on the disturbance magnitude and are determined in accordance with Theorem 3 of [31] as $|z_1 - x_1| <$*

Constant	Value	Unit
K_b	0.001	A/rad
K_m	0.001	Nm/A
L_a	0.1	H
R_a	0.01	Ω
b	0.003	Nms/rad
J	0.005	Nms^2/rad

Table 3.1

$a_1\epsilon^n, |z_2 - x_2| < a_2\epsilon^{n-1}, \cdots |z_n - x_n| < a_n\epsilon$, where ϵ is the disturbance magnitude and $a_1, a_2, \ldots, a_n$ are positive constants. ■

Remark 3.4.2. *The formula for T_F in Theorem above provides uniform (independent of initial conditions) upper bounds for the observer and controller finite convergence times. Furthermore, those convergence times can be made as small as necessary by decreasing negative real parts of eigenvalues of the matrices A, A_1, A', A'_1, i.e., can made less than any fixed positive number. This explains the use of the term "fixed-time convergence."*

3.5 Case Study:Armature-controlled DC motor

In this section, the obtained result is applied to the third-order model benchmark of an industrial armature-controlled DC motor ([104]).

3.5.1 Armature-controlled DC motor

Consider the following third-order model of an industrial armature-controlled DC motor:

$$\begin{aligned} \dot{\theta}(t) &= \omega(t), \\ \dot{\omega}(t) &= \frac{1}{J}(-b\omega + K_m i_a(t) + d_2(t)), \\ \frac{di_a(t)}{dt} &= \frac{1}{L_a}(-R_a i_a(t) - K_b\omega(t) + V_a(t) + d_3(t)), \\ y(t) &= \theta(t). \end{aligned} \tag{3.18}$$

Here, $\theta(t)$ is the rotation angle, $\omega(t)$ is the angular velocity, $i_a(t)$ is the armature current, V_a is the armature voltage (control input), J is the rotor inertia, K_m is the motor constant, K_b is the back electromotive force coefficient, R_a is the armature resistance, L_a is the armature inductance, b is the friction coefficient, and $d_2(t)$ and $d_3(t)$ are unmatched and matched disturbances affecting the angular velocity and the current, respectively. The specific values of the system coefficients are given in Table I. It is assumed that only the state variable $\theta(t)$ can be measured.

3.5.2 Companion Form

Upon introducing another state variable, $w(t) = \dot{\omega}(t)$, the system (3.18) is transformed into the companion form with respect to state variables $\theta(t)$, $\omega(t)$, and $w(t)$:

Initial conditions in terms of θ, ω, i	Initial conditions in terms of θ, ω, w
$(\pi/6, 0, 0)$	$(\pi/6, 0, 0)$
$(\pi/2, 5, 30)$	$(\pi/2, 5, 3)$
$(\pi, 10, 60)$	$(\pi, 10, 6)$
$(4\pi/3, 100, 90)$	$(4\pi/3, 100, -42)$
$(3\pi/2, 500, 120)$	$(3\pi/2, 500, -276)$
$(2\pi, 1000, 150)$	$(2\pi, 1000, -570)$

Table 3.2

$$\begin{aligned}
\dot{\theta}(t) &= \omega(t), \\
\dot{\omega}(t) &= w(t), \\
\dot{w}(t) &= -(\frac{K_b K_m}{JL_a} + \frac{R_a b}{JL_a})\omega(t) - (\frac{b}{J} + \frac{R_a}{L_a})w(t) \\
&+ \frac{K_m}{JL_a}V_a(t) + \frac{K_m}{JL_a}(\frac{R_a}{K_m}d_2(t) + d_3(t)) + \frac{1}{J}\dot{d}_2(t), \\
y(t) &= \theta(t).
\end{aligned} \tag{3.19}$$

Note that the state variables $w(t)$, $\omega(t)$, and $i_a(t)$ are not available, since only the state variable $\theta(t)$ can be measured. Therefore, a fixed-time convergent differentiator is needed to reconstruct values of the output derivatives $y^{(i)}(t)$, $i = 1, 2$, and design a continuous fixed-time convergent controller driving all the state components at the origin or to its certain vicinity.

The control input $V_a(t)$ is represented as

$V_a(t) = V_{a1}(t) + V_{a2}(t)$, *where*

$$V_{a1}(t) = (K_b + \frac{R_a b}{K_m})\omega(t) + \frac{JL_a}{K_m}(\frac{b}{J} + \frac{R_a}{L_a})w(t)$$

is bounded on any compact set, and

$$V_{a2}(t) = \frac{JL_a}{K_m}u(t).$$

As a result, the system (3.19) takes the n-dimensional integrator form

$$\begin{aligned}
\dot{\theta}(t) &= \omega(t), \\
\dot{\omega}(t) &= w(t), \\
\dot{w}(t) &= u(t) + \frac{K_m}{JL_a}(\frac{R_a}{K_m}d_2(t) + d_3(t)) + \frac{1}{J}\dot{d}_2(t), \\
y(t) &= \theta(t).
\end{aligned} \tag{3.20}$$

The equations of the smooth fixed-time convergent differentiator (3.4) take the form

$$\dot{z}_1(t) = z_2(t) - k_1 \mid y(t) - z_1(t) \mid^{\alpha_1} sign(y(t) - z_1(t))$$

$$
\begin{aligned}
& -\kappa_1 \mid y(t) - z_1(t) \mid^{\beta_1} sign(y(t) - z_1(t)), \\
\dot{z}_2(t) &= z_3(t) - k_2 \mid y(t) - z_1(t) \mid^{\alpha_2} sign(y(t) - z_1(t)) \\
& -\kappa_2 \mid y(t) - z_1(t) \mid^{\beta_2} sign(y(t) - z_1(t)), \\
\dot{z}_3(t) &= -k_3 \mid y(t) - z_1(t) \mid^{\alpha_3} sign(y(t) - z_1(t)) \\
& -\kappa_3 \mid y(t) - z_1(t) \mid^{\beta_3} sign(y(t) - z_1(t)),
\end{aligned} \tag{3.21}
$$

and the continuous fixed-time observer-based control law (3.9) for the system (3.20) is given as follows:

$$
\begin{aligned}
u(t) = -m_1 \mid z_1(t) \mid^{\gamma_1} sign(z_1(t)) \\
-m_2 \mid z_2(t) \mid^{\gamma_2} sign(z_2(t)) - m_3 \mid z_3(t) \mid^{\gamma_3} sign(z_3(t)) \\
-M_1 \mid z_1(t) \mid^{\delta_1} sign(z_1(t)) \\
-M_2 \mid z_2(t) \mid^{\delta_2} sign(z_2(t)) - M_3 \mid z_3(t) \mid^{\delta_3} sign(z_3(t)) \\
-\lambda_1 \mid s(t) \mid^{1/2} sign(s(t)) \\
-\lambda_2 \mid s(t) \mid^{3/2} sign(s(t)) - \alpha \int_{t_0}^{t} sign(s(\tau)) d\tau, \\
s(t) = z_3(t) - r(t), \\
\dot{r}(t) = u(t) + \lambda_1 \mid s(t) \mid^{1/2} sign(s(t)) \\
+\lambda_2 \mid s(t) \mid^{3/2} sign(s(t)) + \alpha \int_{t_0}^{t} sign(s(\tau)) d\tau.
\end{aligned} \tag{3.22}
$$

A set of simulations is conducted to reveal properties of the continuous fixed-time observer-based controller designed in Theorem. The examined initial conditions are given in Table 3.2.

3.5.3 Simulation

The smooth fixed-time convergent differentiator (3.21) and the continuous fixed-time observer-based control (3.22) are applied simultaneously from the beginning of the simulation. The exponents for the observer and controller are selected as: $\alpha_3 = 7/10$, $\alpha_2 = 8/10$, $\alpha_1 = 9/10$, $\beta_3 = 12/9$, $\beta_2 = 11/9$, $\beta_1 = 10/9$ *and* $\gamma_3 = 19/21$, $\gamma_2 = 19/22$, $\gamma_1 = 19/23$, $\delta_3 = 21/19$, $\delta_2 = 21/18$, $\delta_1 = 21/17$*; the observer and controller gains are assigned as:* $k_1 = K_1 = 18$, $k_2 = K_2 = 1000$, $k_3 = K_3 = 1000$, *and* $m_1 = M_1 = 4, m_2 = M_2 = m_3 = M_3 = 12$.

The observer settling time parameters are calculated as: $\rho = 1 - \frac{9}{10} = \frac{1}{10}$, $\sigma = \frac{10}{9} - 1 = \frac{1}{9}$, $r = \frac{1}{59.3920}$, $r_1 = \frac{1}{59.3920}$, $\Upsilon = 0.0286$, $\lambda_{min}(P_1) = 0.0286$, $\lambda_{max}(P_1) = 59.3920$, *where*

$$
P_1 = P_1' = \begin{pmatrix} 59.3824 & -0.5000 & -0.5684 \\ -0.5000 & 0.5684 & -0.5000 \\ -0.5684 & -0.5000 & 0.5097 \end{pmatrix}
$$

and $Q_1 = Q_1' = I_3$ *are identity matrices. The upper bound of the observer fixed convergence time is calculated as* $T_{FD} = 1687$ *seconds.*

The controller settling time parameters are calculated as: $\rho = (1 - \frac{19}{21})/\frac{19}{21} = \frac{2}{19}$, $\sigma = (\frac{21}{19} - 1)/\frac{21}{19} = \frac{2}{21}$, $r = \frac{1}{4.2778}$, $r_1 = \frac{1}{4.2778}$, $\Upsilon = 0.0445$, $\lambda_{min}(P) = 0.0445$, $\lambda_{max}(P) = 4.2778$, *where*

Initial conditions	Convergence time by simulation (sec.)	Convergence time estimate (sec.)	Rate
$(\pi/6, 0, 0)$	20	1823.8	92.29
$(\pi/2, 5, 30)$	20	1823.8	92.29
$(\pi, 10, 60)$	20	1823.8	92.29
$(4\pi/3, 100, 90)$	20	1823.8	92.29
$(3\pi/2, 500, 120)$	40	1823.8	45.59
$(2\pi, 1000, 150)$	50	1823.8	36.48

Table 3.3

$$P = P' = \begin{pmatrix} 2.2429 & 1.7286 & 0.1250 \\ 1.7286 & 2.7893 & 0.1857 \\ 0.1250 & 0.1857 & 0.0571 \end{pmatrix}$$

and $Q = Q' = I_3$ are identity matrices. The upper bound for the regulator fixed convergence time is calculated as $T_{FR} = 107.8$ seconds; therefore, $T_{F_1} = T_{FD} + T_{FR} = 1792.77$ seconds.

The unmatched disturbance $d_2(t)$ and matched disturbance $d_3(t)$ are assigned as $d_2(t) = 0.02\sin(0.1t)$ rad/s and $d_3(t) = 0.004 + 0.004\sin(t)$ A/s. Therefore, $\xi(t) = 0.4sin(0.1t) + 0.008 + 0.008sin(t) + 0.4cos(0.1t)$. Accordingly, the upper bound for the Lipschitz constant is set to $L = 0.1$. The compensator control gains are assigned as $\alpha = 0.2$, $\lambda_1 = 10$, $\lambda_2 = 0.1$. The minimum value of $T_f(\epsilon)$ is reached for $\epsilon = (\lambda_1/\lambda_2)^{\frac{1}{p+1/2}} = (10/0.1)^{\frac{1}{3/2+1/2}} = 100^{1/2} = 10$. Thus, $B = \alpha + L = 0.2 + 0.1 = 0.3$, $b = \alpha - L = 0.2 - 0.1 = 0.1$, $h(\lambda_1) = 1/\lambda_1 + (2e/b\lambda_1)^{1/3} = 1/10 + (2e/0.1 \times 10)^{1/3} = 0.1 + (5.4366)^{1/3} = 0.1 + 1.7584 = 1.8584$, and the theorem conditions hold for the compensator control gains: $\alpha > L$: $0.2 > 0.1$, $\lambda_1 h^{-1}(\lambda_1) > B$: $(10)(0.5381) = 5.381 > 0.3$. The upper bound for the disturbance compensator fixed convergence time is calculated as $T_{FC} = 29.06$ seconds; therefore, $T_{F_2} = T_{FR} + T_{FC} = 136.86$ seconds and the upper bound for the complete controller fixed convergence time is calculated as $T_F = T_{FD} + T_{FR} + T_{FC} = 1687 + 107.8 + 29.06 = 1823.86$ seconds.

The treated initial conditions for the rotation angle, angular velocity, and armature current, as well as the corresponding convergence times, are given in Table 3.3. Tables 3.4–3.6 provide information on the convergence accuracy, i.e., the magnitude of the established limit cycle, for the differentiator and controller, which is obtained varying of the magnitudes of disturbance ξ and simulation step τ. For the third-order differentiator and controller, the simulation step varies from 10^{-2} (Table 3.4) to 10^{-3} (Table 3.5) and 10^{-4} (Table 3.6). The disturbance magnitude varies from the full noise $\epsilon = 1$ to the completely absent noise $\epsilon = 0$ (the order of magnitudes for time step and disturbance is reduced by 10^{-1}). The results are given for the initial condition $(\theta = \pi, \omega = 10, w = 6, i_a = 60)$.

Note that the resulting disturbance $\xi(t)$ is present in the lowest relative degree equation of the system (3.1), while the noise in [31] affects the highest relative degree one. Nonetheless, the disturbance magnitude order can be accordingly recalculated by integrating it $n-1$ times to make the accuracy results of [31] applicable to the considered case. This leads to the following results.

- *The steady-state magnitude of the integrator chain variables, $\theta(t)$, $\omega(t)$, and $w(t)$, is less than the disturbance magnitude in all cases.*

$\|\epsilon\|$	1	10^{-1}	10^{-2}	0
Disturbance (ξ)	10^{-1}	10^{-2}	10^{-3}	–
Differentiator	**1**	**2**	**3**	**4**
z_1	10^{-7}	10^{-7}	10^{-7}	10^{-10}
z_2	10^{-5}	10^{-5}	10^{-5}	10^{-8}
z_3	10^{-3}	10^{-3}	10^{-3}	10^{-7}
z_4	10^{-2}	10^{-2}	10^{-2}	10^{-6}
Regulator	**1**	**2**	**3**	**4**
θ	10^{-2}	10^{-3}	10^{-4}	10^{-12}
ω	10^{-4}	10^{-5}	10^{-5}	10^{-9}
w	10^{-3}	10^{-3}	10^{-3}	10^{-7}
i	10^{-1}	10^{-1}	10^{-2}	10^{-6}

Table 3.4: Simulation step $\tau = 10^{-2}$, $T = 200$, Initial conditions $\theta(0) = \pi$, $\omega(0) = 10$, $w(0) = 6$.

$\|\epsilon\|$	1	10^{-1}	10^{-2}	0
Disturbance (ξ)	10^{-1}	10^{-2}	10^{-3}	–
Differentiator	**1**	**2**	**3**	**4**
z_1	10^{-10}	10^{-11}	10^{-11}	10^{-21}
z_2	10^{-10}	10^{-8}	10^{-8}	10^{-16}
z_3	10^{-4}	10^{-4}	10^{-5}	10^{-14}
z_4	10^{-4}	10^{-4}	10^{-4}	10^{-14}
Regulator	**1**	**2**	**3**	**4**
θ	10^{-2}	10^{-3}	10^{-4}	10^{-17}
ω	10^{-4}	10^{-6}	10^{-6}	10^{-19}
w	10^{-4}	10^{-5}	10^{-5}	10^{-16}
i	10^{-1}	10^{-1}	10^{-1}	10^{-15}

Table 3.5: Simulation step $\tau = 10^{-3}$, $T = 200$, Initial conditions $\theta(0) = \pi$, $\omega(0) = 10$, $w(0) = 6$.

- *In the absence of disturbances, the estimation errors for all state variables and the state variables themselves converge to the origin in fixed times bounded by estimates T_{FD} and T_{F_1} of Case 1 in Theorem above. It can be observed from the last columns in Tables 3.4–3.6 that the steady-state magnitudes of the estimation errors for the state variables and the state variables themselves are consistent with the accuracies given in [30] to ensure finite-time (not asymptotical) convergence to the origin.*

- *In the absence of disturbances of magnitude ϵ, the estimation errors for all state variables and the state variables themselves converge to a vicinity of the origin in fixed times bounded by estimates T_{FD} and T_{F_1} of Case 1 in Theorem. It can be observed from the three first columns in Tables 3.4–3.6 that the steady-state magnitudes of the estimation errors for the state variables are consistent with the accuracies given in Theorem 3 of [31]: $|z_1 - x_1| < a_1\epsilon^n, |z_2 - x_2| < a_2\epsilon^{n-1}, \dots |z_n - x_n| < a_n\epsilon$, where $a_1, a_2, \dots, a_n$ are positive constants.*

The following figures demonstrate the behavior of the state variables revealed in Tables 3.4–3.6. The simulation graphs corresponding to the initial values $\theta(0) = \pi$, $\omega(0) = 10$, $w(0) = 6$, $i_a(t) = 60$ are presented in Figs. 3.1–3.7. The simulation step is set to 10^{-3}. The simulation horizon is $T = 200$ seconds.

$\|\epsilon\|$	1	10^{-1}	10^{-2}	0
Disturbance (ξ)	10^{-1}	10^{-2}	10^{-3}	–
Differentiator	**1**	**2**	**3**	**4**
z_1	10^{-10}	10^{-11}	10^{-14}	10^{-30}
z_2	10^{-8}	10^{-9}	10^{-11}	10^{-26}
z_3	10^{-5}	10^{-6}	10^{-6}	10^{-23}
z_4	10^{-4}	10^{-6}	10^{-6}	10^{-22}
Regulator	**1**	**2**	**3**	**4**
θ	10^{-2}	10^{-3}	10^{-4}	10^{-29}
ω	10^{-4}	10^{-6}	10^{-6}	10^{-26}
w	10^{-4}	10^{-5}	10^{-6}	10^{-23}
i	10^{-1}	10^{-2}	10^{-3}	10^{-23}

Table 3.6: Simulation step $\tau = 10^{-4}$, $T = 200$, Initial conditions $\theta(0) = \pi$, $\omega(0) = 10$, $w(0) = 6$.

Figures 3.1–3.4 show zoomed time histories of the states $[\theta(t), \omega(t), w(t), i_a(t)]$ *separately for each variable, which are plotted against the disturbance* ξ *in the entire simulation interval. The observed behavior of the state variables is consistent with the results presented in Table 3.3.*

Figures 3.5 shows zoomed time histories of the states $[\theta(t), \omega(t), w(t), i_a(t)]$ *altogether, which are plotted against the disturbance* ξ *in the entire simulation interval. It can be observed that the disturbance is attenuated in accordance with Table V and consistently with the results of [31], and the convergence of the state variables to a vicinity of the origin is achieved in fixed time.*

Figures 3.6 and 3.7 show time histories of the states $[\theta(t), \omega(t), w(t), i_a(t)]$ *and their estimates* $[z_1(t), z_2(t), z_3(t), z_4(t)]$ *separately for each variable in the simulation interval* $[0, 7]$*. Figures show that the fixed-time differentiator converges rapidly, for no longer than* 7 *seconds.*

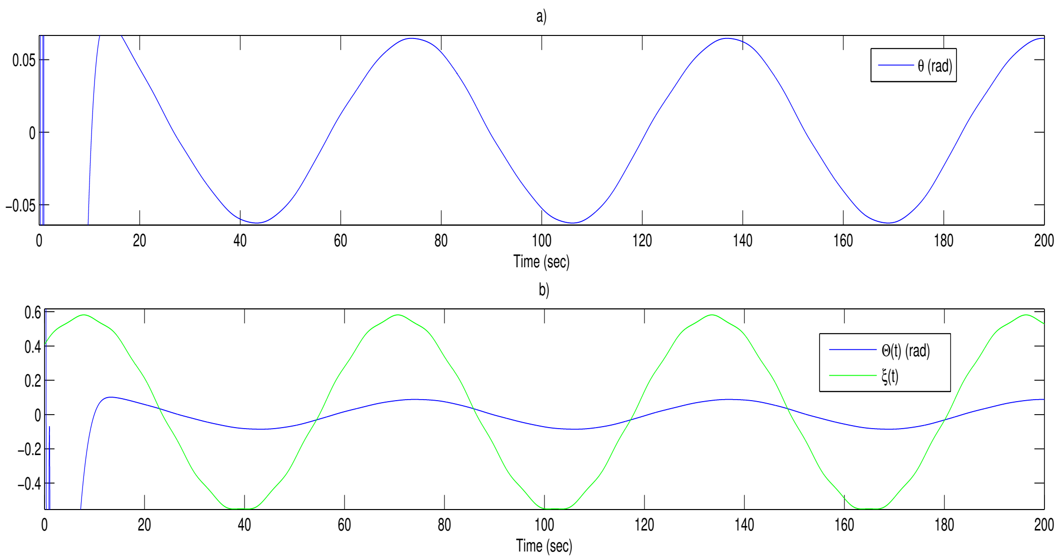

Figure 3.1: Time histories of rotation angle $\theta(t)$(blue) and disturbance $\xi(t)$ (green) in the simulation interval $[0, 200]$.

As follows from Tables 3.4-3.6 and Figures 3.1–3.7, the disturbance noise is attenuated in the process of fixed-time convergence. Therefore, it can be concluded that proposed control law solves the originally stated problem of driving the states of the system (3.18) with unbounded disturbances satisfying

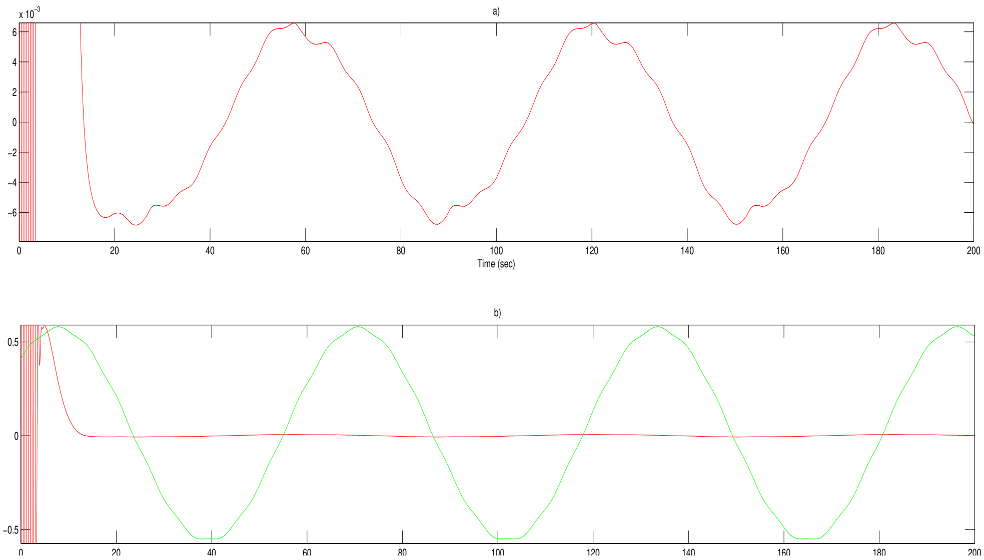

Figure 3.2: Time histories of angular velocity $\omega(t)$ (red) and disturbance $\xi(t)$ (green) in the simulation interval $[0, 200]$.

a Lipschitz condition at the origin/to a vicinity of the origin uniformly in fixed time. The entire controller works in three steps, first, reconstructing real values of the state variables by a smooth fixed-time differentiator, second, compensating for disturbances by a fixed-time super-twisting observer, and finally, providing fixed-time convergence of the state variables at the origin/to a vicinity of the origin by means of a continuous fixed-time convergent control law based on the differentiator variables.

3.6 Conclusions and future work

A continuous fixed-time convergent observer-based controller is designed to drive all states of an n-dimensional integrator at the origin/a vicinity of the origin for a finite time, using a scalar input in the equation for the lowest relative degree state, against unbounded disturbances. The controller design does not assume knowledge of all system states: only the output should be measured. No knowledge or reconstruction of unbounded individual disturbances is assumed as well. This makes the proposed controller more flexible and useful in practice. The presented algorithm makes a considerable advance, since the proposed control laws are continuous and differentiator is smooth, the considered disturbances may be unbounded, and a uniform upper bound (independent of the initial conditions of the system) for the controller convergence time is explicitly calculated. Performance of the designed controller is demonstrated through numerical implementation in a case study of DC-motor, validating the obtained theoretical results. The accuracy of the controller is examined and found consistent with the results obtained in [31].

In future works bounds of settling convergence time estimates can be optimized, and the compensator dynamics of controller designed can be replaced by disturbance observer proposed in [93].

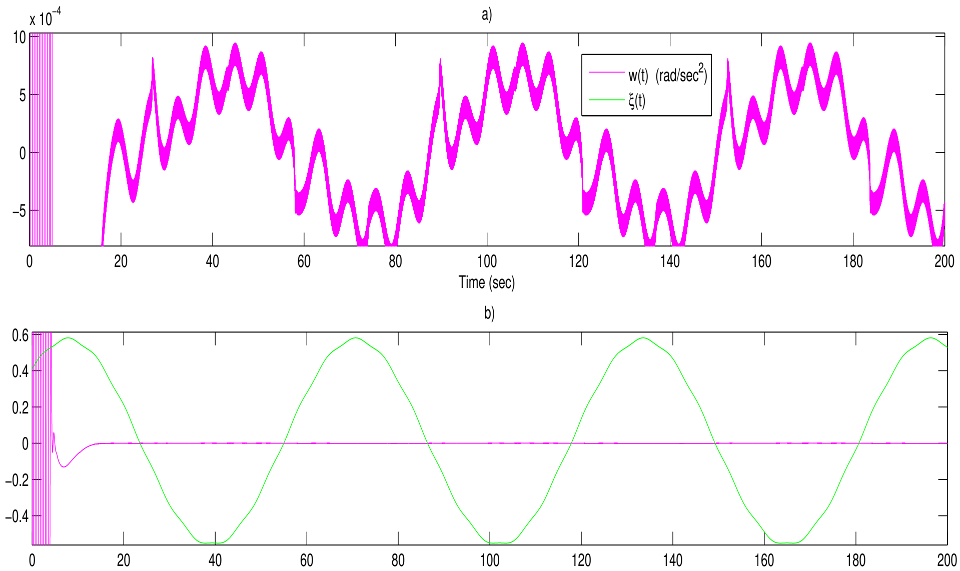

Figure 3.3: Time histories of angular acceleration $w(t)$ (purple) and disturbance $\xi(t)$ (green) in the simulation interval $[0, 200]$.

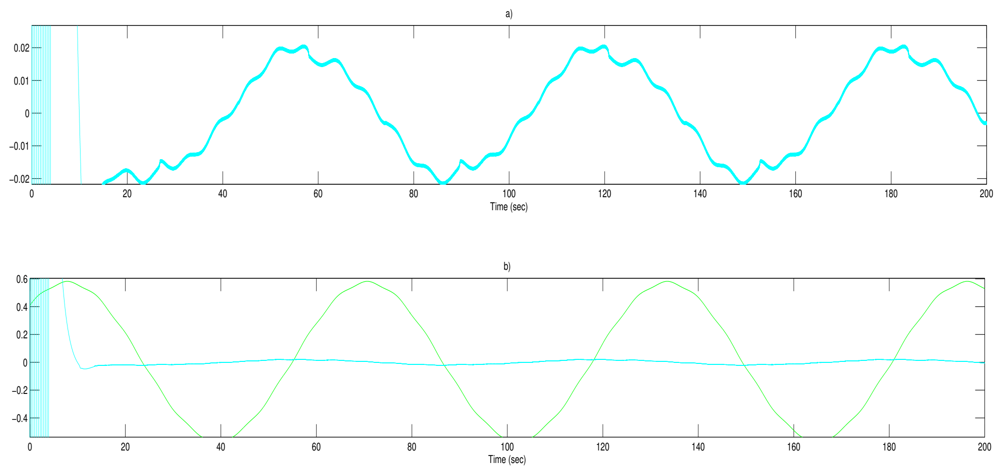

Figure 3.4: Time histories of current $i(t)$ (cyan) and disturbance $\xi(t)$ (green) in the simulation interval $[0, 200]$.

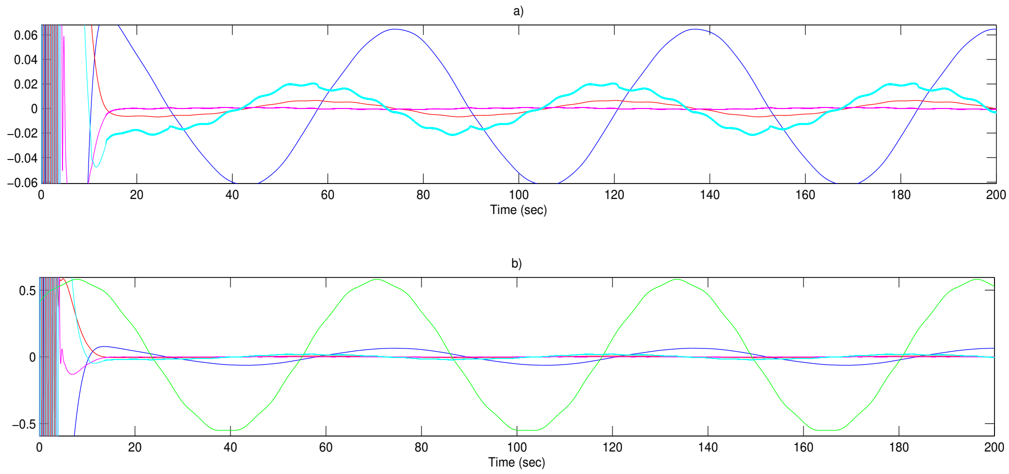

Figure 3.5: Time histories of rotation angle $\theta(t)$ (blue), angular velocity $\omega(t)$ (red), angular acceleration $w(t)$ (purple), current $i(t)$ (cyan), and disturbance $\xi(t)$ (green) in the simulation interval $[0, 200]$.

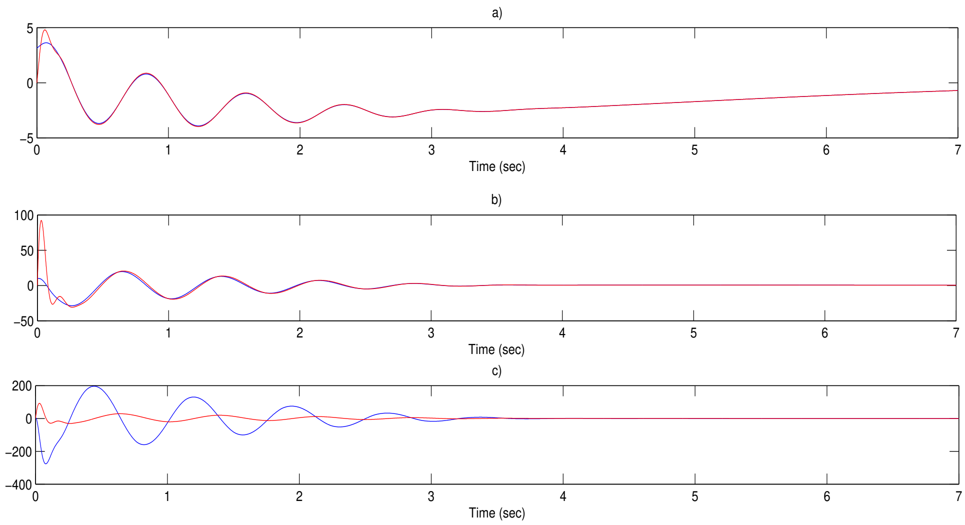

Figure 3.6: Time histories of a) rotation angle $\theta(t)$ (blue) and its estimate $z_1(t)$ (red), b) angular velocity $\omega(t)$ (blue) and its estimate $z_2(t)$ (red), c) angular acceleration $w(t)$ (blue) and its estimate $z_3(t)$ (red) in the simulation interval $[0, 7]$.

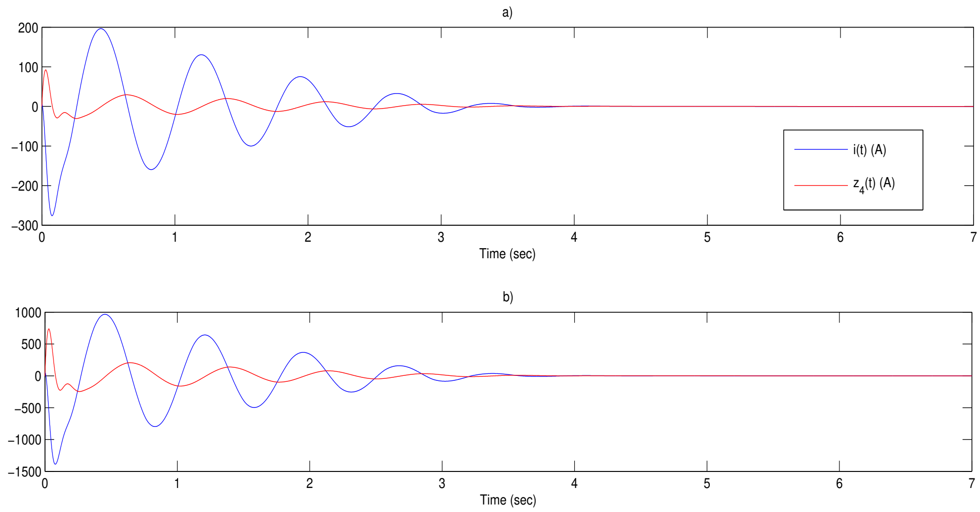

Figure 3.7: Time histories of a) angular acceleration $w(t)$ (blue) and its estimate $z_3(t)$ (red), b) current $i(t)$ and its estimate $z_4(t)$ (red) in the simulation interval $[0, 7]$.

Chapter 4

Adaptive continuous fixed-time controller design

"Premature optimization is the root of all evil."

Donald Knuth (1938–)

4.1 Introduction

This chapter included in [85] presents an adaptive fixed-time convergent continuous controller designed to solve a stock management problem with the objective to drive the states at the reference values, subject to loss rate disturbances whose bounds are unknown. The only measurable state is the state of higher relative degree, whereas the other states should be estimated. The designed controller includes a fixed-time convergent differentiator, an adaptive fixed-time convergent disturbance observer, and a fixed-time convergent regulator. The adaptive fixed-time convergent observer is used to estimate a disturbance without excessively increasing the controller gains. The controller design is based on Sterman's decision rule (see appendix C), and it is validated in a case study of stock management. The calculated upper estimate for the total settling (convergence) time and the obtained simulation results confirm the fixed-time convergence and the robustness of the designed controller.

4.1.1 Antecedents of Stock Management Problem

The research on Stock Management Problem (SMP) starts with the papers of Simon [105] and Forrester [106, 107] and studies the dynamics of the entire supply chain proceeding from the dynamics of a single link and orders representing decision rules or policies. The SMP focuses on how an individual firm manages its inventories and resources as it attempts to balance its production with orders. The decision rule specifies the actions to be carried out so that the current stock level reaches the desired value [108].

Simon [105] introduces the control systems approach to manage the stock system with a single feedback loop, which leads to an Inventory and Order-Based Production Control System (IOBPCS). Sharp and Henry [109] along with Campbell in [110] extend IOBPCS to Proportional (P), Proportional-Integral (PI), and Proportional-Integral-Derivative (PID) policies.

Forrester [106, 107] applies dynamical system methods to the supply chain problem. Christensen and Brogan [111] produce models of supply chains from SD models, Burns and Sivazlian [112] analyze a four echelon supply chain based on Forrester work. Towill [113] uses block diagrams and Laplace *transforms to study an IOBPCS, later reduce the SD models of Forrester [107] and Coyle [114] to find analytical expressions for stability. Axsater [115] gives an state of art about control methods, Edgehill et al. [116] research sensitivity of the system to parameter variations. stability regions are exhibed by Olsmats et al. [117].*

Sterman [118, 119] proposes a policy based on anchoring and adjustment heuristics to simulate behavior of decision makers. The policy is related to fluctuations in other economic models; implications for experimental investigation of dynamic decision making in aggregate systems are explored. John, Naim, and Towill [120] take the structure of IOBPCS and incorporated a work-in-process feedback loop to create the Automatic Pipeline, Inventory and Order Based Production Control System (APIOBPCS) using Laplace transform.

Naim and Towill [121] and Riddalls and Bennett [122] show that the Automatic Pipeline IOBPCS model is equivalent to the anchoring and adjustment heuristic proposed in [118, 119]. Riddalls and Bennett [122] use SD modelling languages for to examine cost structures and find solutions to demand amplification of the APIOBPCS model with a production delay. Disney and Towill [123] and White[124] investigate ways to reduce stockout and bullwhip problems in discrete time models and deduce the stability of APIOBPCS models.

Diehl and Sterman [125] report an experiment where subjects managed an inventory with stochastic sales, increasing time delays and strength of the feedback loops, they obtain that the actors performance is deteriorated, and maps the impact of dysfunctional behavior on the performance in specific tasks. Moxnes [126] summarizes four laboratory experiments to study renewable resource management where the participants over-invested and over-utilized their resources. The explanation offered is systematic misperceptions of stocks, flows and nonlinearities. Through an inventory problem experiment, Sweeney and Sterman [127] find that subjects from variable demographic characteristics shares in common a poor level of understanding of dynamic decision scenarios.

Dejonckheere et al. [128] propose Automatic Pipeline, Variable Inventory and Order Based Production Control System (APVIOBPCS), a new ordering policy regulating the desired inventory and work-in-process according to the demand prediction.

Barlas and Özevin [129] analyse the effects of demand pattern, inventory order decision review, and reception delay in the performance of the players in an inventory management simulation game by evaluating linear and non-linear decision rules used by the subjects. The authors conclude that the common linear anchoring and adjustment rule can represent the smooth and gradually damping type of behaviour, but cannot generate the nonlinear and/or piecewise ordering dynamics behaviors. Axsäter [130] proposes a discrete time of APIOBPCS policy that incorporates an order-up-to (OUT) policy. In these order policies, the target inventory is linked to the market demand.

Laugesen and Mosekilde [131] show how the model of an inventory management simulation game called the Beer Game displays border-collision bifurcations, due to incorporate piecewise linear relations associated with non-negativity conditions related to orders and shipments (just as the heuristic of anchoring and adjustment incorporates). The influence of nonnegative constraint of order quantity in stability of stock dynamics is evidenced in [132, 133].

Tosetti et al. [134] examine the space state control of APIOBPCS using a PID based system simulating the stability and error bounds. They illustrate advantages of the PID based system against the P system. Yasarcan [135] proposes developing a gradual-increase in complexity approach, with multiple *versions of a simulation game that can be used as part of the improvement in the success of the stock management simulation training.*

The use of variable structure techniques for the SMP starts with the works of Ignaciuk and Bartoszewicz for periodic-review inventory systems [136] with uncertain demand [137], and perishable stock [138]. The authors present linear quadratic sliding mode inventory policies which guarantee fast system response and full demand satisfaction from the on-hand stock, and the warehouse capacity is not exceed. The proposed policies are robust against perturbations in demand, guarantees stable system operation for arbitrary lead time and attenuates the bullwhip effect of the ordering signal.

Chakrabarty and Bartoszewicz [139] continues with the design of discrete-time sliding-mode control systems for inventory management strategies showing that robustness can be achieved by choosing the sliding variable, or the output, to be of relative degree two instead of relative degree one. It successfully reduces the ultimate bound of the sliding variable and the reduced order system during sliding becomes finite time stable in absence of disturbance. With disturbance, it becomes finite time ultimately bounded (practically stable).

Latosiiski and Bartoszewicz [140] design a sliding mode controller for inventory management systems with sliding variable of any relative degree. The main contribution is the method for selection of sliding variables with arbitrary relative degrees that guarantee a finite-time response of the system. In [141] introduce a discrete time sliding mode control strategy based on the reaching law approach. The strategy ensures a finite time reaching phase and an upper bounded convergence rate of the system. It has been demonstrated that the available warehouse capacity is never exceeded and that the unpredictable consumers demand can always be satisfied.

Recent research introduces experimental work for to extract characteristics of behavior and hence parameters derivated of agents in dynamic decision making environments. Brehmer [142] indicates the difficulties of that task. [143, 144] emphasize the importance of cognitive resources like finite time availability in decision makers.

There are various applications of variable structure and finite-time convergent algorithms to switched systems [145, 146, 147, 148], electric circuit design [149], Markov jump systems [150], state estimation [151, 152], fuzzy systems [145], and many other technical and economic problems.

The main contributions of this chapter are listed as follows

- *An adaptive fixed-time convergent continuous sliding mode controller is designed to solve a Stock Management Problem with the objective to drive stock and supply chain levels at the reference values, subject to loss rate disturbances bounds are unknown. The only measurable state of the supply chain is the inventory retailer stock level, whereas the supply line inventory level should be estimated. The designed controller includes a fixed-time convergent differentiator, an adaptive fixed-time convergent disturbance observer, and a fixed-time convergent regulator.*
- *A new version of the adaptive fixed-time convergent observer is proposed and used to estimate a disturbance, without excessively increasing the controller gains, in the situation when the disturbance Lipschitz constant and initial value are unknown, thus generalizing the result of [47], where the disturbance initial value is assumed equal to zero.*
- *The controller design is validated in a case study of stock management, driving the stock and supply chain levels at the reference values for a fixed time. An upper estimate for the total convergence time is calculated.*

The fixed-time convergence of the state and disturbance observer errors enables one to safely employ the separation principle, that is, to activate a continuous fixed-time convergent control after a fixed-time convergent differentiator converges, given that the fixed convergence time of the differentiator can be calculated a priori. In a particular case of stock measurement, the fixed-time convergence property enables one to determine after what time exact values of supply line become known to the stock manager and to calculate the time when the stock level variable achieves a given set-point. Furthermore, an unknown value of the disturbance Lipschitz constant and an unknown disturbance initial value are taken care of by the designed adaptive super-twisting compensator. Examples of the technical systems, where the developed adaptive fixed-time convergent controller is directly applicable, including cart inverted pendulum, single machine infinite bus turbo generator with main stream valve control, permanent magnet synchronous motor servo system, hypersonic missile powered by an air-breathing jet engine, and many others.

The chapter content is organized as follows. The adaptive fixed-time convergent continuous control law for a super-twisting system with a non-zero disturbance initial value is proposed in Section 4.2. The stock management problem statement is given in Section 4.3. Section 4.4 presents a case study of stock management and designs the corresponding adaptive fixed-time convergent continuous controller. Section 4.5 presents and discusses the obtained simulation results. Section 4.6 provides conclusions.

4.2 Adaptive continuous control law

Consider the dynamical system

$$\dot{r}(t) = v(t) + \xi(t), \quad r(t_0) = r_0. \tag{4.1}$$

where $r(t) \in R$ *is a system state,* $v(t) \in R$ *is a control input,* $\xi(t)$ *is an external disturbance with an initial value bounded by a constant* K

$$|\xi(t_0)| \leq K \tag{4.2}$$

and satisfying the Lipschitz condition

$$|\xi(t_1) - \xi(t_2)| \leq L(t_1 - t_2), \tag{4.3}$$

so that its derivative exists almost everywhere and bounded by a constant L, $|\ \dot{\xi}(t)\ | \leq L$. *The equation* (4.1) *represents the input-output dynamics of the scalar sliding variable* $r(t)$*, after appropriate pretransformations of the original full order dynamics and a (smooth) feedback control law are applied to "cancel" known terms, for example feedback linearisation. In this case* $\dot{r}(t) = \sigma^{(r)}$ *(see HOSM in chapter 2).*

The control objective is to design an adaptive continuous control law, so that the resulting system is fixed-time convergent at the origin or to a vicinity of the origin (fixed time attractive) in the sense of the definitions given in (2.4.3) *,and find an upper estimate of the corresponding convergence time.*

In chapter 3 a continuous fixed time control law that drives $\sigma, \sigma^{(1)}, \cdots, \sigma^{(r)} \to 0$ *for the full order dynamics of perturbed system* (4.1) *was given. Specifically in case that disturbance* $\xi(t) \neq 0$ *satisfies a Lipschitz condition* (4.3)*, a sliding mode Super-twisting disturbance observer to reconstruct* $\xi(t)$ *in fixed time is provided, and then compensate for it by means of control. However the main drawback of this procedure is that super-twisting algorithm contains a sign term whose derivative is a discontinuous high frequency switching function with gain* $\beta > L$. *In order to reduce chattering, it is desirable to make* β

as close to L as possible whilst ensuring $\beta > L$. Therefore, adapt the gains λ_1 and β in equation super-twisting observer so that β is close to L is the challenge. This reduces the amplitude of the high frequency part of the super-twisting term, which mitigates chattering.

So, the first aim is to design a sliding mode adaptive disturbance observer to reconstruct $\xi(t)$ in fixed time, and then compensate for it by means of control.

The adaptive uniform fixed-time convergent control law $v(t)$ for system (4.1) is assigned as

$$\begin{aligned} v(t) &= -\lambda_1(r,\dot{r},t)\lfloor r(t)\rceil^{1/2} - \lambda_2\lfloor r(t)\rceil^{p} \\ &- \frac{\alpha(r,\dot{r},t)}{2}\int_{t_0}^{t} sign(r(\tau))d\tau. \end{aligned} \tag{4.4}$$

Here, $\lfloor r(t)\rceil^{p} = |r(t)|^{p} sign(r(t))$, the control gain λ_2 is fixed, $\lambda_1(r,\dot{r},t) = \lambda_1$ and $\alpha(r,\dot{r},t) = \alpha$ are adaptive gains to be defined, λ_1, λ_2, $\alpha > 0$, $p > 1$.

After substituting the control law (4.4) into (4.1), the resulting system can be represented as

$$\begin{aligned} \dot{r}(t) &= -\lambda_1\lfloor r(t)\rceil^{1/2} - \lambda_2\lfloor r(t)\rceil^{p} + y(t), \quad r(t_0) = r_0, \\ \dot{y}(t) &= -\frac{\alpha}{2} sign(r(t)) + \dot{\xi}(t), \quad y(t_0) = y_0, \end{aligned} \tag{4.5}$$

where $\dot{\xi}(t)$ exists almost everywhere and bounded by the constant L, $|\dot{\xi}(t)| \leq L$.

An upper estimate of the fixed settling time of both states of the system (4.5) to a certain vicinity of the origin (real sliding mode) is given as follows. The following result generalizes the result obtained in [47], where the disturbance initial value is assumed equal to zero, $K = 0$.

Theorem 4.2.1. *Let the dynamic system (4.1),(4.4) be affected by disturbances $\xi(t)$ satisfying conditions (4.2), (4.3) for unknown constants L and K. The states $r(t)$ and $y(t)$ converge to a certain vicinity of the origin (real sliding mode) for a fixed time less or equal than T_{FC} determined as*

$$\begin{aligned} T_{FC} &\leq \frac{1}{\lambda_2(p-1)\epsilon_1^{p-1}} + \frac{2}{\rho}[(\lambda + 4\epsilon^2)\epsilon_1 \\ &+ (\frac{M}{\lambda_2^2(p-1)^2\epsilon_1^{2(p-1)}} + K) \\ &\times (\frac{M}{\lambda_2^2(p-1)^2\epsilon_1^{2(p-1)}} + K - 4\epsilon\epsilon_1^{1/2})]^{1/2}, \end{aligned} \tag{4.6}$$

where $\epsilon_1 > \mu$,

$$\begin{aligned} M &= \frac{\omega}{2}\sqrt{\frac{\gamma}{2}}\frac{1}{\lambda_2(p-1)\epsilon_1^{p-1}} + \frac{\lambda_0}{2} \\ &+ (\lambda + 4\epsilon^2)\lambda_2\epsilon_1^{p-1/2} + L, \end{aligned} \tag{4.7}$$

$\rho = \epsilon\frac{\sqrt{\mu_{min}(P)}}{\mu_{max}(P)}$, and $\mu_{max}(P)$ and $\mu_{min}(P)$ are the maximum and minimum eigenvalues of the matrix

$$P = \begin{pmatrix} \lambda + 4\epsilon^2 & -2\epsilon \\ -2\epsilon & 1 \end{pmatrix},$$

using the adaptive parameters λ_1, $\alpha > 0$ *in (4.4), (4.5) that satisfy the equations*

$$d\lambda_1(t)/dt = \omega(\gamma/2)sign(r(t) - \mu), \tag{4.8}$$

when $\lambda_1 - \lambda_{min} > 0$ *or* $|r(t)| > \mu$,

$$d\lambda_1(t)/dt = 0,$$

$$\lambda_1(0) = \lambda_0,$$

$$\alpha(t) = 2\epsilon\lambda_1(t) + 2(\lambda + 4\epsilon^2)\lambda_2\epsilon_1^{p-0.5},$$

when $\lambda_1 - \lambda_{min} \leq 0$ *and* $|r(t)| \leq \mu$, *if the following conditions*

$$\lambda_1 > \lambda_{lim} = -\frac{(4L+1)\epsilon}{(1-\gamma)\lambda} + \frac{[-4\epsilon^2 - \lambda - 2L]^2}{12\lambda(1-\gamma)\epsilon}, \tag{4.9}$$

$$\alpha > 2L,$$

hold for $|r_0| > \mu$. *Here,* $0 < \gamma < 1$ *and* ω, ϵ, λ, λ_0, *and* λ_{min} *are some positive constants. The real sliding mode vicinity is given by* $\{(r, y) : |r| \leq \mu, |y| \leq \varrho\}$, $\varrho = \int_{t_0}^{T_{FC}} (\frac{\alpha(\tau)}{2} + L)d\tau$.

Proof of Theorem

A. *Consider* $|r_0| > \epsilon_1$, *where* $\epsilon_1 > \mu$, $\mu > 0$ *are given constants. The first equation in system (4.5) yields*

$$\begin{aligned}\frac{d|r(t)|}{dt} &= sign(r(t))\frac{dr(t)}{dt} \\ &= -\lambda_1|r(t)|^{1/2} - \lambda_2|r(t)|^p + y(t)sign(r(t)) \\ &\leq -\lambda_2|r(t)|^p.\end{aligned} \tag{4.10}$$

taking into account that $sign(y(t))sign(r(t)) < 0$ *for* $t > t_0$, *while* $r(t)$ *does not cross* $r(t) = 0$, *and* $\alpha > L$. *Solving (4.10) with an initial condition* $r(t_0) = r_0$ *yields*

$$\frac{|r(t)|^{1-p}}{1-p} \leq -\lambda_2(t - t_0) + \frac{|r(0)|^{1-p}}{1-p} \leq -\lambda_2(t - t_0)$$

and

$$|r(t)|^{p-1} \leq \frac{1}{\lambda_2(p-1)(t-t_0)}.$$

Therefore, $|r(t)|$ *decreases and reaches the value* $|r(t)| = \epsilon_1$ *for a time* $T_1 \leq \frac{1}{\lambda_2(p-1)\epsilon_1^{p-1}}$, *which corresponds to the first term in (4.6) and is independent of an unknown initial condition* $r(t_0) = r_0$.

Step ***A*** *ends with* $|r(T_1)| = \epsilon_1 > 0$. *Given that* $sign(y(t))$ *is opposite to* $sign(r(t))$ *for* $t \in [t_0, T_1]$, $y(t)$ *increases for* $t \in [t_0, T_1]$ *and* $|y(T_1)| < K + M(T_1 - t_0) = K + \frac{M}{\lambda_2(p-1)\epsilon_1^{p-1}}$, *where* M *is the maximum velocity of* $y(t)$ *according to the second equation in (4.5), which can be calculated as*

$$\begin{aligned}M &= \frac{1}{2}\alpha(\frac{1}{\lambda_2(p-1)\epsilon_1^{p-1}}) + L \\ &= \frac{\omega}{2}\sqrt{\frac{\gamma}{2}}\frac{1}{\lambda_2(p-1)\epsilon_1^{p-1}} + \frac{\lambda_0}{2} \\ &+ (\lambda + 4\epsilon^2)\lambda_2\epsilon_1^{p-1/2} + L.\end{aligned} \tag{4.11}$$

The maximum value of $\alpha(t)$ *en* $t = \frac{1}{\lambda_2(p-1)\epsilon_1^{p-1}}$ *is calculated in view of the equations for* $\lambda_1(t)$ *and* $\alpha(t)$ *in (4.8).*

If $\mu \le |r_0| \le \epsilon_1$*, then step* ***A*** *is not executed; therefore the term* $\frac{1}{\lambda_2(p-1)\epsilon_1^{p-1}}$ *would be absent in (4.6).*

B. *Consider now that* $sign(y_0) = sign(r_0)$. *Then,* $r(t)$ *cannot reach zero until* $sign(y(t))$ *becomes opposite to* $sign(r(t))$*, since* $y(t)$ *has to reach zero first. In view of the second equation in (4.5), the convergence time of* $y(t)$ *to zero can be estimated from above as* $T_2 = \frac{K}{M}$ *and added to the time* T_1 *calculated in step* ***A***.

C. *Upon introducing a new state vector*

$$a(t) = [a_1(t), a_2(t)] = [\lfloor r(t) \rceil^{1/2}, y(t)],$$

so that $sign(a_1(t)) = sign(r(t))$*, the system (4.5) can be rewritten as*

$$\begin{aligned} \dot{a}_1(t) &= \frac{1}{2|a_1(t)|}(-\lambda_1 a_1(t) - \lambda_2 \lfloor a_1(t) \rceil^{2p} + a_2(t)) \\ \dot{a}_2(t) &= \frac{1}{2|a_1(t)|} a_1(t) + \dot{\xi}(t), \end{aligned} \tag{4.12}$$

or in the compact form $\dot{a}(t) = \frac{1}{2|a_1(t)|} A(a,t)a(t)$*, where*

$$A(a,t) = \begin{pmatrix} -(\lambda_1 + \lambda_2 |a_1(t)|^{2p-1}) & 1 \\ -(\alpha - \rho_1(t)) & 0 \end{pmatrix}, \tag{4.13}$$

yielding $\dot{\xi}(t) = \frac{1}{2|a_1(t)|}\rho_1(t)a_1(t)$, $|\rho_1(t)| \le 2L$.

Note that if $a_1(t)$ *and* $a_2(t)$ *converge to the origin for a fixed time, then* $r(t)$ *and* $y(t)$ *converge to the origin for the same fixed time.*

Consider the following Lyapunov function candidate [43]

$$V(a) = (\lambda + 4\epsilon^2)a_1^2 + a_2^2 - 4\epsilon a_1 a_2 = a^T P a, \tag{4.14}$$

where

$$P = \begin{pmatrix} \lambda + 4\epsilon^2 & -2\epsilon \\ -2\epsilon & 1 \end{pmatrix}. \tag{4.15}$$

The full time derivative is calculated as

$$\dot{V}(a) = \frac{1}{2|a_1(t)|} a^T [A^T(a)P + PA(a)] a = \frac{1}{2|a_1(t)|} a^T Q(a) a,$$

where the symmetric matrix $Q(a)$ *takes the form*

$$\begin{aligned} Q(a) &= A^T(a)P + PA(a) \\ &= \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12} & 4\epsilon \end{pmatrix}, \end{aligned} \tag{4.16}$$

with $Q_{11} = 2\lambda\lambda_1 + 4\epsilon(2\epsilon\lambda_1 - \alpha) + 4(\lambda + 4\epsilon^2)\lambda_2|a_1(t)|^{2p-1} + 4\epsilon\rho_1(t)$ *and* $Q_{12} = \alpha - 2\epsilon\lambda_1 - 2(\lambda + 4\epsilon^2)\lambda_2|z_1(t)|^{2p-1} - \lambda - 4\epsilon^2 - \rho_1(t)$.

To ensure positive definiteness of the matrix $Q(a)$*, the parameter* α *is assigned as*

$$\alpha(t) = 2\epsilon\lambda_1 + 2(\lambda + 4\epsilon^2)\lambda_2\epsilon_1^{p-1/2},$$

then $Q(a)$ *is positive definite with the minimum eigenvalue* $\mu_{min}(Q) \geq 2\epsilon$*, if the condition (4.9) holds.*

In view of Raleigh's inequality $a^T Q(a)a \geq \mu_{min}(Q)||a||^2$*,*

$$\dot{V}(a) = \frac{1}{2|a_1(t)|}a^T Q(a)a \leq -\frac{\epsilon}{a_1(t)}||a||^2.$$

Using Raleigh's inequalities $V(a) \leq \mu_{max}(P)||a||^2$ *and* $V^{1/2}(a) \geq \sqrt{\mu_{min}(P)}||a||$*, with* $\rho = \epsilon\frac{\sqrt{\mu_{min}(P)}}{\mu_{max}(P)}$*, yields*

$$\dot{V}(a) \leq -\rho V^{1/2}(a).$$

Based on Theorem 12 in [24], direct integration of the latter formula provides an estimate of the settling time for the vector state $a(t)$ *governed by the system (4.12):*

$$T_{FC}(a(T_1)) \leq \frac{2\mu_{max}(P)V^{1/2}(a(T_1))}{\epsilon\sqrt{\mu_{min}(P)}} \tag{4.17}$$

Having into account that at the end of the step ***A*** *the values of* $a_1(T_1)$ *and* $a_2(T_1)$ *are bounded by* $|a_1(T_1)| \leq \epsilon_1^{1/2}$ *and* $a_2T_1 \leq \frac{M}{\lambda_2(p-1)\epsilon_1^{p-1}} + K$*, one obtains*

$$V(a) \leq (\lambda + 4\epsilon^2)\epsilon_1 + (\frac{M}{\lambda_2(p-1)\epsilon_1^{p-1}} + K)$$

$$\times(\frac{M}{\lambda_2(p-1)\epsilon_1^{p-1}} + K - 4\epsilon\epsilon_1^{1/2}).$$

The last expression implies formula (4.6). Note that the system (4.5) converges to the domain $|a_1(t)| \leq \mu^{1/2}$ *or* $|r(t)| \leq \mu$ *and* $|a_2(t)| \leq \eta(\mu, L)$*, with* $\eta = \int_{t_0}^{T_{FC}}(\frac{\alpha(t)}{2} + L)ds$*.* ■

Remark 4.2.2. *Theorem above holds from the beginning, if the conditions (4.9) are effective at* $t = 0$*. Otherwise, the adaptive parameter* $\lambda_1(t)$ *increases in accordance with (4.8) until the conditions (4.9) hold at a certain* t_0 *bounded by*

$$t_0 \leq \frac{1}{\omega}\sqrt{\frac{2}{\gamma}}[\lambda_{lim} - \lambda_0].$$

If L *is known, the estimate for* t_0 *can be calculated. Therefore, the system (4.1),(4.4) is fixed-time convergent to the origin, when* $\mu = 0$*, or to a certain vicinity of the origin (real sliding mode), when* $\mu \neq 0$*. In both cases, the upper estimate* T_{FC} *can be explicitly calculated.*

Consider now that the state $r(t)$ *outgoes from a vicinity of the origin (real sliding mode) in view of variations of adaptive parameters or disturbance effects. The fixed-time convergence property of the proposed adaptive control (4.4), (4.8) is given as follows.*

Theorem 4.2.3. *Let the dynamic system (4.1),(4.4) be affected by disturbances* $\xi(t)$ *satisfying conditions (4.2), (4.3) for unknown constants* L *and* K*. Let the system state* $r(t)$ *outgo from the vicinity* $\{(r, y)| \;|$

$x \mid\leq \mu, \mid y \mid\leq \varrho\}$ at time t_1. The states $r(t)$ and $y(t)$ converge back to the same vicinity of the origin (real sliding mode) for a fixed time less or equal than T_{ret} determined as

$$T_{ret} \leq T_m + \frac{2}{\rho}[(\lambda + 4\epsilon^2)r_m + y_m(y_m - 4\epsilon r_m^{1/2})]^{1/2}, \tag{4.18}$$

where $r_m = \mu + LT_m$,

$$y_m = \eta + (\epsilon[\lambda_{lim} + \omega\sqrt{\frac{\gamma}{2}}T_m] + (\lambda + 4\epsilon^2)\lambda_2\epsilon_1^p + L)T_m,$$

$$T_m = \frac{1}{\omega}\sqrt{\frac{2}{\gamma}}[\lambda_{lim} - \lambda_{min}].$$

Proof of Theorem.

A. *Given $|r(t)| = \mu$ and $|r(t)|$ is increasing. Consider the worst case $\xi = Ltsign(r(t))$. Since the current value of $\lambda_1(t)$ is no less than λ_{min}, it reaches the value λ_{lim} for a time no greater than $T_m = \frac{1}{\omega}\sqrt{\frac{2}{\gamma}}[\lambda_{lim} - \lambda_{min}]$, where $\omega\sqrt{\frac{\gamma}{2}}$ is the adaptation speed for $\lambda_1(t)$, in view of the first equation in (4.8).*

Consider now the system (4.1) with the initial condition $r(t_1) = \mu$, and $\xi = Ltsign(r(t))$:

$$\begin{aligned}
\frac{d|r(t)|}{dt} &= sign(r(t))\frac{dr(t)}{dt} \\
&= -\lambda_1|r(t)|^{1/2} - \lambda_2|r(t)|^p + y(t)sign(r(t)) \\
&\leq -\lambda_1|r(t)|^{1/2} - \lambda_2|r(t)|^p + Ltsign(r(t))sign(r(t)) \\
&\leq Lt \\
&\leq L(t_1 + T_m).
\end{aligned} \tag{4.19}$$

Integrating the differential equation $\frac{d\psi(t)}{dt} = L(t_1+T_m)$ with the initial condition $\psi(t_1) = \mu$, and evaluating $\psi(t)$ at $t = t_1 + T_m$ yields the relation

$$\psi(t) \leq \mu + L(T_m),$$

which implies that the maximum value of $|r(t)|$ after the time $t = t_1 + T_m$ is not greater than r_m.

On other hand, since $|y(t_1)| \leq \eta$, the maximum value of $|y(t)|$ after time T_m is no greater than $y_m = \eta + (\alpha(T_m)/2 + L)T_m$, in view of the second equation in (4.5). The values of $\alpha(T_m)$ and $\lambda_1(T_m)$ are calculated in view of equations for $\alpha(t)$ and $\lambda_1(t)$ in (4.8) and the fact that the current value of the $\lambda_1(t)$ is no greater than λ_{lim}.

*Hence, step **A** ends with values of $r(t)$ and $y(t)$ bounded by r_m and y_m, respectively, and a value of $\lambda_1(t)$ no less than λ_{lim}, i.e., the condition (4.9) becomes valid at the end of step **A**.*

B. *Since the condition (4.9) holds, in view of Theorem 1, the control law (4.4) with the adaptive gains $\alpha(t)$ and $\lambda_1(t)$ drives both states $r(t)$ and $y(t)$ back to the origin for a fixed time via the Lyapunov function (4.14) given by the expression (4.17):*

$$T_{FC}(a(T_1)) \leq \frac{V^{1/2}(a_m)}{\rho} \tag{4.20}$$

where $a_m = [\lfloor r_m\rceil^{1/2}, y_m]$ and

$$V(a_m) = (\lambda + 4\epsilon^2)r_m + y_m(y_m - 4\epsilon r_m^{1/2}).$$

*Substituting the obtained value of $V(a_m)$ into (4.20) yields the second term in (4.18). Adding the time T_m found at step **A** implies the complete formula (4.18).* ■

Hereafter, a Stock Management Problem with the objective to drive stock and supply chain levels at the reference values, subject to loss rate disturbances whose bounds are unknown, is addressed via an adaptive fixed-time convergent continuous controller including a fixed-time convergent differentiator. In the examined case, the only measurable state of the supply chain is the inventory retailer stock level, whereas the supply line inventory level should be estimated.

4.3 Stock Management Problem Statement

Consider the conventional stock management model ([105, 108])

$$\begin{aligned} \dot{x}_1(t) &= -\frac{x_1(t)}{lt} + \frac{x_2(t)}{al} - d(t), \quad x_1(t_0) = x_{10}, \\ \dot{x}_2(t) &= -\frac{x_2(t)}{al} + O(X,t), \qquad\qquad x_2(t_0) = x_{20}, \\ y &= x_1(t). \end{aligned} \tag{4.21}$$

where $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$ is the system state, $x_1(t) \in \mathbb{R}$ is the stock level, which is the only measurable state, $x_2(t) \in \mathbb{R}$ is the supply line, $O(t) \in \mathbb{R}$ is the order rate (control input), $d(t) \in \mathbb{R}$ is an external disturbance satisfying Lipschitz condition, which represents nonlinear effects in the stock variation, $lt \in \mathbb{R}^{>0}$ is the lifetime of each stock unit, $al \in \mathbb{R}^{>0}$ is the acquisition lag of each unit in the supply line. The fractions $\frac{x_1(t)}{lt}$, $\frac{x_2(t)}{al}$ are called loss rate and acquisition rate, respectively.

The stock $x_1(t)$, accumulates the acquisition rate less the loss rate. The loss rate depends on the stock and approaches zero when the stock is depleted. The supply line $x_2(t)$ accumulates the incoming orders less the acquisition rate, which represents the orders that have been placed but not yet received. For this reason, the acquisition rate depends on the supply line and acquisition lag.

The order rate is the decision rule that indicates the rate at which managers wish to add units to the supply line. Two considerations are fundamental: managers should replace losses from the stock as well as reduce the discrepancy between the desired and current stock and the discrepancy between the desired and current supply lines by acquiring more than losing, when the stock is below the desired level, and less than losing, when there is a surplus [108].

The problem consists in designing a decision rule $O(t)$ that drives the states of the system (4.21) at the desired stock and supply chain levels x_{1r}, x_{2r} for a fixed time independent of initial conditions, in the presence of an external disturbance satisfying Lipschitz condition. Since the only measurable state is $x_1(t)$, a fixed-time convergent differentiator is used to estimate the state $x_2(t)$.

4.4 Adaptive continuous fixed-time convergent controller for SMP

4.4.1 Companion Form

Let us introduce the new variables for the system (4.21), $s_1(t) = x_1(t)$ and $s_2(t) = -\frac{x_1(t)}{lt} + \frac{x_2(t)}{al} - d(t)$, then $x_2(t) = al(s_2(t) + \frac{s_1(t)}{lt} + d(t))$, where $s_2(t)$ represents the variation of stock $x_1(t)$. The introduced *variables satisfy the equations*

$$\dot{s}_2(t) = -\frac{\dot{s}_1(t)}{lt} + \frac{\dot{x}_2(t)}{al} - \dot{d}(t),$$

$$\dot{s}_2(t) = -\frac{\dot{s}_1(t)}{lt} + \frac{1}{al}(-s_2(t) - \frac{s_1(t)}{lt} - d(t) + O(t)) - \dot{d}(t),$$

and therefore,

$$\begin{aligned} \dot{s}_1(t) &= s_2(t), \quad s_1(t_0) = s_{10}, \\ \dot{s}_2(t) &= -\frac{s_1(t)}{(al)(lt)} - (\frac{1}{al} + \frac{1}{lt})s_2(t) \\ &+ \frac{O(t)}{al} - \frac{d(t)}{al} - \dot{d}(t), \\ s_2(t_0) &= s_{20}. \end{aligned} \tag{4.22}$$

4.4.2 Fixed-Time Convergent Differentiator

Consider the system (4.22) in the companion form. A fixed-time convergent differentiator should be used to estimate the state $s_2(t)$, given that only the state $s_1(t)$ is measurable. Then, a fixed-time convergent continuous controller is designed to drive the system states at a reference point or its certain vicinity (real sliding mode).

Denote for $i = 1, 2$

$$\lceil z_i(t) - s_{ir} \rceil^{\alpha} = |z_i(t) - s_{ir}|^{\alpha} sign(z_i(t) - s_{ir}).$$

where $z_i(t)$ is the estimate of the variable $s_i(t)$ and s_{ir} is the reference value for the variable $s_i(t)$, respectively.

The smooth fixed-time convergent differentiator (see details in [84]) is employed to estimate the system states (4.22):

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) - k_1\lfloor z_1(t) - s_1(t)\rceil^{\alpha_1} - \kappa_1\lfloor z_1(t) - s_1(t)\rceil^{\beta_1}, \\ \dot{z}_2(t) &= -k_2\lfloor z_1(t) - s_1(t)\rceil^{\alpha_2} - \kappa_2\lfloor z_1(t) - s_1(t)\rceil^{\beta_2}. \end{aligned} \tag{4.23}$$

Here, the exponents α_i, $i = 1, 2$, and β_i, $i = 1, 2$, are selected as $\alpha_1 \in (1 - \epsilon, 1)$, $\beta_1 \in (1, 1 + \epsilon_1)$, where $\epsilon > 0$, $\epsilon_1 > 0$ are small positive numbers, $\alpha_2 = 2\alpha_1 - 1$, and $\beta_2 = 2\beta_1 - 1$. Differentiator gains k_i, $i = 1, 2$, and κ_i, $i = 1, 2$, are chosen so that the characteristic polynomials of the matrices A_1 and A'_1

$$A_1 = \begin{pmatrix} -k_1 & 1 \\ -k_2 & 0 \end{pmatrix}, \quad A'_1 = \begin{pmatrix} -\kappa_1 & 1 \\ -\kappa_2 & 0 \end{pmatrix}$$

are Hurwitz. The positive definite symmetric matrices P_1 and P'_1 satisfy the Lyapunov equations

$$P_1 A_1 + A_1^T P_1 = -Q_1, \qquad P'_1 A'_1 + A'^T_1 P'_1 = -Q'_1,$$

where $Q_1, Q'_1 \in R^{n \times n}$ are positive definite symmetric matrices.

4.4.3 Fixed-Time Convergent Regulator

Consider the continuous fixed-time convergent control input (decision rule) $O(t)$ for the systems (4.21) and (4.22) (see details in [84]):

$$O(t) = O_1(t) + O_2(t),$$

where $O_1(t) = al(\frac{1}{al} + \frac{1}{lt})z_2(t)$ *and* $O_2(t) = \max\{0, \frac{z_1(t)}{lt} + u_1(t)\} + u_2(t)$,

$$\begin{aligned} u_1(t) &= -\frac{1}{sat_1}\lfloor z_1(t) - s_{1r}\rceil^{\gamma_1} \\ &- \frac{al}{slat_1}\lfloor z_2(t) - s_{2r}\rceil^{\gamma_2} \\ &- \frac{1}{sat_2}\lfloor z_1(t) - s_{1r}\rceil^{\delta_1} - \frac{al}{slat_2}\lfloor z_2(t) - s_{2r}\rceil^{\delta_2}, \\ u_2(t) &= -\lambda_1\lfloor r(t)\rceil^{1/2} \\ &- \lambda_2\lfloor r(t)\rceil^{3/2} - \alpha\int_{t_0}^{t} sign(r(\tau))d\tau, \\ r(t) &= z_2(t) - s_{2r} - \vartheta(t), \\ \dot{\vartheta}(t) &= u(t) - u_2(t). \end{aligned} \tag{4.24}$$

The exponents γ_i and δ_i, $i = 1, 2$, are chosen as $\gamma_2 \in (1 - \epsilon_2, 1)$, $\delta_2 \in (1, 1 + \epsilon_3)$, where $\epsilon_2 > 0$, $\epsilon_3 > 0$ are small positive numbers, $\gamma_1 = \gamma_2/(2 - \gamma_2)$, $\delta_1 = \delta_2/(2 - \delta_2)$, $sat_i > 0$, $i = 1, 2$, are stock adjustment times, $slat_i > 0$, $i = 1, 2$, are supply line adjustment times, and $\alpha > 0$. Parameters $\frac{1}{sat_i}$ and $\frac{al}{slat_i}$, $i = 1, 2$, are selected so that $\lambda^2 + \frac{1}{sat_2}\lambda + \frac{1}{sat_1}$ and $\lambda^2 + \frac{al}{slat_2}\lambda + \frac{al}{slat_1}$ are Hurwitz polynomials.

As a result, if $\frac{z_1(t)}{lt} + u_1(t) > 0$ and the differentiator estimates $z_i(t)$ converge to the variables $s_i(t)$, $i = 1, 2$, the system (4.22) is represented as

$$\begin{aligned} \dot{s}_1(t) &= s_2(t), \qquad s_1(t_0) = s_{10}, \\ \dot{s}_2(t) &= u(t) + \xi(t), \quad s_2(t_0) = s_{20}, \end{aligned} \tag{4.25}$$

where $u(t) = u_1(t) + u_2(t)$, $\xi(t) = -\frac{d(t)}{al} - \dot{d}(t)$.

Remark 4.4.1. *Note that the required inequality $u_1(t) > 0$ becomes valid for all $t \geq t_1$, where t_1 is a time moment close to zero, in view of the term $\frac{z_1(t)}{lt}$ in (4.24) and the fact that the reference point s_{1r} for the stock level $s_1(t) = x_1(t)$ is a positive number. Thus, the control law (4.24) remains fixed-time convergent to a vicinity of the origin, according to Theorem 1 in [84].*

Remark 4.4.2. *In practice, to maintain the acquisition rate consistent with the acquisition lag and loss rate, the desired supply line x_{2r} is formulated as a function of the acquisition lag al and loss rate [153]: $x_{2r} = al(\frac{x_1(t)}{lt})$.*

4.5 Simulations

Consider a firm managing its stock. The stock is the total quantity of finished products, and the supply line is the amount of products on unfilled orders, i.e., the orders that have been placed but not yet closed. The loss rate represents the product sales with an average lifetime of $lt = 8$ days. The acquisition process goes on with the acquisition lag set to $al = 1.5$ days and there are no capacity constraints.

Two numerical simulations are conducted. In Simulation 1, all the system (4.22) states are assumed measurable, and in Simulation 2, $x_1(t)$ *is considered the only measurable state. Therefore, only the continuous fixed-time regulator (4.24) is applied in Simulation 1, while both, the continuous fixed-time regulator (4.24) and the smooth fixed-time differentiator (3.4), are employed in Simulation 2.*

The time step is set to 0.0001 *days (practically a continuous review at every* 8.64 *seconds). The planning horizon is set to* 50 *days. The initial condition is assigned to* 1000 *units for the stock level and* 500 *units for the supply line. The reference stock level is set to* 80 *units, and the reference supply line is set to* 15 *units.*

Two kinds of disturbances representing nonlinear effects in the stock variation are considered. The first disturbance $d_1(t)$ *is assigned as* $d_1(t) = 5\sin(2t) + \frac{\sin(100t)}{1000} + 10t$*, therefore,*

$$\xi_1(t) = -\frac{10}{3}\sin(2t) - \frac{2\sin(100t)}{3000}$$

$$-\frac{20}{3}t - 10\cos(2t) - \frac{\cos(100t)}{10} - 10.$$

The Lipschitz constant for $\xi_1(t)$ *can be estimated as* $L_1 = 44$*; therefore, the corresponding* $\lambda_{lim} \leq 1293.7$ *and* λ_0 *can be assigned as* $\lambda_0 = 1294$.

The second disturbance $d_2(t)$ *differs from* $d_1(t)$ *in the amplitude of its high-frequency component:* $d_2(t) = 5\sin(2t) + \sin(100t) + 10t$*. Therefore,*

$$\xi_2(t) = -\frac{10}{3}\sin(2t) - \frac{2\sin(100t)}{3}$$

$$-\frac{20}{3}t - 10\cos(2t) - 100\cos(100t) - 10.$$

The Lipschitz constant for $\xi_2(t)$ *can be estimated as* $L_2 = 10100$*; therefore, the corresponding* λ_{lim} *is about* 68000000*. Nonetheless, the controller (differentiator and regulator) parameter values are intentionally not changed to demonstrate robustness of the algorithm.*

The exponents for the differentiator and regulator are assigned as $\alpha_2 = 8/10$, $\alpha_1 = 9/10$, $\beta_2 = 11/9$, $\beta_1 = 10/9$ *and* $\gamma_2 = 19/22$, $\gamma_1 = 19/23$, $\delta_2 = 21/18$, $\delta_1 = 21/17$*. The differentiator and regulator gains are set to* $k_1 = \kappa_1 = 100$, $k_2 = \kappa_2 = 1000$, $m_1 = M_1 = 2$*, and* $m_2 = M_2 = 4$*, where* $m_i = \frac{1}{sat_i}$, $M_i = \frac{al}{slat_i}$, $i = 1, 2$.

The differentiator convergence time parameters are calculated [37] as: $\rho = 1 - \frac{9}{10} = \frac{1}{10}$, $\sigma = \frac{10}{9} - 1 = \frac{1}{9}$, $r = \frac{1}{5.0550}$, $r_1 = \frac{1}{5.0550}$, $\Upsilon = 0.050$, $\lambda_{min}(P_1) = 0.050$, $\lambda_{max}(P_1) = 5.0550$*, where*

$$P_1 = P_1' = \begin{pmatrix} 5.0050 & -0.5000 \\ -0.5000 & 5.0050 \end{pmatrix}$$

and $Q_1 = Q_1' = I_3$ *are identity matrices. The differentiator convergence time upper estimate is calculated as* $T_{FD} = 141.399$ *days.*

The regulator convergence time parameters are calculated [26] as: $\rho = (1 - \frac{19}{21})/\frac{19}{21} = \frac{2}{19}$, $\sigma = (\frac{21}{19} - 1)/\frac{21}{19} = \frac{2}{21}$, $r = \frac{1}{1.4255}$, $r_1 = \frac{1}{1.4255}$, $\Upsilon = 0.1370$, $\lambda_{min}(P) = 0.1370$, $\lambda_{max}(P) = 1.4255$*, where*

$$P = P' = \begin{pmatrix} 1.3750 & 0.2500 \\ 0.2500 & 0.1875 \end{pmatrix}$$

and $Q = Q' = I_3$ *are identity matrices. The regulator convergence time upper estimate is calculated as* $T_{FR} = 32.1439$ *days. Thus,* $T_{F_1} = T_{FD} + T_{FR} = 173.5429$ *days.*

For the compensator adaptive control law $u_2(t)$*, the non-adaptive gain is set to* $\lambda_2(t) = 1$*, and* $p = 3/2$*. The adaptive control gains* $\lambda_1(t)$*, and* $\alpha(t)$ *are assigned according to the equation (4.8) and the parameter values are* $\epsilon = 1$*,* $\epsilon_1 = 1$*,* $\omega = 70$*,* $\gamma = 0.5$*,* $\mu = 0.002$*,* $\lambda_{min} = 0.02$*.*

In accordance with (4.6), the disturbance compensator convergence time upper bound is calculated as $T_{FC} = 159.2128$ *days. Therefore, the uniform convergence time upper bound for the entire controller is calculated as* $T_F = T_{FD} + T_{FR} + T_{FC} = 141.399 + 32.1439 + 159.2128 = 332.7557$ *days.*

4.5.1 Simulation 1: All states are measurable

Since all states are measurable, only the continuous fixed-time regulator (4.24), including the adaptive fixed-time disturbance compensator $u_2(t)$*, is employed in this simulation. The simulation graphs shown in Figs. 4.1–4.6 correspond to the initial values* $s_1(0) = 1000$*,* $s_2(0) = 500$*, with the simulation step equal to* 10^{-4}*. The reference points are set as* $x_{1r} = 80$ *and* $x_{2r} = 15$*. The simulation horizon is* $T = 50$ *days. The upper bound for the settling time is calculated as* $T_F = T_{FR} + T_{FC} = 32.1439 + 159.2128 = 191.3567$ *days.*

Figure 4.1 displays the disturbance $d_1(t)$*. Figures 4.2–4.4 show the time histories of state variables* $x_1(t) = s_1(t)$*,* $x_2(t)$*, and* $s_2(t)$*. The finite convergence time for all state variables is about 35 days. Figures 4.5–4.6 present the behavior of the adaptive gains* $\lambda_1(t)$ *and* $\alpha(t)$*.*

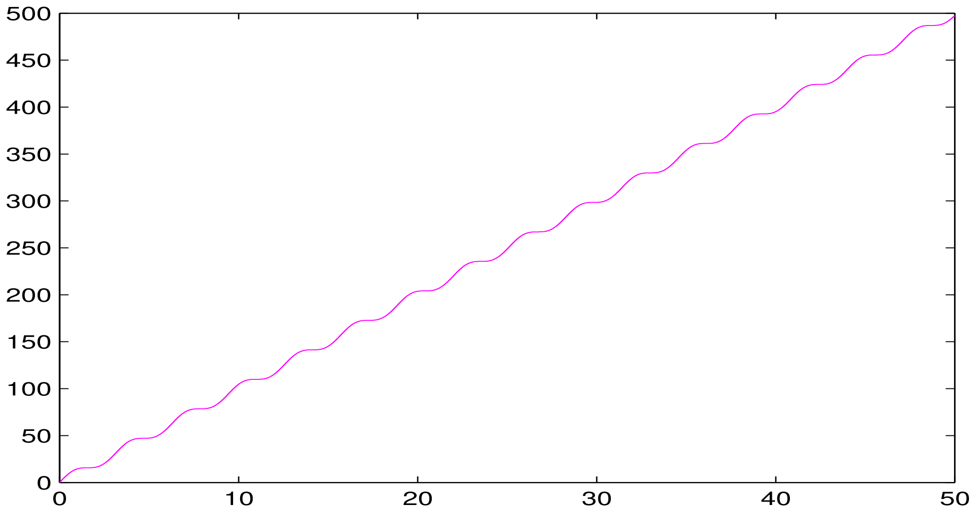

Figure 4.1: Time history of disturbance $d_1(t)$.

4.5.2 Simulation 2: Only the stock level $x_1(t)$ is measurable

The designed controller operates with three steps: estimating the state variables by the fixed-time convergent smooth differentiator (3.4), compensating for disturbances by the adaptive fixed-time convergent continuous disturbance compensator $u_2(t)$*, and assuring fixed-time convergence of the state variables at*

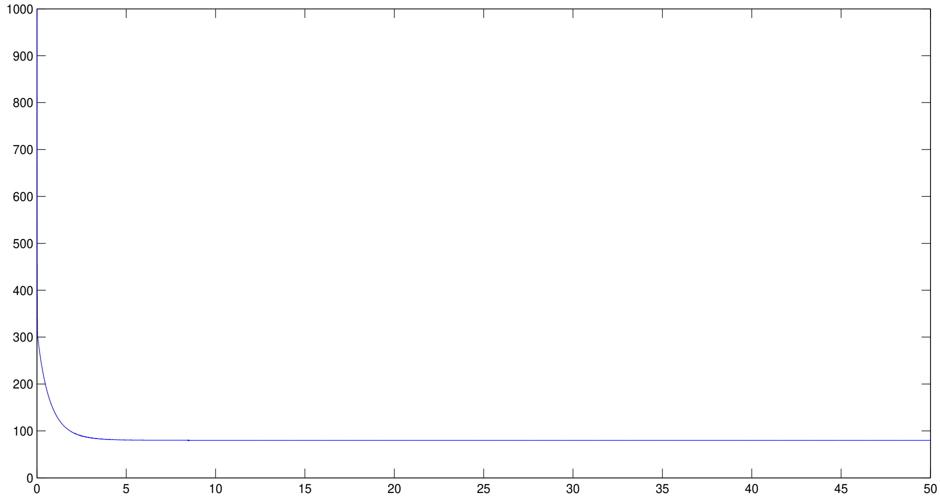

Figure 4.2: Time history of stock level $x_1(t)$.

the origin/to a vicinity of the origin by the fixed-time convergent continuous regulator (4.24) using the estimates provided by the differentiator. This set of simulations also demonstrates robustness of designed controller, when the high-frequency component of the disturbance $d_2(t)$ has an amplitude 1000 times higher than the high-frequency component of the disturbance $d_1(t)$. The initial conditions, reference points, and simulation step and horizon are the same as in Simulation 1.

Simulation 2a

The disturbance $d_1(t)$ with a low amplitude of the high-frequency component is considered. The smooth fixed-time convergent differentiator (3.4) and the continuous fixed-time convergent regulator (4.24) are both applied simultaneously from the beginning of the simulation. Figures 4.7–4.9 show the time histories of the state variables $x_1(t) = s_1(t)$ and $s_2(t)$, with their respective estimates $z_1(t)$ and $z_2(t)$, and $x_2(t)$. Figures 4.10–4.11 present the behavior of the adaptive gains $\lambda_1(t)$ and $\alpha(t)$ in this case. The convergence time for all state variables is also about 35 *days. The convergence time of the estimates $z_1(t)$ and $z_2(t)$ to the states $x_1(t)$ and $x_2(t)$ does not exceed* 10 *days.*

Simulation 2b

The disturbance $d_2(t)$ with a high amplitude of the high-frequency component is considered. The smooth fixed-time convergent differentiator (3.4) and the continuous fixed-time convergent regulator (4.24) are both applied simultaneously from the beginning of the simulation. Figure 4.12 displays the disturbance $d_2(t)$. Figure 4.13 shows the time history of the state variable $s_2(t)$. Figures 4.14–4.15 present the behavior of the adaptive gains $\lambda_1(t)$ and $\alpha(t)$ in this case. Given that the adaptive gains $\lambda_1(t)$ and $\alpha(t)$ of the disturbance compensator $u_2(t)$ remain on a higher level in view of a larger disturbance, the convergence time for all state variables is about 15 *days, which is even shorter than the convergence time in Simulation 2a, as it can be observed in Figs. 4.9 and 4.13.*

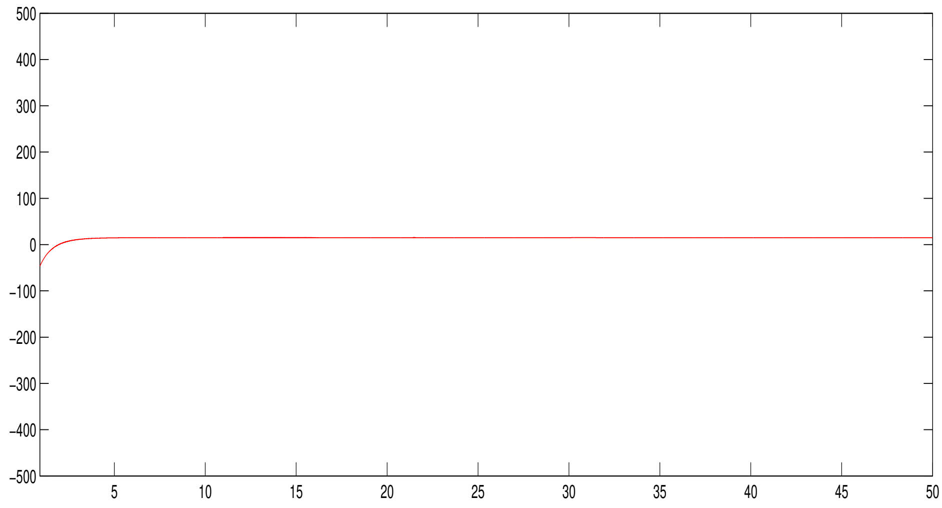

Figure 4.3: Time history of supply line $x_2(t)$.

As follows from the simulation results, an unbounded disturbance with either small or large magnitude of the high-frequency component is suppressed in the process of uniform fixed-time convergence of the state variables of the systems (4.21) and (4.22) to given reference points or its vicinities in uniform fixed time. Therefore, it can be concluded that the designed controller solves the originally stated problem of uniform fixed-time stabilization of the stock level and supply line at desired values. The steady-state magnitude of the stock variation $s_2(t)$ is uniformly bounded and is much less than the original disturbance magnitude in all cases.

4.6 Conclusions and future work

An adaptive fixed-time convergent continuous sliding mode controller is proposed for a class of dynamic systems affected by disturbances satisfying a Lipschitz condition with an unknown Lipschitz constant and an unknown initial value, whose only measurable state is the highest relative degree one. The adaptive controller design presented is validated in a Stock Management Problem, with the objective to drive the stock and supply chain levels at the reference values, subject to loss rate disturbances whose bounds are unknown. The adaptive controller component is based on APVIOBPCS policy combined with the super-twisting algorithm and employed to compensate for the disturbances. Two cases are studied: first, both, the stock level and supply chain, are assumed measurable and available for control design, and second, only the stock level is measurable. In the second case, unbounded disturbances with both, small and large, magnitudes of the high-frequency component are considered. The obtained simulation results show that the designed controller solves the uniform fixed-time stabilization problem for the stock level and supply line, driving all state variables to given reference points or its vicinities in uniform fixed time, and demonstrates robustness with respect to an abrupt increase in magnitude of the high-frequency disturbance component.

Future work will consider to replace Compensator with a controller based on Utkin equivalent control with an adaptive law like in [46], including fixed time convergence.

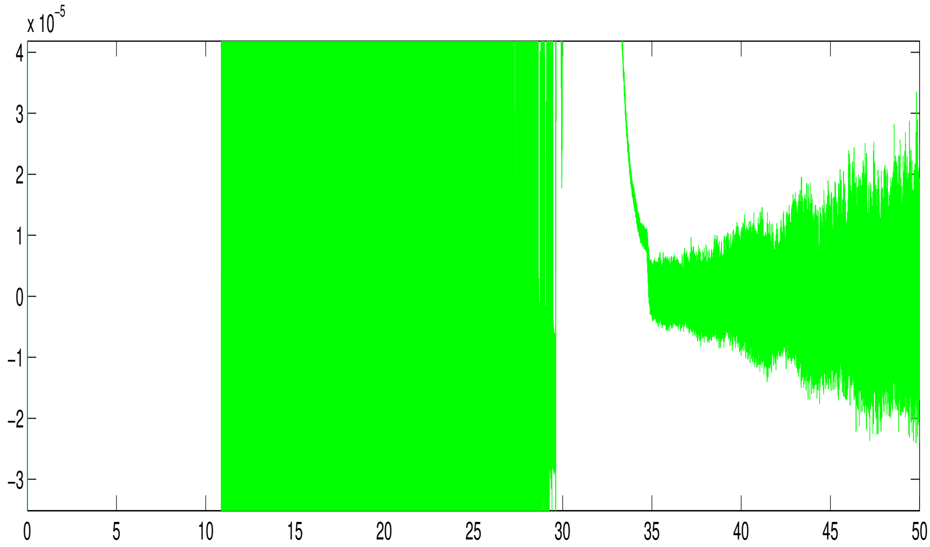

Figure 4.4: Time history (zoom) of stock variation $s_2(t)$.

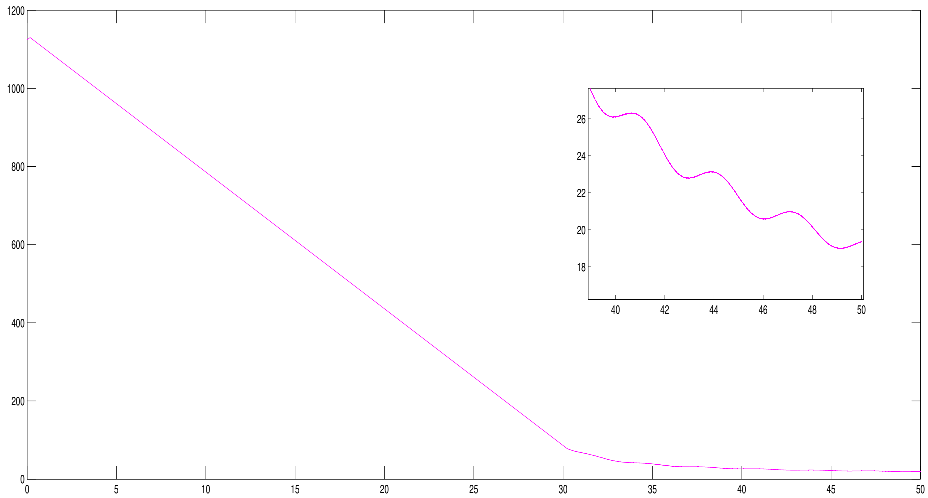

Figure 4.5: Time history of adaptive gain $\lambda_1(t)$.

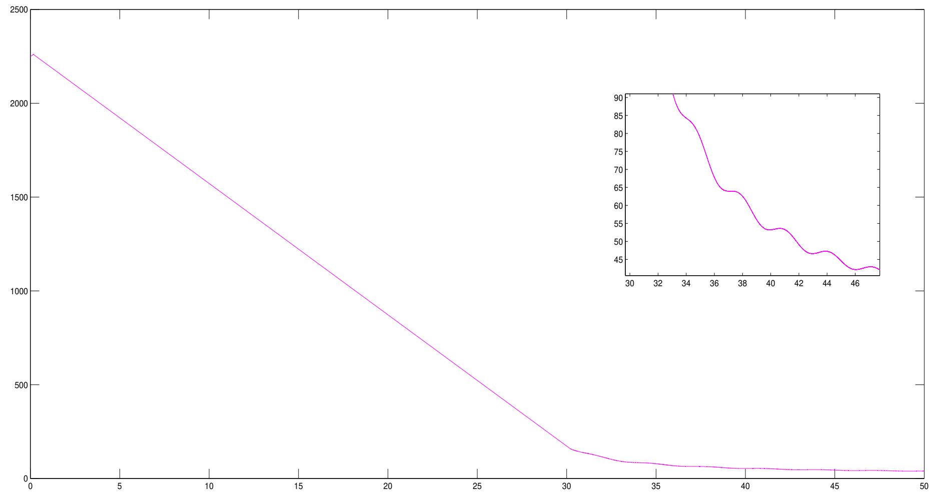

Figure 4.6: Time history of adaptive gain $\alpha(t)$.

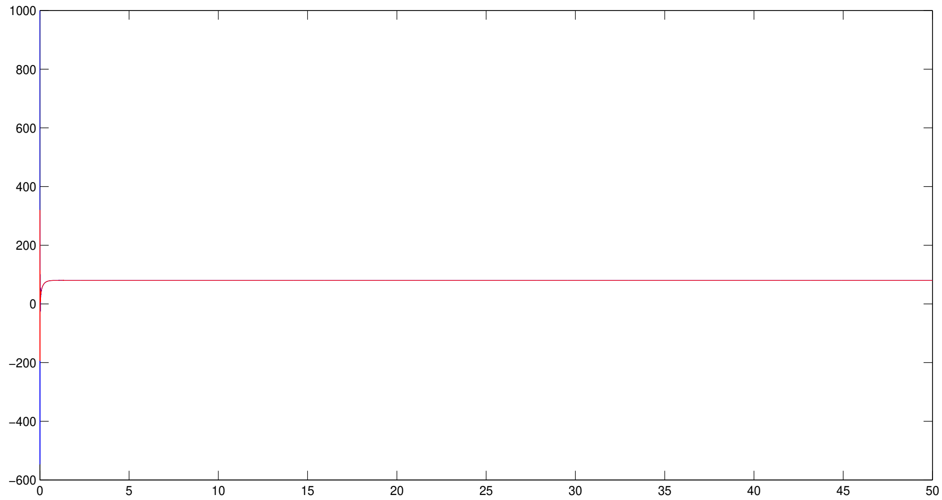

Figure 4.7: Time histories of stock level $x_1(t)$ (blue) and its estimate $z_1(t)$ (red).

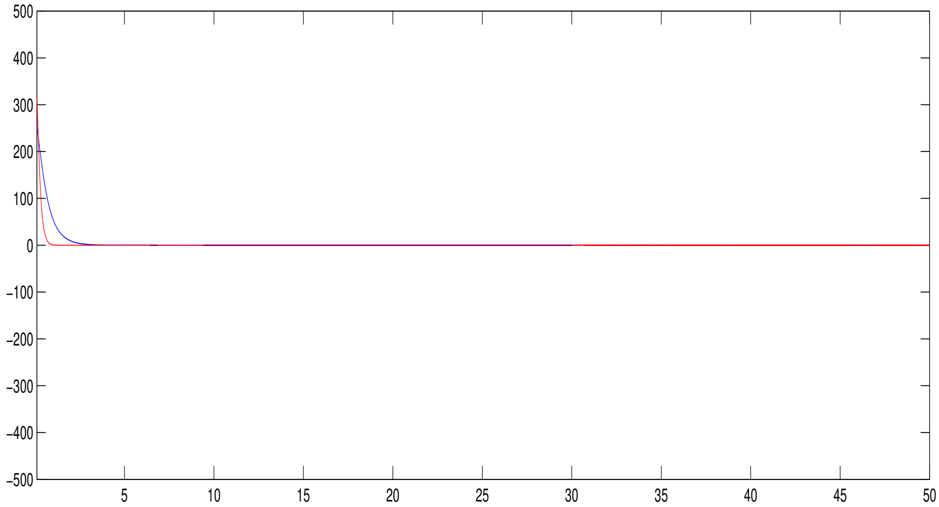

Figure 4.8: Time histories of stock variation $s_2(t)$ (blue) and its estimate $z_2(t)$ (red).

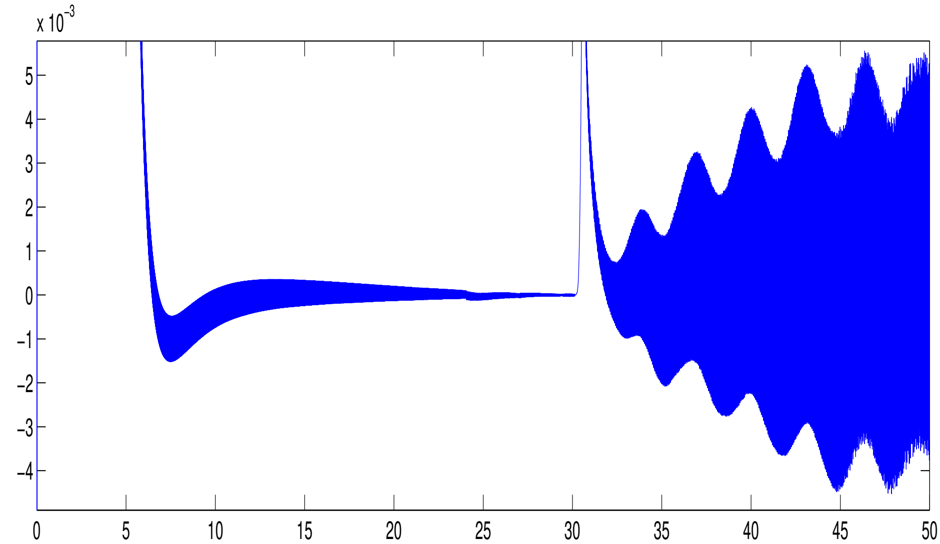

Figure 4.9: Time history (zoom) of stock variation $s_2(t)$.

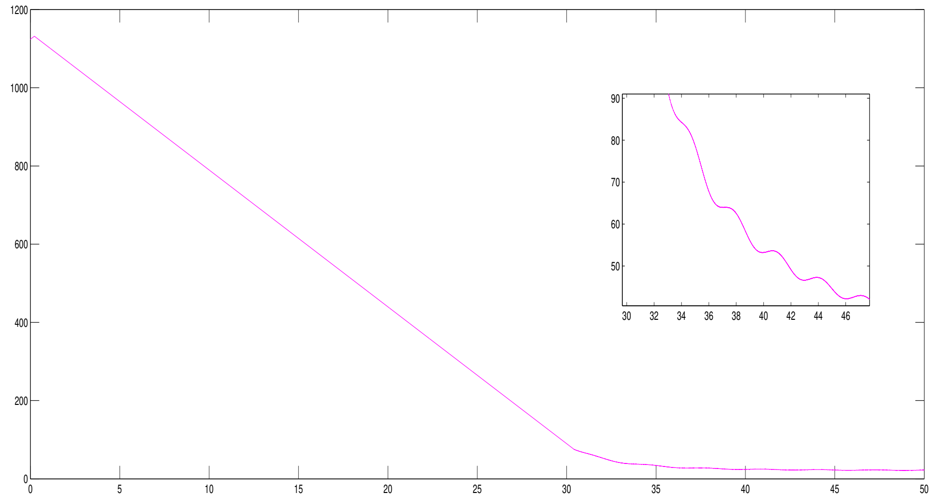

Figure 4.10: Time history of adaptive gain $\lambda_1(t)$.

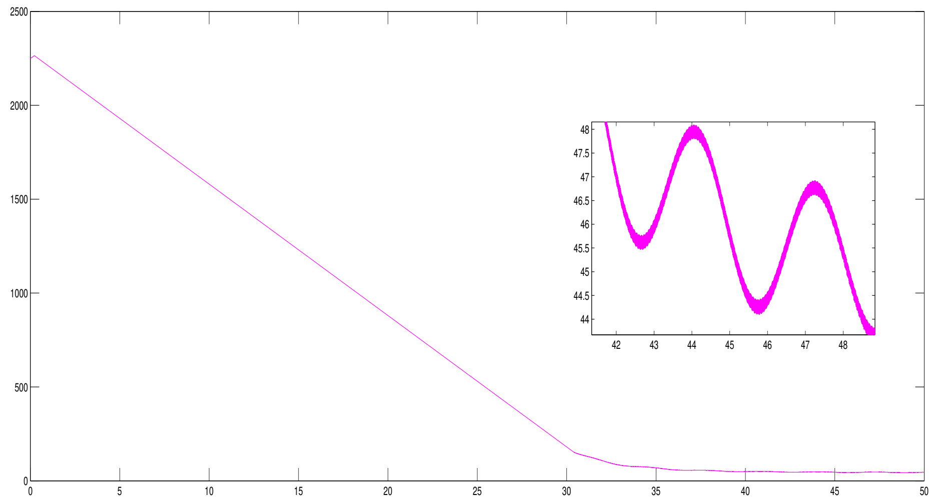

Figure 4.11: Time history of adaptive gain $\alpha(t)$.

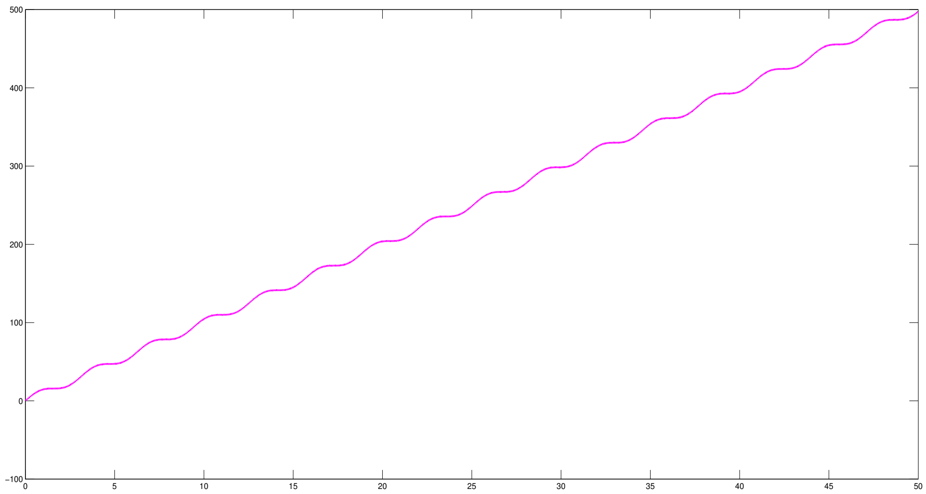

Figure 4.12: Time history of disturbance $d_2(t)$.

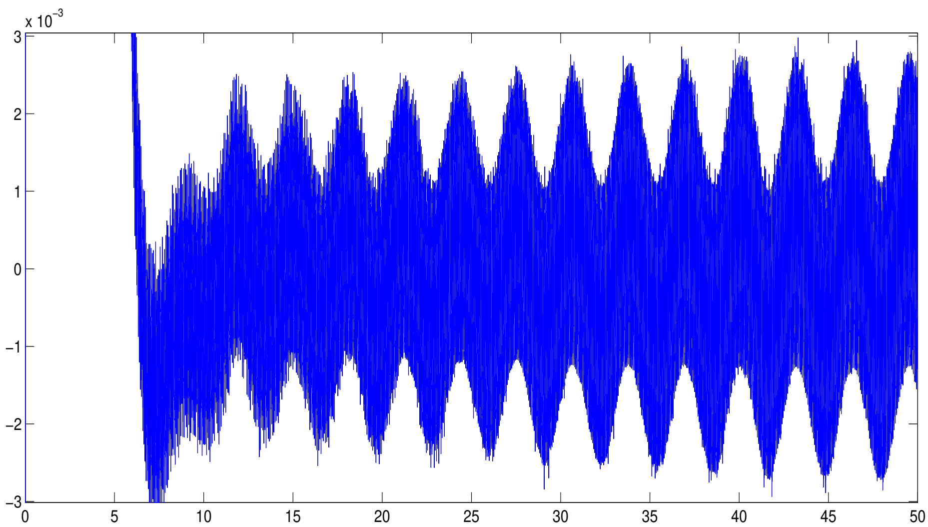

Figure 4.13: Time history (zoom) of stock variation $s_2(t)$.

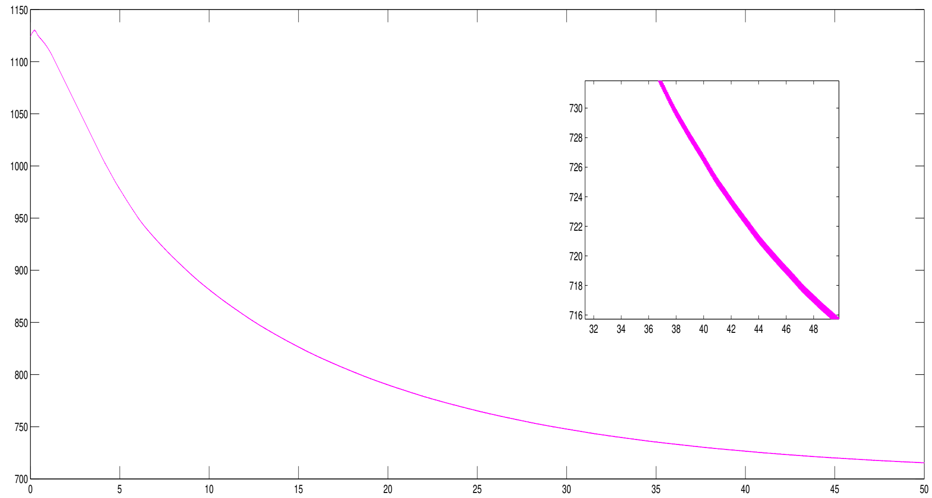

Figure 4.14: Time history of adaptive gain $\lambda_1(t)$.

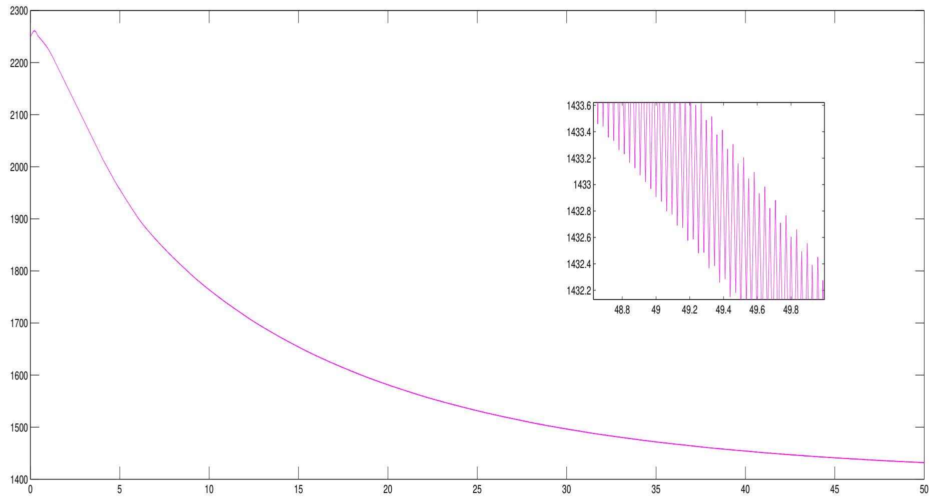

Figure 4.15: Time history of adaptive gain $\alpha(t)$.

Chapter 5

A Fixed-Time Convergent Stochastic Super-twisting Algorithm

"Some things benefit from shocks; they thrive and grow when exposed to volatility, randomness, disorder, and stressors and love adventure, risk, and uncertainty."

Nassim Nicholas Taleb (1960–)

5.1 Introduction

The main contribution of this chapter whose content is included in [86], consists in designing a continuous time convergent in ρ-mean control law driving the states of a stochastic Super-twisting system at the origin for a finite pre-established (fixed) time. The stochastic Super-twisting system includes both a stochastic white noise and an unbounded deterministic disturbance satisfying a Lipschitz condition. Two convergence theorems are given and numerical simulations are conducted to validate the obtained theoretical results. To the best of our knowledge, this is the first result of stochastic stabilization in ρ-mean for fixed-time convergent super-twisting systems.

Content of chapter is organized as follows. Preliminaries about fixed time convergence for stochastic systems are given in section 5.2. The problem statement is presented in section 5.3. Two theorems about fixed time convergence of Super-twisting algorithm are provided in Section 5.4. Numerical experiments to illustrate validity of theoretical results are reported in Section 5.5. Analysis of obtained results, conclusions, and future work directions are given in Section 5.6.

5.2 Preliminaries

Consider an n-dimensional stochastic dynamical system

$$\begin{aligned} dX(t) &= b(t, X(t))dt + \sigma(t, X(t))dW \\ X(0) &= X_0 \in \mathbb{R}^n \quad t \geq 0. \end{aligned} \tag{5.1}$$

where the stochastic process $X(t) = X_t(\omega)$, $X(t) =\in \mathbb{R}^n$ is the vector state with initial condition $X(0) = X_0 \in \mathbb{R}^n$, $W(t) \in \mathbb{R}$ is the Wiener process defined on the complete probability space (Ω, F, P), Ω is *the sample space, F is a σ-field with a filtration $\{F_t\}_{t\geq 0}$, and P a probability measure. Assuming the coefficients of (5.1) are Borel measurable, continuous in X and satisfy, $b(t,0)=0$, $\sigma(t,0)=0$, for all $t\geq 0$.*

The following concept of stochastic ρ-mean convergence can be viewed as an extension of mean-square convergence, which corresponds to $\rho=2$.

Definition 5.2.1. *The trivial solution of the system in Equation (5.1) is said globally fixed-time convergent in ρ-mean, if for any initial state $X_0\in\mathbb{R}^n$ exists a positive constant $T_{max}>0$ (independent of X_0), such that $E|X(t)|^\rho=0$ for all $t\geq T_{max}$.*

Definition 5.2.2. *[82] For system in Equation (5.1), define $T_0(t_0,X_0,W)=\inf\{T\geq 0: E|X(t)|^\rho = 0, \forall t\geq t_0+T\}$ which is called the stochastic settling time function.*

Note that settling time function $T_0(t_0,X_0,W)$ is bounded and the upper bound is a positive constant T_{max}, which is independent of the initial state of system (5.1).

Convergence cases most frequently considered in the literature is that of convergence in $\rho=1$ (convergence in the mean) and for $\rho=2$ (convergence in mean square).

5.2.1 Lyapunov Functions

Let consider Lyapunov functions sufficiently smooth in t and and x in a neighbourhood of $x=0$, except possibly at the point $x=0$ itself.

If $U\subseteq(0,\infty)\times\mathbb{R}$ is a connected open set (a domain), with closure $\overline{U}$, we shall say that a function $V(t,x)$ is in class $C_0^2(U)$ ($V(t,x))\in C_0^2(U)$ if it is twice continuously differentiable with respect to x and continuously differentiable with respect to t throughout U, except possibly for the set $x=0$, and continuous in the closed set $\overline{U}$.

The infinitesimal generator $\mathcal{L}$ associated with system in equation (5.1) is given by

$$\mathcal{L}V(t,x)=\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)b(t,x)+\frac{1}{2}Tr(\frac{\partial^2 V}{\partial x^2}(t,x)\sigma(t,x)\sigma(t,x)^T) \tag{5.2}$$

for a function $V(t,x))\in C_0^2(U)$.

5.2.2 Ito formulas

The Ito formula for $V(x)$ takes the form

$$\begin{aligned} dV(x) &= \frac{\partial V}{\partial x}(x)dx+\frac{1}{2}Tr(\frac{\partial^2 V}{\partial x^2}(x)\sigma(t,x)\sigma(t,x)^T)dt \\ &= \frac{\partial V}{\partial x}(x)[b(t,x)dt+\sigma(t,x)dW] \\ &+ \frac{1}{2}Tr(\frac{\partial^2 V}{\partial x^2}(x)\sigma(t,x)\sigma(t,x)^T)dt \\ &= \mathcal{L}V(x)+\frac{\partial V}{\partial x}(x)[\sigma(t,x)dW] \end{aligned} \tag{5.3}$$

Hereafter, consider a non-negative scalar function V satisfies $V\leq\mu(V)$, $V(t)\in C_0^2(U)$, with μ differentiable function, $\mu(V)>0$ when $V>0$, and $\mu(0)=0$ in other case, and $\mu'(V)>0$.

5.3 Problem statement

Consider a stochastic super-twisting system

$$\begin{aligned} dx(t) &= [-\lambda_1\lfloor x(t)\rceil^q - \lambda_2\lfloor x(t)\rceil^p + y(t)]dt \\ &+ \sigma(t, X(t))dW(t), \quad x(t_0) = x_0, \\ dy(t) &= [-\alpha sign(x(t)) + \dot{\xi}(t)]dt, \quad y(t_0) = 0. \end{aligned} \tag{5.4}$$

Here, $[x(t), y(t)] \in \mathbb{R}^2$ *is a two-dimensional stochastic process,* $W(t) \in \mathbb{R}$ *is a Wiener process. A deterministic disturbance* $\xi(t)$ *is a measurable function satisfying a Lipschitz condition* $|\ \xi(t)\ | \leq L(t - t_0)$ *with a certain constant L. Therefore, its derivative* $\dot{\xi}(t)$ *in the second equation in (2) is uniformly bounded,* $|\dot{\xi}(t)| \leq L$. *The control gains* λ_1, λ_2, $\alpha > 0$ *are positive, the exponents satisfy the conditions* $0 < q < 1$, $p > 1$, *and the diffusion term is equal to zero at the origin,* $\sigma(t, 0) = 0$.

The objective is to establish conditions under which the system (5.4) is globally fixed-time convergent at the origin in ρ*-mean,* $\rho > 1$.

It is worth noting that the deterministic disturbance $\xi(t)$ *in* (5.4) *may be unbounded.*

Table 5.1: Lyapunov functions with admissible parameters for super-twisting convergence.

Lyapunov Function	Value of parameters with disturbances
[29] without Lyapunov function,	$\alpha > L$, $\frac{2(\alpha+L)^2}{\lambda_1^2(\alpha-L)} < 1$
[94] $V_1 = 2\sqrt{0.5y^2 + \alpha\|x\|}$,	$\alpha > L$, $\lambda_1 h^{-1}(\lambda_1) > M$, $M = \alpha + L$, $m = \alpha - L$, $h(\lambda_1) = \frac{1}{\lambda_1} + (\frac{2e}{\lambda_1 m})^{2/3}$
[36] $V_2 = (\|x\|^{0.5}sign(x), y)P(\|x\|^{0.5}sign(x), y)^T$ $P = \frac{1}{2}\begin{pmatrix} 4\beta + \alpha^2 & -\alpha \\ \alpha & 2 \end{pmatrix} > 0$	$\alpha > 3L$, $2\alpha - L > (\lambda_1 + \frac{2}{\lambda_1})^2$
[34] $V_3 = \begin{cases} \frac{k^2}{4}(\frac{y}{\gamma}sign(x) + k_0 e^{m(x,y)\sqrt{s(x,y)}})^2, & xy \neq 0, \\ 2\lambda_1^{-2}\hat{k}^2y^2, & x = 0, \\ \frac{\|x\|}{2} & y = 0 \end{cases}$ $s(x, y) = y^2 - \lambda_1\|x\|^{0.5}ysign(x) + 2\gamma\|x\|$, $m(x, y) = \frac{\arctan(\frac{\lambda_1 g\|x\|^{0.5}sign(x) + 2\gamma\|x\| - 2y}{y\sqrt{g-1}})}{\sqrt{g-1}}$	$k_0 > 0$, $\hat{k}$ large enough, $g > 1$, $\gamma \geq \alpha + L$, $\alpha > 5L$, $\lambda_1^2 \in (32L, 8(\alpha - L))$
[25] $V_4 = 2\alpha\|\|x\|\| + \frac{1}{2}(\|\|y\|\|^2 + \|\|\lambda_1\frac{x}{\|\|x\|\|^{1/2}} - y\|\|^2)$	$\alpha > 4L$ or $\alpha > L$, $\lambda_1 > \sqrt{2\alpha}$, $\lambda_2 > 0$

5.4 Fixed-time convergence of stochastic super-twisting system

5.4.1 Fixed-time convergence without deterministic disturbances

Consider a scalar stochastic system

$$dx(t) = u(t)dt + \sigma(t, x(t))dW(t), \quad x(t_0) = x_0, \tag{5.5}$$

satisfying the conditions imposed on (5.4). The following theorem establishes sufficient conditions for fixed-time convergence in ρ*-mean of the stochastic system (5.5), if the control input* $u(t)$ *takes the form*

$$u(t) = -\lambda_1 \lceil x \rfloor^q - \lambda_2 \lceil x \rfloor^p, \tag{5.6}$$

where $\lceil x \rfloor^q := |x|^q sgn(x)$, $\lambda_1 > 0$, $\lambda_2 > 0$, $0 < q < 1$, $p > 1$.

The closed-loop system is given by

$$\begin{aligned} dx(t) = [-\lambda_1 \lfloor x(t) \rceil^q - \lambda_2 \lfloor x(t) \rceil^p] dt + \sigma(t, x(t)) dW(t), \\ x(t_0) = x_0. \end{aligned} \tag{5.7}$$

Theorem 5.4.1. *The system (5.7) is fixed-time convergent at the origin in* ρ*-mean,* $\rho > 1$, *if* $2\lambda_1 > \rho - 1 > 0$, $2\lambda_2 > \rho - 1 > 0$, *and*

$$\sigma(t, x(t)) = |x|^r, \text{ where } (1+q) \le 2r \le (1+p). \tag{5.8}$$

Proof of theorem. *Consider a Lyapunov function* $V(x) = E[|x|^\rho]$, $\rho > 1$. *Therefore,* $\frac{\partial V}{\partial x} = E[\rho |x|^{\rho-1} sign(x)]$, $\frac{\partial^2 V}{\partial x^2} = E[\rho(\rho-1)|x|^{\rho-2}]$.

Applying Ito formula yields

$$\begin{aligned} dV &= E[\rho|x|^{\rho-1} sign(x)[-\lambda_1 |x|^q sign(x) - \lambda_2 |x|^p sign(x) \\ &\quad + \frac{1}{2}\sigma^2(t, x(t))\rho(\rho-1)|x|^{\rho-2}]]dt \\ &\quad + E[\rho|x|^{\rho-1} sign(x)\sigma(t, x(t))]dW \\ &= E[-\rho\lambda_1 |x|^{\rho-1+q} - \rho\lambda_2 |x|^{\rho-1+p} \\ &\quad + \frac{1}{2}\sigma^2(t, x(t))\rho(\rho-1)|x|^{\rho-2}]dt \\ &\quad + E[\rho|x|^{\rho-1} sign(x)\sigma(t, x(t))dW]. \end{aligned}$$

Denoting $V_t = V(x(t))$ *and using the integral form of (5.1), one obtains*

$$\begin{aligned} V_{t+\Delta t} - V_t &= \int_t^{t+\Delta t} E[\phi(\tau)] d\tau \\ &\quad + E[\int_t^{t+\Delta t} \frac{\partial V}{\partial x} \sigma(\tau, x(\tau)) dW(\tau)], \end{aligned}$$

where

$$\begin{aligned} \phi(t) = -\rho\lambda_1 |x|^{\rho-1+q} - p\lambda_2 |x|^{\rho-1+p} \\ + \frac{1}{2}\sigma(t, x(t))^2 \rho(\rho-1)|x|^{\rho-2}]. \end{aligned}$$

Dividing by Δt, *tending* $\Delta t \to 0$, *and assuming that*

$$\int_t^{t+\Delta t} E[\sigma^2(t, x(t))(\frac{\partial V}{\partial x})^2] d\tau < \infty, \tag{5.9}$$

one obtains

$$\frac{dV(x)}{dt} = E[-\rho\lambda_1|x|^{\rho-1+q} - \rho\lambda_2|x|^{\rho-1+p} + \frac{1}{2}\sigma^2(t,x(t))\rho(\rho-1)|x|^{\rho-2}], \quad (5.10)$$

taking into account that

$$E[\int_t^{t+\Delta t} \frac{\partial V}{\partial x}\sigma(\tau,x(\tau))dW(\tau)] = 0.$$

Note that the integral given in (5.9) is finite for all $\sigma(t,x(t))$ satisfying (5.8).

Expressing the right-hand side of (5.10) in terms of $|x|$ yields

$$\frac{dV(x)}{dt} \leq E[-\rho\lambda_1|x|^{\rho-1+q} - \rho\lambda_2|x|^{\rho-1+p} + \frac{1}{2}\rho(\rho-1)|x|^{\rho-2+2r}],$$

where $0 < \frac{\rho-1+q}{\rho} = 1 - \frac{1-q}{\rho} < 1$ *and* $\frac{\rho-1+p}{\rho} = 1 + \frac{\rho-1}{\rho} > 1$.

Consider two cases:

a. $|x| \leq 1$. *Using the condition* $(1+q) \leq 2r$ *and* $2\lambda_1 > \rho - 1 > 0$ *implies that* $\rho\lambda_1|x|^{\rho-1+q} > \frac{1}{2}\rho(\rho-1)|x|^{\rho-2+2r}$ *and, therefore,* $\frac{1}{2}\rho(\rho-1)|x|^{\rho-2+2r} - \rho\lambda_1|x|^{\rho-1+q} \leq (\frac{1}{2}(\rho-1)-\lambda_1)\rho|x|^{\rho-1+q} < 0$.

b. $|x| \geq 1$. *Using the condition* $(1+p) \geq 2r$ *and* $2\lambda_2 > \rho - 1 > 0$ *implies that* $\rho\lambda_2|x|^{\rho-1+p} > \frac{1}{2}\rho(\rho-1)|x|^{\rho-2+2r}$ *and, therefore,* $\frac{1}{2}\rho(\rho-1)|x|^{\rho-2+2r} - \rho\lambda_2|x|^{\rho-1+p} \leq (\frac{1}{2}(\rho-1)-\lambda_2)\rho|x|^{\rho-1+p} < 0$.

Hence, the full time derivative of the Lyapunov function satisfies the inequality

$$\frac{dV(x)}{dt} \leq -aV(x)^b - cV(x)^d,$$

where $a = \rho(\lambda_1 - \frac{1}{2}(\rho-1))) > 0$, $c = \rho(\lambda_2 - \frac{1}{2}(\rho-1)) > 0$, $b = \frac{\rho-1+q}{\rho} \in (0,1)$, $d = \frac{\rho-1+p}{\rho} > 1$. *Thus, in view of Lemma 1 in [23], the system (5.7) is globally fixed-time convergent at the origin in ρ-mean.* ∎

5.4.2 Fixed-time convergence with deterministic disturbances

Consider a stochastic super-twisting system (5.4), where the control law (5.6) can be represented as

$$u(t) = -\lambda_1\lceil x\rfloor^q - \lambda_2\lceil x\rfloor^p - \alpha\int_{t_0}^t sign(x(\tau)d\tau. \quad (5.11)$$

Theorem 5.4.2. *Let the conditions of Theorem above hold. Then, the system (5.4) is globally fixed-time convergent at the origin in ρ-mean, $\rho > 1$, provided that any of the conditions for gains $\lambda_1, \lambda_2, \alpha$ obtained in [29, 94, 36, 34, 25] (see Table 5.1) are additionally assumed.*

Proof of theorem. *Consider two cases:*

a. *If $\sigma(t,X(t)) = 0$, the system (5.4) reduces to a deterministic super-twisting system*

$$\begin{aligned} dx(t) &= [-\lambda_1\lfloor x(t)\rceil^q - \lambda_2\lfloor x(t)\rceil^p + y(t)]dt, \quad x(t_0) = x_0, \\ dy(t) &= [-\alpha sign(x(t)) + \dot{\xi}(t)]dt, \quad y(t_0) = 0. \end{aligned} \quad (5.12)$$

The system (5.12) is globally fixed-time convergent at the origin, provided that any of the conditions for gains $\lambda_1, \lambda_2, \alpha$ *obtained in [29, 94, 36, 34, 25] (see Table 5.1) hold.*

b. *If* $\sigma(t, X(t)) \neq 0$ *and a deterministic disturbance is absent,* $\xi(t) = 0$*, the system (5.4) is fixed-time convergent at the origin in* ρ*-mean for* $\alpha = 0$ *in view of Theorem 1. If a deterministic disturbance is present,* $\xi(t) \neq 0$*, it is compensated assuming additionally any of the conditions for gains* $\lambda_1, \lambda_2, \alpha$ *obtained in [29, 94, 36, 34, 25] (see Table 5.1). The corresponding proof follows the lines drawn in the corresponding papers, for instance, applying geometrical considerations related to a "majorant curve" in [29]. Following [29], in view of* $\alpha > L$*, the addition to the velocity vector of system (5.4) given by the term* $-\alpha sign(x(t)) + \dot{\xi}$ *would always be directed (see Chapter 4 in [3] for similar arguments) inside the curve defined by the right-hand side of the system* (5.7)*, which converges in fixed time due to Theorem 1. Therefore, the system (5.4) still converges to the origin in fixed time.* ■

Note that if $\sigma(t, 0) \neq 0$ *in* (5.4)*, the fixed-time convergence at the origin would be lost.*

5.5 Numerical simulations

Two set of simulations are conducted: simulations of the stochastic system (5.7) *including a stochastic noise only and simulations of the stochastic system* (5.4) *including both a stochastic noise and a deterministic disturbance.*

5.5.1 Simulations without deterministic disturbance

The system (5.7) is simulated with initial condition $x(0) = 10$*, time step* $\tau = 10^{-3}$*, time horizon* $T = 1$ *and control gains* $\lambda_1 = 23$*,* $\lambda_2 = 100$*,* $q = 0.5$*,* $p = 1.5$*; therefore, according to Theorem 1,* $\sigma(t, x(t)) = |x|^r$*, where* $1.5 \leq 2r \leq 2.5$*.*

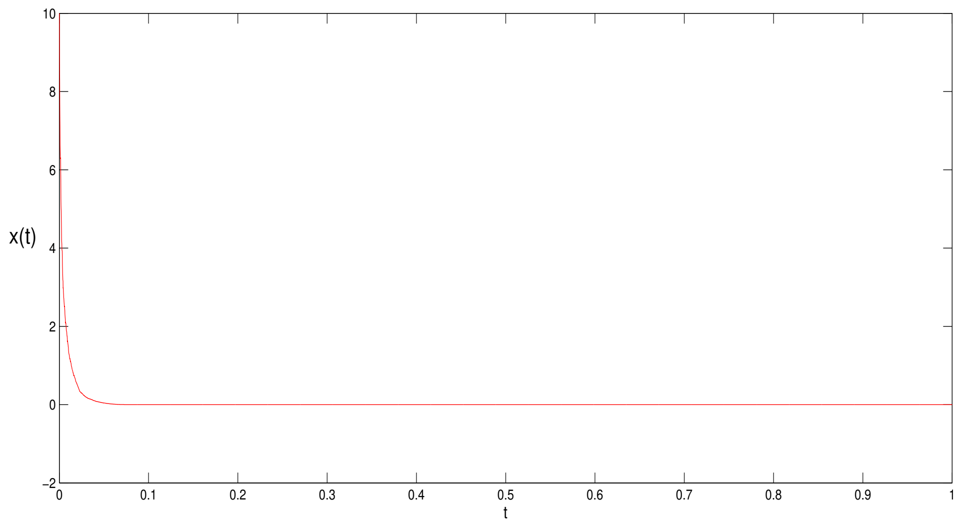

Figure 5.1: Time history of $x(t)$ (red) for $\sigma^2 = |x|^{2.5}$ in simulation interval $[0, 1]$.

For $\sigma^2 = |x|^{2.5}$*, the accuracy of* $1.3228e-07$ *is achieved at the end of the simulation interval. Figures 5.1 and 5.2 show the corresponding time history of* $x(t)$*.*

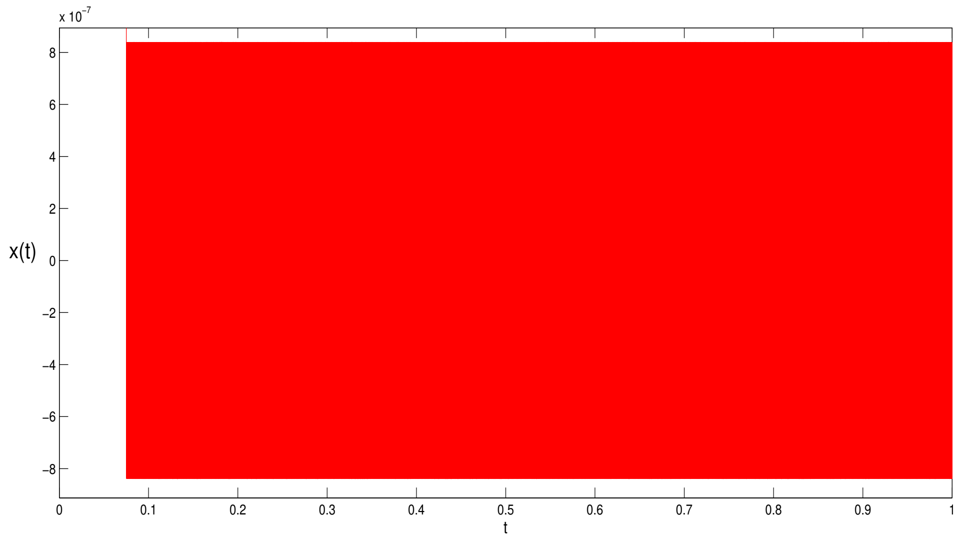

Figure 5.2: Time history (zoom) of $x(t)$ (red) for $\sigma^2 = |x|^{2.5}$ in simulation interval $[0, 1]$.

For $\sigma^2 = |x|^{1.5}$, the accuracy of $1.5462e-06$ at the end of the simulation interval. Figures 5.3 and 5.4 show the corresponding time history of $x(t)$.

For $\sigma^2 = |x|^{5.4}$ and $\sigma^2 = |x|^{0.3}$, the corresponding trajectories of $x(t)$ diverge at the end of the simulation interval, as shown in Fig 5.5, or oscillate around zero with magnitudes of order 10^{-1}, as shown in Figures 5.6 and 5.7.

As observed, the achieved accuracy becomes better at the end of the simulation interval as σ grows.

5.5.2 Simulations with deterministic disturbance

Consider a deterministic disturbance $\xi(t) = \sin(10t) + 10t$; therefore, $L = 20$ is a Lipschitz constant for $\xi(t)$, and the control gain α is set to $\alpha = 100$. Other parameters are the same as in the preceding set of simulations.

According to [25], the continuous fixed-time convergent control law (5.11) drives the states $[x(t), y(t)]$ of the system (5.4) at the origin uniformly for a fixed time no greater than T_{F_s} given by

$$\begin{aligned} TF_s &\leq \left(\frac{1}{\lambda_2 (p-1)\epsilon^{p-1}} + \frac{2\epsilon^{1/2}}{\lambda_1} \right) \\ &\times \left(1 + \frac{1}{m\left(\frac{1}{M} - \frac{h(\lambda_1)}{\lambda_1}\right)} \right), \end{aligned}$$

Here, $\epsilon > 0$, $M = \alpha + L = 120$, $m = \alpha - L = 80$, $h(\lambda_1) = 1/\lambda_1 + (2e/m\lambda_1)^{1/3} = 1/23 + (2e/80 \times 23)^{1/3} \approx 0.187$, and e is the base of natural logarithms, provided that the following conditions hold for control gains: $\alpha > L$ $(100 > 20)$, $\lambda_1 h^{-1}(\lambda_1) > M$, $(23(5.35) > 120)$. The minimum value of $T_f(\epsilon)$ is reached for $\epsilon = (\lambda_1/\lambda_2)^{\frac{1}{p+1/2}} = 0.23^{1/2} = 0.48$.

The convergence time obtained using formula above is $TF_s \leq 3.78$ seconds.

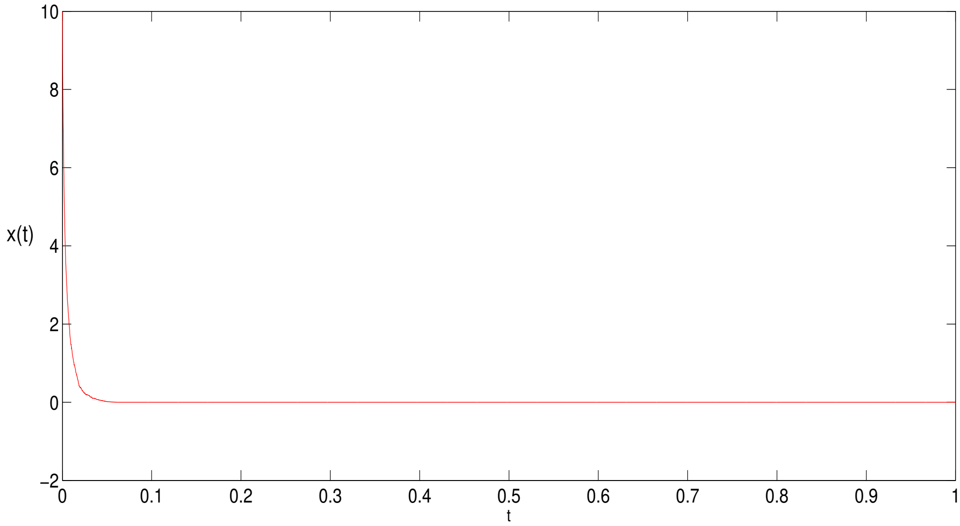

Figure 5.3: Time history of $x(t)$ (red) for $\sigma^2 = |x|^{1.5}$ in simulation interval $[0, 1]$.

Initial conditions x_0	Convergence time by simulation with $\sigma^2 = 0$ (sec.)	Convergence time by simulation with $\sigma^2 = |x|^{2.5}$ (sec.)	Convergence time estimate
1	0.10	0.10	3.78
10	0.10	0.10	3.78
-10	0.12	0.12	3.78
100	0.10	0.10	3.78
-100	0.12	0.12	3.78
1000	0.10	0.10	3.78
-1000	0.12	0.12	3.78
10000	0.10	0.10	3.78
-10000	0.10	0.10	3.78

Table 5.2: Convergence times for different initial conditions and diffusion terms $\sigma^2 = 0$ and $\sigma^2 = |x|^{2.5}$.

Table 5.2 shows convergence times for different initial conditions and diffusion terms $\sigma^2 = 0$ *and* $\sigma^2 = |x|^{2.5}$*. It can be concluded that the convergence time is not affected by diffusion terms.*

For $\sigma^2 = |x|^{2.5}$*, the accuracy of* $5.9371e - 07$ *is achieved at the end of the simulation interval. Figures 5.8 and 5.9 show the corresponding time history of* $[x(t), y(t)]$*. For* $\sigma^2 = |x|^{1.5}$*, the achieved accuracy is* $9.115e{-}07$ *at the end of the simulation interval. Figures 5.10 and 5.11 show the corresponding time history of* $[x(t), y(t)]$*. As observed, the achieved accuracy is actually not affected by diffusion terms as well, if the* σ^2 *remains within the established convergence limits.*

5.6 Conclusions and future work

A continuous fixed-time convergent in ρ*-mean control law is designed to drive the states of a stochastic super-twisting system at the origin for a fixed time. The stochastic super-twisting system includes both*

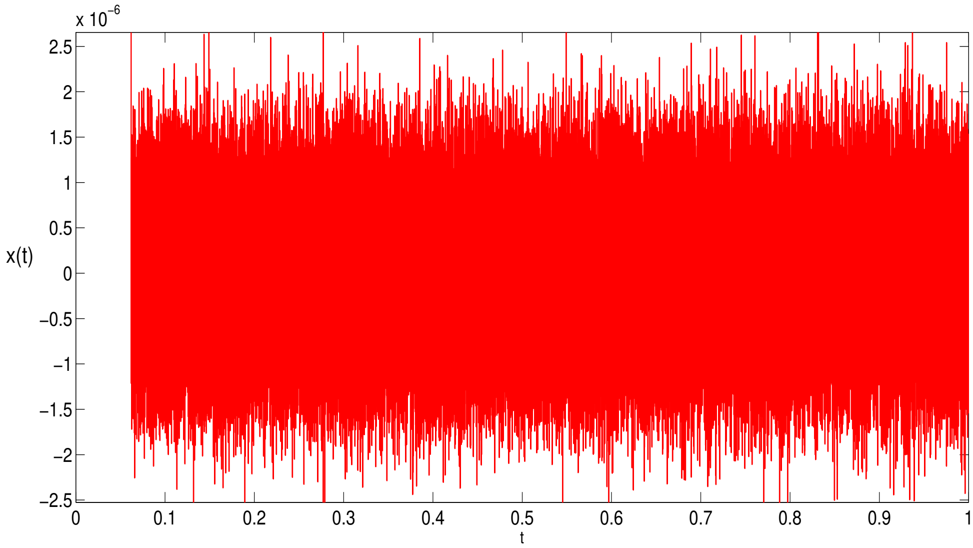

Figure 5.4: Time history (zoom) of $x(t)$ (red) for $\sigma^2 = |x|^{1.5}$ in simulation interval $[0, 1]$.

a stochastic white noise and an unbounded deterministic disturbance satisfying a Lipschitz condition. Sufficient conditions of the fixed-time convergence are obtained separately in cases when a deterministic disturbance is absent or present. Performance of the developed algorithm is verified with numerical simulations, which demonstrate that the fixed convergence time does not change if the super-twisting system is additionally affected by a stochastic noise. Our ongoing research is dedicated to:

1. *Designing a fixed-time convergent in ρ-mean scalar control law driving all states of a n-dimensional stochastic linear system at the origin.*

2. *The incidence of white noise in robustness of stochastic systems in presence of matched deterministic disturbances will be conducted.*

3. *Proposing Continuous Fixed-Time Convergent observers based on Stochastic Super-Twisting System. Designed Observers will take into account unbounded variation of a white noise.*

4. *Replace continuous fixed time convergent injections by predefined time convergence injections like in [154].*

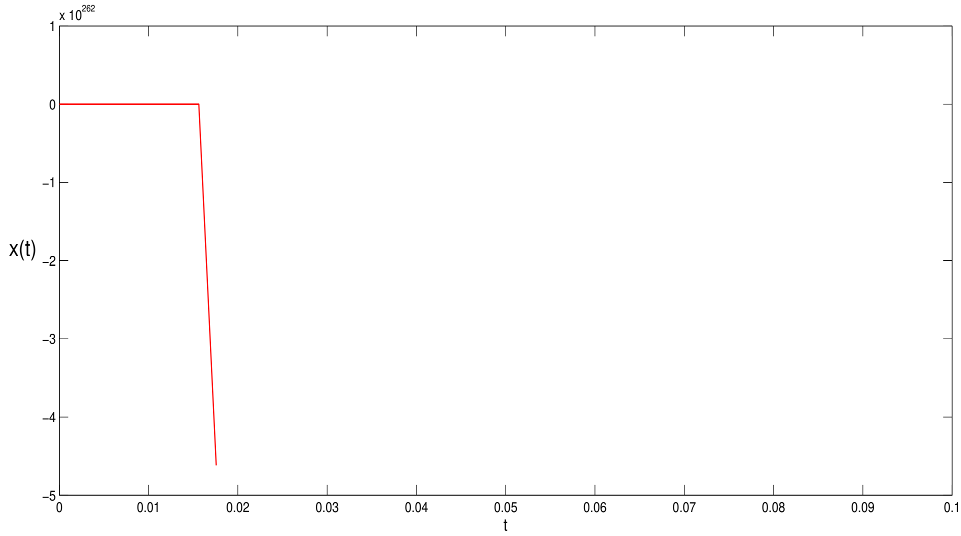

Figure 5.5: Time history of $x(t)$ (red) for $\sigma^2 = |x|^{5.4}$ in simulation interval $[0, 1]$.

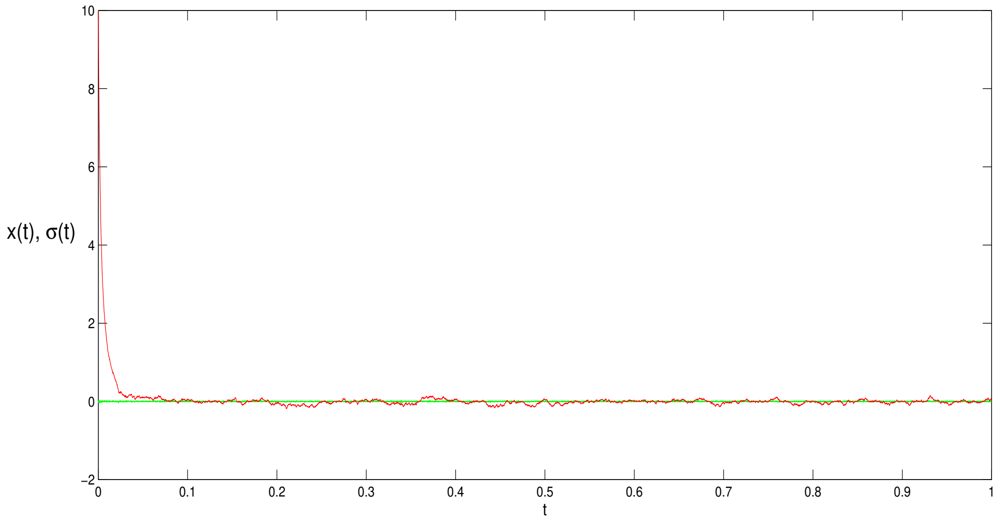

Figure 5.6: Time history of $x(t)$ (red) for stochastic disturbance $\sigma^2 = |x|^{0.3}$ (green) in simulation interval $[0, 1]$.

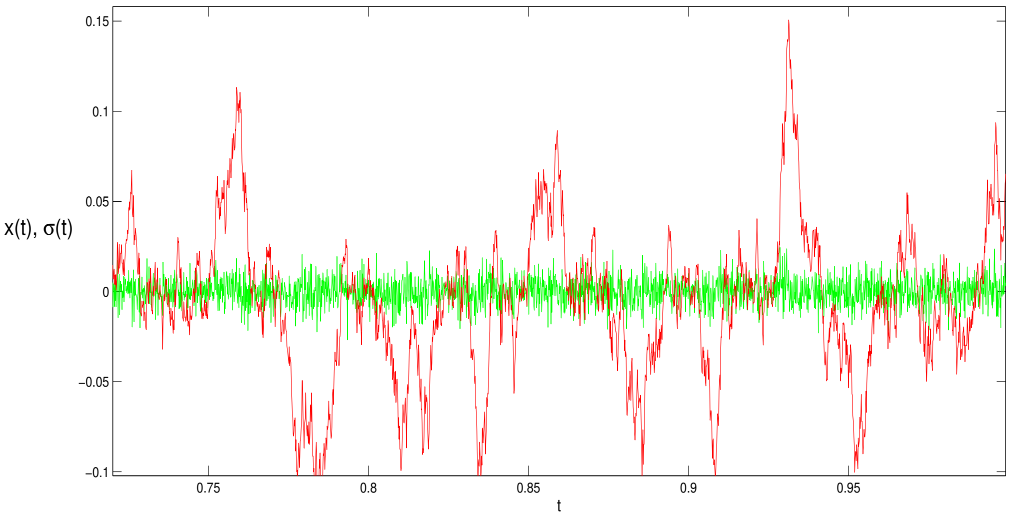

Figure 5.7: Time history (zoom) of $x(t)$ (red) for stochastic disturbance $\sigma^2 = |x|^{0.3}$ (green) in simulation interval $[0, 1]$.

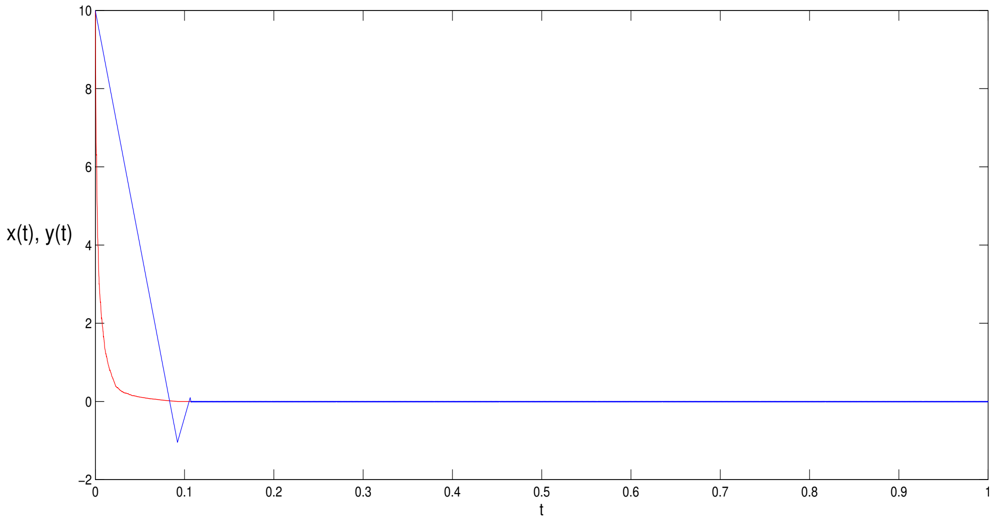

Figure 5.8: Time history of $x(t)$ (red) and $y(t)$ (blue) for $\sigma^2 = |x|^{2.5}$ in simulation interval $[0, 1]$.

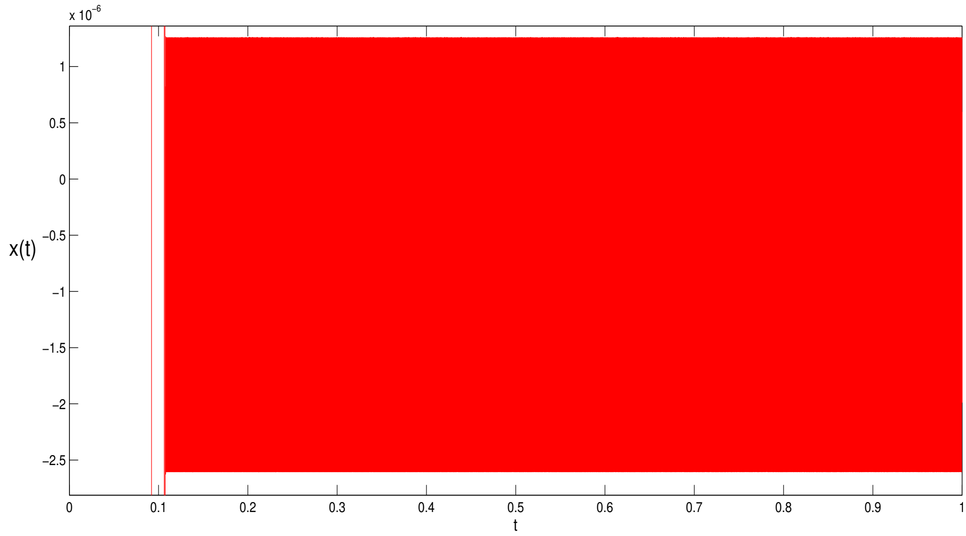

Figure 5.9: Time history (zoom) of $x(t)$ (red) for $\sigma^2 = |x|^{2.5}$ in the simulation interval $[0, 1]$.

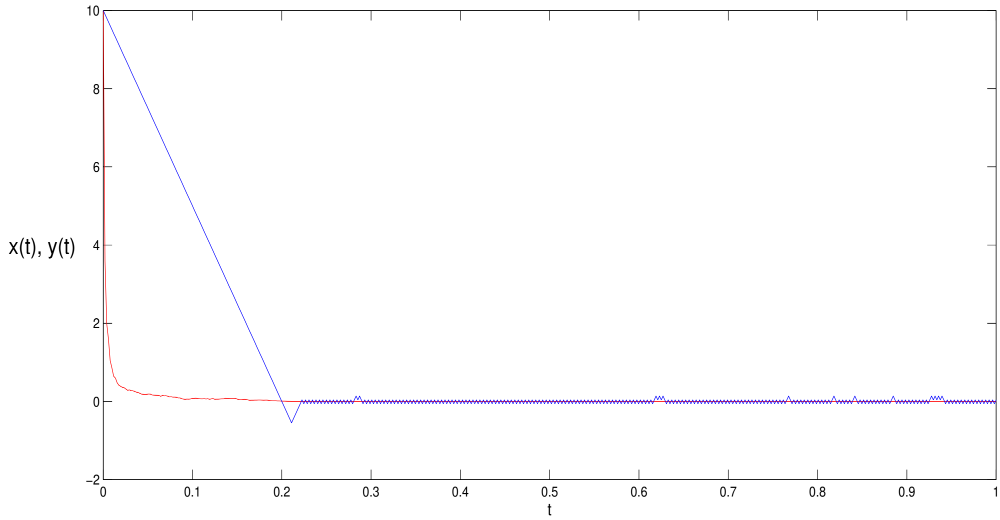

Figure 5.10: Time history of $x(t)$ (red) and $y(t)$ (blue) for $\sigma^2 = |x|^{1.5}$ in simulation interval $[0, 1]$.

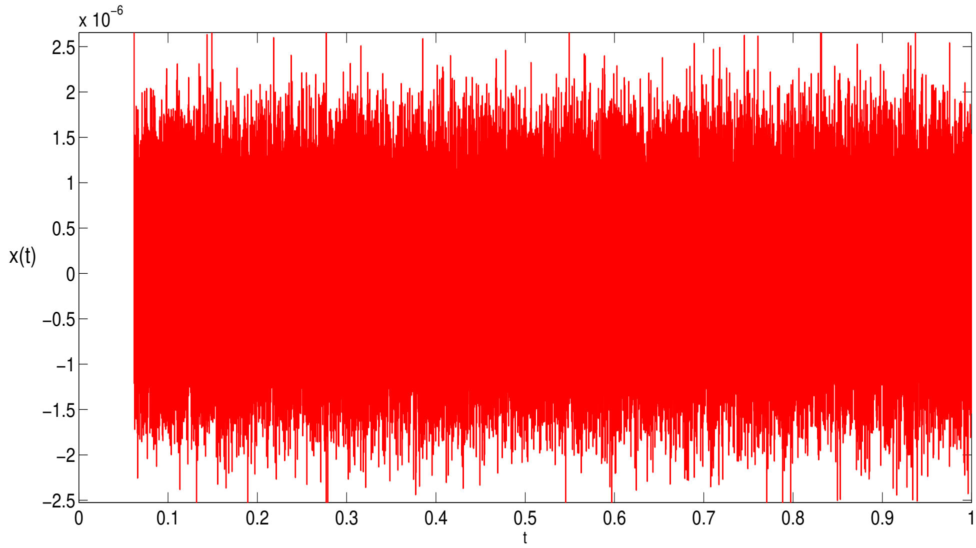

Figure 5.11: Time history (zoom) of $x(t)$ (red) for $\sigma^2 = |x|^{1.5}$ in simulation interval $[0, 1]$.

Chapter 6

Conclusions and Future Work

6.0.1 Conclusions

1. *A fixed-time convergent observer-based controller is designed to drive all states of an n-dimensional chain of integrators at the origin/a vicinity of the origin for a finite pre-established (fixed) time, using a scalar input in the equation for the lowest relative degree state, against unbounded disturbances. The controller design does not assume knowledge of all system states: only the output should be measured. The presented algorithm uses continuous injections in their control law, and consists of a smooth fixed-time convergent observer (differentiator), a fixed-time convergent regulator, and a disturbance compensation mechanism.*

2. *. The uniform upper bound (independent of the initial conditions of the system) for the convergence time of the continuous fixed-time convergent observer-based controller is explicitly calculated.*

3. *Performance of the designed controller is demonstrated through numerical implementation in a case study of DC motor, validating the obtained theoretical results. The accuracy of the controller is examined and found consistent with the results obtained in [31].*

4. *An adaptive fixed-time convergent continuous sliding mode controller is proposed for a class of dynamic systems affected by disturbances satisfying a Lipschitz condition with an unknown Lipschitz constant and an unknown initial value, whose only measurable state is the highest relative degree one.*

5. *The adaptive controller design is validated in a Stock Management Problem, with the objective to drive the stock and supply line levels at the reference values, subject to loss rate disturbances whose bounds are unknown. The adaptive controller component is based on APVIOBPCS policy combined with the supertwisting algorithm and employed to compensate for the disturbances. Two cases are studied:first, both, the stock level and supply line, are assumed measurable and available for control design and, second, only the stock level is measurable. In the second case, unbounded disturbances with both, small and large, magnitudes of the high-frequency component are considered.*

6. *The obtained simulation results show that the designed controller solves the uniform fixed-time stabilization problem for the stock level and supply line, driving all state variables to given reference points or its vicinities in uniform fixed time, and demonstrates robustness with respect to an abrupt increase in magnitude of the high-frequency disturbance component.*

7. *A continuous fixed-time convergent in ρ-mean control law driving the states of a stochastic super-twisting system at the origin for a fixed time. The stochastic super-twisting system includes both a stochastic white noise and an unbounded deterministic disturbance satisfying a Lipschitz condition.*

Sufficient conditions of the fixed-time convergence are obtained separately in cases when a deterministic disturbance is absent or present. Performance of the developed algorithm is verified with numerical simulations, which demonstrate that the fixed convergence time does not change if the super-twisting system is additionally affected by a stochastic noise.

6.0.2 Future Work

1. *Bounds of Settling convergence time estimates can be optimized.*
2. *Replace continuous fixed time convergent injections by predefined time convergence injections like a [154].*
3. *Compensator designed in Chapters 3 and 4 can be replaced by the disturbance observer proposed in [93] or the one based on Utkin equivalent control law [46], including fixed-time convergence, respectively.*
4. *Study the incidence of white noise in robustness of stochastic systems in presence of matched deterministic disturbances.*
5. *Propose Continuous Fixed-Time Convergent observers based on Stochastic Super-Twisting System. Designed Observers will take into account unbounded variation of a white noise.*
6. *Designing a fixed-time convergent in ρ-mean scalar control law driving all states of a n-dimensional stochastic linear system at the origin.*

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Appendix A

About surface design

Summary: Sliding surface design based on integral sliding mode control consider a continuous control law $v(t,x) = v_1(t,x) + v_2(t,x)$ that ensures fixed-time convergence. Control component $v_1(t,x)$ yields to fixed-time convergence in the absence of uncertainties. The second component $v_2(t,x)$ called the integral sliding mode control provides fixed-time convergence in the presence of uncertainties following a disturbance compensation mechanism.

A.1 Problem statement

Consider the uncertain non-linear system obtained from (2.31)

$$\sigma^r = h(t,x) + g(t,x)u(t,x) \tag{A.1}$$

where $h(t,x) = L^r_{\hat{a}}\sigma$ *and,* $g(t,x) = L_{\hat{b}}L^{r-1}_{\hat{a}}(\sigma(x))$.

It is assumed that the following relations hold (always true at least locally) $h(t,x) = h_1(t,x) + h_2(t,x)$*, such that* $h_1(t,x) \leq \delta_1|s(t)|^{1/2}$ *($s(t)$ will be defined below),* $|\dot{h}_2(t,x)| \leq \delta_2$*,* δ_1*,* $\delta_2 > 0$ *are unknown finite boundaries,* $g(t,x) = g_0(t,x) + \Delta g(t,x)$*,* $g_0(t,x) > 0$ *is a known function,* $\Delta g(t,x)$ *is a bounded perturbation such that* $\frac{\Delta g(t,x)}{g_0(t,x)} \leq \gamma(t,x) \leq \gamma_1 \leq 1$*,* $1-\gamma_1 \leq (1+\frac{\Delta g(t,x)}{g_0(t,x)}) \leq 1+\gamma_1$ *[43].*

Replacing $u(t,x) = \frac{v(t,x)}{g_0(t,x)}$*, and* $g_1(t,x) = (1+\frac{\Delta g(t,x)}{g_0(t,x)})$ *yields*

$$\begin{aligned}\sigma^{(n)}(t) &= h(t,x) + (1+\frac{\Delta g(t,x)}{g_0(t,x)})v(t,x)\\ \sigma^{(n)}(t) &= h(t,x) + g_1(t,x)v(t,x)\end{aligned} \tag{A.2}$$

where $1-\gamma_1 \leq g_1(t,x) \leq 1+\gamma_1$.

Also assume that the multiplicative disturbance is absent i.e., $\Delta g(t,x) = 0$*, therefore*

$$\dot{x}_n(t) = \sigma^{(n)}(t) = h(t,x) + v(t,x) \tag{A.3}$$

Then the r-sliding mode control of (A.3) *respect to sliding variable* σ *is equivalent at least locally to finite/fixed time stabilization of n-dimensional chain of integrators ($r = n$)*

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t), & x_1(t_0) = x_{10},\\
\dot{x}_2(t) &= x_3(t), & x_2(t_0) = x_{20},\\
\vdots \;&\vdots\; \vdots & \vdots\\
\dot{x}_n(t) &= h(t,x) + v(t,x), & x_n(t_0) = x_{n0},\\
y(t) &= x_1(t),
\end{aligned}
\tag{A.4}
$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T = [\sigma, \dot{\sigma}, \dots \sigma^{(n-1)}] \in \mathbb{R}^n$ *is the system state,* $y(t) \in \mathbb{R}$ *is the state measurement (observation),* $v(t,x) = g_0(t,x)u(t,x)$, $u(t,x) \in \mathbb{R}$ *is the control input.*

The control objective is to design a continuous control law $v(t,x)$ *that drives all state variables of the system (A.4) at the origin (or a vicinity of the origin) for a finite pre-established (fixed) time. Since only the scalar output* $y(t) = x_1(t)$ *is measured, a fixed-time convergent differentiator should first be employed to estimate values of all state components for a finite pre-established (fixed) time. Then, based on the obtained estimates, a continuous fixed-time convergent controller with a compensation term should be designed to drive all the states at the origin (a vicinity of the origin) for a finite pre-established (fixed) time using a scalar control input.*

Remark A.1.1. *Supposing that the relative degree of system* (2.31) *with respect to sliding variable* σ *is* r*, the problem of higher order sliding mode control for* $n > r$ *is an extension of the current work, through the extension of system* (2.31) *by* $n - r$ *-length integrators chain. All the results displayed below can then be applied to the extended system. For the purpose of clarity, the current work is only focused to* $r = n$ *case.*

A general procedure for design of controller is given hereafter [155].

A.1.1 General procedure for the controller design

Consider the system given in equation (A.4) *The aim can be rewritten as*

$$\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = \sigma^{(r)} = 0, \tag{A.5}$$

in fixed time.

The aim is to design a control law $v(t,x)$ *that produces fixed-time convergence of the states* $\sigma, \dot{\sigma}, \dots \sigma^{(r-1)}$ *of the uncertain system* (A.3) *under conditions given in above section. Note that exact knowledge of the parameters bounds is not necessary in practice, because they only influence the magnitude of the control to be designed.*

The strategy consists in design a continuous control law $v(t,x) = v_1(t,x) + v_2(t,x)$ *that ensures fixed-time convergence. the control objective consists of two components, the first one* $v_1(t,x)$ *yields to fixed-time convergence in the absence of uncertainties. Moreover, this control component generate signals that the system has to track. The second component* $v_2(t,x)$ *called the integral sliding mode control provides fixed-time convergence in the presence of uncertainties following a compensation mechanism.*

1. ***Design of controller component*** $v_1(t,x)$. *Consider the case* $h(t,x) = 0$*, then* $v(t,x) = v_1(t,x)$ *because* $v_2(t,x) = 0$*. The control objective is to drive the state of system* (A.6) *to* $x = 0$ *at the fixed*

time $t = t_F$, *with* $x(0)$ *being a bounded initial state vector.*

$$\begin{array}{rcll} \dot{x}_1(t) & = & x_2(t), & x_1(t_0) = x_{10}, \\ \dot{x}_2(t) & = & x_3(t), & x_2(t_0) = x_{20}, \\ \vdots & \vdots & \vdots & \vdots \\ \dot{x}_n(t) & = & v_1(t,x), & x_n(t_0) = x_{n0}, \\ y(t) & = & x_1(t), & \end{array} \tag{A.6}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T = [\sigma, \dot{\sigma}, \ldots \sigma^{(n-1)}] \in \mathbb{R}^n$ *is the system state,* $y(t) \in \mathbb{R}$ *is the state measurement (observation),* $v_1(t,x) \in \mathbb{R}$ *is the control input. This objective can be done through transforming the dynamics of system given in* (A.6)*, to the dynamics of the fixed-time convergent system*

$$\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t), \\ \dot{x}_2(t) & = & x_3(t), \\ & \vdots & \\ \dot{x}_n(t) & = & -\sum_{i=1}^{n}[k_i\lfloor s_i(t)\rceil^{\gamma_i} + \hat{k}_i\lfloor s_i(t)\rceil^{\delta_i}] \end{array} \tag{A.7}$$

where $x_i(t_0) = x_{i0}$ *and* $\dot{x}_i(t) = \sigma^{(i)}(x)$ *for* $i = 1, 2, \cdots, n$*. Therefore setting* $v_1(t,x) = -\sum_{i=1}^{n}[k_i\lfloor x_i(t)\rceil^{\gamma_i} + \hat{k}_i\lfloor x_i(t)\rceil^{\delta_i}]$*. Fixed-time convergence is guarantee with appropriate selection of gains* k_i*, and* $\hat{k}_i$ *(see [95]).*

2. ***The integral sliding manifold design*** $(h(t,x) \neq 0)$ *The idea consists in determining a sliding manifold such that the state trajectories of system* (A.4) *start on this manifold at the initial time* $t = 0$*, which induces a sliding mode without reaching phase [91], transforming the dynamics of system given in* (A.4)*, to the dynamics of the certain fixed-time convergent system.*

   Design of control law v_2*: Let consider the sliding manifold defined as*

$$s(t,x) = s_1(x) + \vartheta(t),$$

where $s_1(x)$ *depends of the states, and term* $\vartheta(t)$ *is the integral term and will be determined below. Suppose that, using a control law* v_2*, a sliding mode is established on the sliding manifold defined by the set* $\Sigma = \{x \in X | s(t,x) = 0\}$ *from* $t = 0$*. In order to determine the dynamics on the sliding manifold, consider auxiliary variable* $r(t)$ *complying with* $\dot{r}(t) = h(t,x) + v_2(t,x) = 0$*, and* $r(t) = 0$ *when* $v_2(t,x) = stw(t,x)$ *is any of the fixed-time convergent super-twisting control laws given in chapters 3 and 4 (non adaptive or adaptive). The time derivative of* s *is*

$$\dot{s} = \frac{\partial s_1}{\partial x}(Ax + Bv_1) + \frac{\partial s_1}{\partial x_n}(h(t,x) + v_2(t,x)) + \dot{\vartheta}(t) \tag{A.8}$$

A sufficient condition ensuring the sliding condition $\dot{s} = 0$ *is* $stw(t,x) = -h(t,x)$ *and*

$$\dot{\vartheta}(t) = -\frac{\partial s_1}{\partial x}(Ax + Bv_1)$$

with $\vartheta(0) = -s_1(x(0))$*.*

where $\vartheta(0)$ is determined based on the requirement $s_1(0) = 0$, i.e. sliding mode occurs from the initial time. Then, the motion equation of the system in sliding mode is

$$\dot{x}(t) = Ax + Bv_1$$

which is the motion equation of the ideal system (A.7)

3. ***The sliding mode establishment***. *Viewing the structure of system* (A.4), *consider the particular case $s_1(x) = x_n(t) = \sigma^{(n-1)}(t)$, $\dot{s}_1(x) = \dot{x}_n(t) = \sigma^{(n)}(t)$, then $\dot{s}(t,x) = \dot{x}_n(t) + \dot{\vartheta}(t) = h(t,x) + v(t,x) + \dot{\vartheta}(t) = h(t,x) + v_1(t,x) + v_2(t,x) + \dot{\vartheta}(t)$. The condition $stw(t,x) = -h(t,x)$ complies if $\dot{\vartheta}(t) = -v_1(t,x)$ and $\vartheta(0) = -x_n(0)$. Therefore the sliding variable is*

$$s(t,x) = \sigma^{(n-1)}(t) + \vartheta(t),$$

$\vartheta(t) = -\int_{t_0}^{t} v_1(\tau,x)d\tau$, *where* $\dot{\vartheta}(t) = -v_1(t,x)$.

4. *Using the fact that the auxiliary variable r with dynamics $\dot{r}(t) = v_2(t,x) + h(t,x)$, satisfies $\dot{r}(t) = 0$, and $r(t) = 0$ when $v_2(t,x) = stw(t,x)$ is the fixed time convergent super-twisting (adaptive or non adaptive) control law given in chapters 3 and 4. $\dot{s}(t,x) = \sigma^{(n)}(t) + \dot{\vartheta}(t) = h(t,x) + v(t,x) + \dot{\vartheta}(t) = h(t,x) + v(t,x) - v_1(t,x) = v_1(t,x) + v_2(t,x) + h(t,x) - v_1(t,x) = h(t,x) + v_2(t,x)$, if $v_2(t,x) = stw(t,x)$ it induces a 2SM in fixed time in which $\dot{r} = r = 0$,and during the 2SM the control term $v_2(t,x)$ exactly compensates for the uncertainty: i.e. $h(t,x) = -v_2(t,x)$ and $\dot{s}(t,x) = \sigma^{(n)}(t) + \dot{\vartheta}(t) = 0$ i.e.,*

$$\sigma^{(n)}(t) = -\dot{\vartheta}(t)$$

5. *Finally, fixed time convergence is proved put $\sigma^{(n)}(t) = \dot{x}_n(t) = -\dot{\vartheta}(t) = v_1(t,x) = -\sum_{i=1}^{n}[k_i\lfloor x_i(t)\rceil^{\gamma_i} + \hat{k}_i\lfloor x_i(t)\rceil^{\delta_i}]$ and then*

$$\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = \sigma^{(r)} = 0, \tag{A.9}$$

is a fixed time stable equilibrium.

It is worth nothing that reaching condition $s\dot{s} \leq -\eta|s|$ is obtained by appropriate selection of gains.

Remark A.1.2. *The continuous controller $v(t,x)$ can be considered a continuous HOSM controller for the system in* (2.31), *since it drives σ, $\dot{\sigma}$, $\cdots$, $\sigma^{(r)} \to 0$ in fixed-time in the presence of the smooth disturbance $a(t,x)$ with bounded derivatives [156].*

Appendix B

Homogeneity

Summary: The importance of homogeneity is because many of properties known for linear systems can be hold in homogeneous systems. For instance, asymptotically stable homogeneous systems with respect to dilation* (B.1) *at a vicinity of the origin implies global asymptotic stability, and also, the existence of a homogeneous Lyapunov function with respect to dilation. Furthermore, if a vector field can be rewritten as a sum of vector fields each one homogeneous respect to the same dilation (B.1), then the asymptotic stability of the vector field with the lower homogeneity degree implies asymptotical stability of the original vector field.

B.1 Homogeneity

Homogeneity is a property of the certain objects to scaling in a consistent way with respect to a scaling operation (dilation) defined on a underlying space. the scaling operation is the action of the real numbers as multiplicative group on the state space. The familiar operation of multiplication by scalar in $\mathbb{R}^n$ yields to the standard dilation $\Delta_\lambda(x) = \lambda x$, where $\lambda > 0$ and $x \in \mathbb{R}^n$.

A incresasing interest in homogeneity with respect to dilations

$$\Delta : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \Delta_\lambda(x) = (\lambda^{r_1} x_1, \lambda^{r_2} x_2, ..., \lambda^{r_n} x_n) \tag{B.1}$$

$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ where r_i are positive real numbers. Note that standard dilation is the particular case which $r_i = 1$ for all i.

B.1.1 Homogeneous Function

Definition B.1.1. *A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree d respect to weights $(r_1, r_2, ..., r_n) \in \mathbb{R}^n_+$ if*

$$V(\Delta_\lambda x) = V(\lambda^{r_1} x_1, \lambda^{r_2} x_2, ..., \lambda^{r_n} x_n) = \lambda^d V(x_1, x_2, ..., x_n) = \lambda^d V(x) \tag{B.2}$$

$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$

where $\lambda > 0$. the standard dilation is obtained with $r_i = 1$ for all i.

Example 13. *Consider the function*

$$V(y) = -k sign(y(t))|y(t)|^\alpha$$

for $\alpha > 0$ the function $V(y)$ satisfies $V(\Delta_\lambda(x)) = V(\lambda x) = -ksgn(\lambda x)|\lambda x|^\alpha = -k\lambda^\alpha sgn(y(t))|y(t)|^\alpha$ and then, $V(y)$ is homogeneous with homogeneity degree α respect to standard dilation $\Delta_\lambda(x) = \lambda x$.

B.1.2 Homogeneous Vector Field

Definition B.1.2. *A vector field g is said to be homogeneous of degree m respect to the weights $(r_1, r_2, ..., r_n) \in \mathbb{R}^n_+$ if, for all $1 \le i \le n$ the i-component g_i is a homogeneous function of degree $r_i + m$, i.e.,*

$$g_i(\lambda^{r_1}x_1, \lambda^{r_2}x_2, ..., \lambda^{r_n}x_n) = \lambda^{r_i+m}V(x_1, x_2, ..., x_n) \tag{B.3}$$

$x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ for all $\lambda > 0$.

In this way the dynamical system is homogeneous of degree m if their vector field is homogeneous of degree m.

Example 14. *Consider the vector field*

$$f = f_1(x_1, x_2)\frac{\partial}{\partial x_1} + f_2(x_1, x_2)\frac{\partial}{\partial x_2}$$

on $\mathbb{R}^2$ con $f_1(x_1, x_2) = x_2$, $f_2(x_1, x_2) = -\frac{1}{m}sign(x_2)|x_2|^\alpha - \frac{1}{m}sign(x_1)|x_1|^{\frac{\alpha}{2-\alpha}}$, con $\alpha \in (0, 1)$, $m > 0$.

Noting that $f_1(\lambda^{2-\alpha}x_1, \lambda x_2) = \lambda x_2 = \lambda^{(2-\alpha)+\alpha-1}f_1(x_1, x_2)$, $f_2(\lambda^{2-\alpha}x_1, \lambda x_2) = \lambda^\alpha f_2(x_1, x_2) = \lambda^{1-(\alpha-1)}f_2(x_1, x_2)$. then, the vector field has negative degree $\alpha-1$ respect to dilation $\Delta_\lambda(x) = (\lambda^{2-\alpha}x_1, \lambda x_2)$.

Appendix C

Sterman's Decision Rule For SMP

Summary: Sterman's Decision Rule based on anchoring and adjustment heuristic for SMP is explained. Decision rule is able to represent the non-negative ordering decision process and is defined based on two conditions: sufficient to cover the expected inventory losses, and reduce the discrepancy between desired and actual inventory.

C.0.1 Preliminaries

A supply chain is the set of structures and process involving multiple echelons (links) that an organization uses to deliver a tangible or intangible output to a customer; each echelon maintains and controls stocks of materials and finished product, receives orders and adjusts the production to cope with changes in demand [108].

Decision-making process in SMP is dynamic, in the sense that it requires a series of dependent decisions in real time as a consequence of the decision maker's actions. Simon argued in 1952 that the behavior of decision makers should be viewed as bounded rational: they are individuals who are satisficed with a good option (instead of an optimal one) in managing dynamic systems [105].

The anchoring and adjustment heuristic is an example of a dynamic decision rule proposed by Tversky and Kahneman [157]; in the absence of available information and time, the agents use the anchoring and adjustment heuristic. In case of the SMP, it is suggested that a simple anchoring and adjustment heuristic is able to represent the ordering decision process whose expected loss rate is the anchor; a reference point for place an order, and deviation of desired stock from actual stock is the adjustment. This ordering policy must be non-negative and is defined based on two conditions: sufficient to cover the expected inventory losses, and reduce the discrepancy between desired and actual inventory. The decision parameter of the anchoring and adjustment heuristic is the stock-adjustment time; the average time to close the discrepancy between the actual and desired stock. The selection of this parameter value determines the success of the heuristic, and thus, managerial decisions.

A decision rule is based on anchoring and adjustment heuristic attempts to mimic the behavior of people in stock management taking into account limited cognitive resources and robustness under extreme conditions [108]. Consequently, it is desirable to incorporate finite-time availability as a cognitive resource [143, 144], generate nonlinear ordering dynamics behavior patterns based on nonnegative constraints (see [129]), ensuring robustness to disturbances and therefore stability of the SMP. All this supposing that the design of policies differs from the dynamics of linear asymptotic decision rules.

C.0.2 Sterman's Decision Rule based in Anchoring and Adjustment Heuristic

Consider the dynamic system given in Equation (4.21)*. This dynamic system is the simplest model that describes an individual link in a supply chain. The SMP for the system in Equation* (4.21) *consists in determining the dynamics of this individual link in the presence of the an inflow rate specifying a decision rule or policy (the inflow rate must be designed).*

For example, if the acquisition rate $O(X, t)$ *in Equation* (4.21) *takes the form*

$$O_s(X(t), t) = \max\{0, \frac{1}{lt}x_1 + \sum_{i=1}^{2} \frac{1}{sat_i}(x_{ir} - x_i)\}, \tag{C.1}$$

then $O(X, t)$ *is the decision rule based in anchoring and adjustment heuristic (denoted* O_s*). The dynamic system in Equation* (4.21) *is converted into*

$$\begin{aligned} \dot{x}_1(t) &= -\frac{x_1(t)}{lt} + \frac{x_2(t)}{al}, \quad x_1(t_0) = x_{10}, \\ \dot{x}_2(t) &= -\frac{x_2(t)}{al} + \max\{0, \frac{1}{lt}x_1 + \sum_{i=1}^{2} \frac{1}{sat_i}(x_{ir} - x_i)\}, \\ x_2(t_0) &= x_{20}, \end{aligned} \tag{C.2}$$

here, $x_1(t) \in \mathbb{R}$ *is the current stock of retailer,* $x_2(t) \in \mathbb{R}$ *is the stock level of supply line. For* $i = 1, 2$, $x_{ir}(t) \in \mathbb{R}$ *are the stock reference values for the retailer and the supply line, respectively.* sat_1 *is the stock adjustment time of the retailer and* sat_2 *is the stock adjustment time of the supply line (both constants), and* $\xi(t)$ *is an external disturbance satisfying the Lipschitz condition with a constant* L.

The acquisition rate indicates the rate at which managers wish to adds units to the stock. Two considerations are fundamental: managers should replace expected losses from the stock as well ass reduce the discrepancy between the desired and current stock by acquiring more than the expected losses when the stock is below the desired level, and lesser than expected losses when there is a surplus.

The formulation is interpreted as an example of the anchoring and adjustment (Tversky and Kahneman [157]). Here the anchor is the expected loss rate. Adjustments are then made to correct discrepancies between the desired and current stock. Sterman [119, 118] suggests an anchoring and adjustment heuristic as a representation of the managerial decision-making process for a SMP, a decision rule used by managers to control the acquisition of new units. The anchoring and adjustment heuristic incorporated in $O(X, t)$ *assumes that, when controlling stocks, the subjects anchor their decisions in the expected losses* $\frac{1}{lt}x_1$ *and adjust for the stock and supply line discrepancies*

$$\sum_{i=1}^{2} \frac{1}{sat_i}(x_{1r} - x_1)$$

in asymptotic way. For the SMP, managers seek to maintain a stock — the state of the system $X(t)$ *— of a particular target level* x_{1r}, x_{2r}*, within an acceptable range. Stocks are altered only by changes in their acquisition rate* $u(x_1, t)$ *and loss (outflow) rate* $\frac{1}{lt}x_1$*. The* max *function ensures that order rate cannot be negative.*

Particularly, in the SMP, disturbances affect the loss rate, preventing from reaching the desired stock and supply line values.