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Sliding Mode Regulator as Solution to Optimal Control Problem for Nonlinear Polynomial Systems

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Abstract— **This paper presents the optimal control problem for a nonlinear polynomial system with respect to a Bolza-Meyer criterion with a non-quadratic non-integral term. The optimal solution is obtained as a sliding mode control, whereas the conventional polynomial-quadratic regulator does not lead to a causal solution and, therefore, fails. Performance of the obtained optimal controller is verified in the illustrative example against the conventional polynomial-quadratic regulator that is optimal for the quadratic Bolza-Meyer criterion. The simulation results confirm an advantage in favor of the designed sliding mode control.**

I. Introduction

Since the sliding mode control was invented in the beginning of 1970s (see a historical review in [1]), the sliding mode control technique is recently used in stabilization [2], [3], tracking [4], observer design [5], identification [6], frequency domain analysis [7], and other control problems. Other promising modifications of the original sliding mode concept, such as integral sliding mode [8], are developed. Application of the sliding mode method is extended even to stochastic systems ([9]–[13]) and stochastic filtering problems [14], [15]. However, although it is possible to design a sliding manifold so that an infinite-horizon quadratic cost functional including the system state only is minimized [1], it seems, to the best of authors' knowledge, that no optimal sliding mode algorithms for nonlinear polynomial systems, similar to the optimal polynomial-quadratic regulator [16] or linear-quadratic feedback control [17], [18], have been designed. Meanwhile, simply the fact that the sliding mode control has a transparent physical sense [1] and is successfully applied to many technical problems [19] leads to a conjecture that the optimal control problems whose solution is given by a sliding mode control should exist. One of those optimal control problems for nonlinear systems is considered in this paper.

This paper presents the solution to the optimal control problem for a nonlinear polynomial system with a Bolza-Meyer criterion where the integral control and state energy terms are quadratic and the non-integral term is of the first degree. That type of criteria would be useful in the joint control and parameter identification problems where the objective should be reached for a finite time. It is shown that optimal solution is given by a causal sliding mode control, whereas the conventional polynomial-quadratic regulator does not lead to a causal solution and, therefore, fails. The theoretical result is complemented with an illustrative example verifying performance of the designed control algorithm. The optimal sliding mode regulator is compared to the polynomial-quadratic regulator corresponding to the quadratic Bolza-Meyer criterion ([16]). The simulation results confirm an advantage in favor of the designed sliding mode control. For comparison purposes, both sliding mode and polynomial-quadratic regulators are applied to minimizing the quadratic Bolza-Meyer criterion without the non-integral term. In accordance with the developed theory, the simulation results confirm coincidence of both, sliding mode and polynomial-quadratic, optimal regulator algorithms in this case.

The paper is organized as follows. Section 2 states the optimal control problem for a nonlinear polynomial system with a non-quadratic Bolza-Meyer criterion. The sliding mode solution to the optimal control problem is given in Section 3. The proof of the obtained results is given in Appendix. Section 4 contains an illustrative example.

II. Optimal Control Problem Statement

Consider a polynomial time-varying system with linear control input

$$\dot{x}(t) = f(x,t) + B(t)u(t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^l$ is the control input. Without loss of generality, the system (1) is assumed to be controllable, i.e, the uncontrollable state components are removed from the consideration.

The nonlinear function $f(x,t)$ is considered polynomial of n variables, components of the state vector $x(t) \in R^n$, with time-dependent continuous coefficients. Since $x(t) \in R^n$ is a vector, this requires a special definition of the polynomial for $n > 1$. In accordance with [20], a p-degree polynomial of a vector $x(t) \in R^n$ is regarded as a p-linear form of n components of $x(t)$

$$f(x,t) = a_0(t) + a_1(t)x + a_2(t)xx^T + \ldots + a_p(t)x \ldots_{p\ times} \ldots x,$$

where a_0 is a vector of dimension n, a_1 is a matrix of dimension $n \times n$, a_2 is a 3D tensor of dimension $n \times n \times n$, a_p is an $(p+1)$D tensor of dimension $n \times \ldots_{(p+1)\ times} \ldots \times n$, and $x \times \ldots_{p\ times} \ldots \times x$ is a pD tensor of dimension $n \times \ldots_{p\ times} \ldots \times n$ obtained by p times spatial multiplication of the vector $x(t)$ by itself. Such a polynomial can also be expressed in the summation form

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$$f_k(x,t) = a_{0\ k}(t) + \sum_i a_{1\ ki}(t)x_i(t) + \sum_{ij} a_{2\ kij}(t)x_i(t)x_j(t) + \ldots$$
$$+ \sum_{i_1 \ldots i_p} a_{p\ ki_1\ldots i_p}(t)x_{i_1}(t)\ldots x_{i_p}(t), \quad k,i,j,i_1\ldots\ i_p = 1,\ldots,n.$$

In the classical linear optimal control problem [17], [18], the criterion to be minimized is defined as a quadratic Bolza-Meyer function:

$$J_2 = \frac{1}{2}[x(T)]^T \psi [x(T)] + \tag{2}$$
$$\frac{1}{2}\int_{t_0}^{T} (u^T(s)R(s)u(s) + x^T(s)L(s)x(s))ds,$$

where $R(t)$ is positive and ψ, $L(t)$ are nonnegative definite continuous symmetric matrix functions, and $T > t_0$ is a certain time moment. The solution to this problem is obtained recently [16].

In this paper, the criterion to be minimized includes a non-quadratic terminal term and is defined as follows:

$$J_1 = \sum_{i=1}^{n} \psi_{ii} \mid x_i(T) \mid + \tag{3}$$
$$\frac{1}{2}\int_{t_0}^{T} (u^T(s)R(s)u(s) + x^T(s)L(s)x(s))ds,$$

where $R(s)$ is positive and $L(s)$ is a nonnegative definite continuous symmetric matrix functions, ψ is a diagonal nonnegative definite matrix, and $\mid x_i \mid$ is the absolute value of the component x_i of the vector $x \in R^n$.

The optimal control problem is to find the control $u^*(t)$, $t \in [t_0,T]$, that minimizes the criterion J_1 (3) along with the trajectory $x^*(t)$, $t \in [t_0,T]$, generated upon substituting $u^*(t)$ into the state equation (1).

A solution to the stated optimal control problem is given in the next section and then proved in Appendix. As demonstrated, the obtained solution is a sliding mode control that is optimal with respect to the criterion (3).

III. Optimal Control Problem Solution

The solution to the optimal control problem for the polynomial system (1) and the criterion (3) is given as follows. The optimal control law takes the sliding mode control form

$$u^*(t) = (R(t))^{-1}B^T(t)Q(t)Sign[x(t)], \tag{4}$$

where the Signum function of a vector $x = [x_1,\ldots,x_n] \in R^n$ is defined as $Sign[x] = [sign(x_1),\ldots,\ sign(x_n)] \in R^n$, the signum function of a scalar x is defined as $sign(x) = 1$, if $x > 0$, $sign(x) = 0$, if $x = 0$, and $sign(x) = -1$, if $x < 0$.

The matrix function $Q(t)$ satisfies the matrix equation with time-varying coefficients

$$\dot{Q}(t) = L(t) * \mid x(t) \mid - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \ldots \tag{5}$$
$$+ pa_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)]^T Q(t),$$

where $\mid x \mid = [\mid x_1 \mid,\ldots,\mid x_n \mid] \in R^n$ is defined as the vector of absolute values of the components of the vector $x \in R^n$, and $A * b$ denotes a product between a matrix $A \in R^{n\times n}$ and a vector $b \in R^n$, that results in the matrix defined as follows: all entries of the j-th column of the matrix A are multiplied by the j-th component of the vector b, $j = 1,\ldots,n$.

The terminal condition for the equation (5) is defined as $Q(T) = -\psi I_n$, where I_n is the $n\times n$ identity matrix, if the state $x(t)$ does not reach the sliding manifold $x(t) = 0$ within the time interval $[t_0,T]$, $x(t) \neq 0$, $t \in [t_0,T]$. Otherwise, if the state $x(t)$ reaches the sliding manifold $x(t) = 0$ within the time interval $[t_0,T]$, $x(t) = 0$ for some $t \in [t_0,T]$, then the $Q(t)$ is set equal to a matrix function $Q_0(t)$ that is such a solution of (5) that $x(t)$ reaches the sliding manifold $x(t) = 0$ under the control law (4) with the matrix $Q_0(t)$ exactly at the final time moment $t = T$, $x(T) = 0$, but $x(t) \neq 0$, $t < T$. The trivial case $x(t_0) = 0$ and, therefore, $x(t) = 0$, $t \in [t_0,T]$, is not considered here. Indeed, if $x(t) = 0$, then $u(t) = 0$; therefore, the value of $Q(t)$ is not needed.

Upon substituting the optimal control (4) into the state equation (1), the optimally controlled state equation is obtained

$$\dot{x}(t) = f(x,t) + B(t)(R(t))^{-1}B^T(t)Q(t)Sign[x(t)],\ x(t_0) = x_0. \tag{6}$$

Consequently, the main result is formulated in the following theorem and proved in Appendix.

Theorem 1. The optimal regulator for the linear system (1) with respect to the criterion (3) is given by the sliding mode control law (4) and the gain matrix differential equation (5). The optimally controlled state of linear system (1) is governed by the equation (6).

Remark 1. It is not difficult to see that the solution $Q_0(t)$ really exists and can be calculated. Indeed, if $\psi = 0$ in the criteria (2) and (3) and the non-integral term is absent, then the optimal solutions with respect to both criteria coincide. In this case, as follows from the optimal polynomial-quadratic regulator theory [17], [18], [16], the optimal gain matrix $Q(t)$ has zero terminal value, $Q(T) = 0$, however, the state terminal value may be different from zero, $x(T) \neq 0$. Then, decreasing the value of $-\psi$ as the terminal condition for the equation (5) and, consequently, increasing the energy of the control (4), the zero terminal state value would be reached for a certain negative value of $-\psi_0$, taking into account that each manifold $x_i = 0$, $i = 1,\ldots,n$, is sliding for the corresponding component x_i and the system (1) is assumed controllable. Finally, the solution of the equation (5) with the terminal condition $-\psi_0$ would be the desired solution $Q_0(t)$.

Remark 2. Note that Theorem 1 suggests a feasible algorithm for numerical solution of the gain matrix equation (5). Indeed, first, the system of equations (1),(4),(5) is solved with a given initial condition x_0 and the terminal condition $-\psi$ corresponding to the non-integral term in the criterion (3). Any known numerical method, such as "shooting," which consists in varying initial conditions for (5) until a given terminal condition is satisfied, could be used. If the system state $x(t)$ does not reach zero in the interval $[0,T]$ or reaches exactly at the final moment $t = T$, then the optimal trajectory and the optimal control are found. If $x(t)$ reaches zero at any point $t < T$, the system of equations (1),(4),(5) is solved again with the initial condition x_0 and the terminal condition $-\psi_0$, yielding the solution $Q_0(t)$. The corresponding solution of the equation (1) yields the optimal trajectory. The formula (4) with substituted $Q_0(t)$ and the optimal trajectory yields the optimal control as a function of time.

Remark 3. As follows from Theorem 1, application of the sliding mode control (4) leads to a causal terminal condition for the gain matrix equation (5), which makes the optimal control problem numerically solvable. In contrast, application of the linear feedback control $u^*(t) = K(t)x(t)$ leads to the terminal condition $Q(T) * \mid x(T) \mid = -\psi$, which explicitly depends on the unknown value $x(T)$, and, therefore, is non-causal. As well-known, non-causal problems are not numerically solvable and unusable in practice. Thus, in case of a criterion (3), the sliding mode control allows one to obtain a feasible solution to the optimal control problem, whereas the linear feedback control fails.

IV. Example

This section presents an illustrative example of designing the optimal regulator for a system (1) with a criterion (3), using the scheme (4)–(6).

Consider a scalar linear system

$$\dot{x}(t) = 0.1x^2(t) + u(t), \quad x(0) = 1. \tag{7}$$

The control problem is to find the control $u(t)$, $t \in [0,T]$, $T = 5$, that minimizes the criterion

$$J_1 = 25 \mid x(T) \mid + \frac{1}{2}[\int_0^T u^2(t)dt + \int_0^T x^2(t)dt], \tag{8}$$

where $\mid x \mid$ denotes the absolute value of a scalar variable x.

Applying the optimal regulator (4)-(6), the control law (4) takes the form

$$u^*(t) = Q^*(t)sign[x(t)], \tag{9}$$

where $Q^*(t)$ satisfies the equation

$$\dot{Q}^*(t) = \mid x(t) \mid -0.2x(t)Q^*(t), \tag{10}$$

with the initial condition $Q^*(5) = -50$, if $x(t) \neq 0$ for any $t < 5$, and $Q^*(5) = 0$, otherwise.

Upon substituting the control (9) and the obtained expression for $Q^*(t)$ into (7), the optimally controlled system takes the form

$$\dot{x}(t) = 0.1x^2(t) + Q^*(t)sign[x(t)], \quad x(0) = 1. \tag{11}$$

The obtained system (10),(11) can be solved using simple numerical methods, such as "shooting." This method consists in varying initial conditions of (10) until the given terminal condition is satisfied.

The system (10),(11) is first simulated with the terminal condition $Q^*(5) = -50$. As the simulation shows, the state $x(t)$ reaches zero before the final moment $T = 5$. Accordingly, the terminal condition for the equation (10) is reset to $Q^*(5) = 0$, and the system (10),(11) with (10),(11) is simulated again. The results of applying the regulator (9)–(11) to the system (7) are shown in Fig. 1, which presents the graphs of the gain matrix (10) $Q^*(t)$, the control (9) $u^*(t)$, the state (11) $x(t)$, and the criterion (8) $J_1(t)$ in the interval $[0,5]$. The value of the criterion (8) at the final moment $T = 5$ is $J_1(5) = 1.0693$.

The optimal regulator (9)–(11) is compared to the optimal polynomial regulator for the criterion (2) with the quadratic non-integral term

$$J_2 = 25x^2(T) + \frac{1}{2}[\int_0^T u^2(t)dt + \int_0^T x^2(t)dt]. \tag{12}$$

As follows from the optimal polynomial-quadratic theory [16], the control law takes the form

$$u(t) = Q(t)x(t), \tag{13}$$

where the gain matrix $Q(t)$ satisfies the equation

$$\dot{Q}(t) = L(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \dots$$
$$+ pa_p(t)x(t) \dots_{p-1 \ times} \dots x(t)]^T Q(t) -$$
$$Q(t)[a_1(t) + a_2(t)x(t) + a_3(t)x(t)x^T(t) + \dots$$
$$+ a_p(t)x(t) \dots_{p-1 \ times} \dots x(t)] - Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T) = -\psi$. Substituting numerical values from (7),(12) for the parameters, the last equation turns to

$$\dot{Q}(t) = 1 - 0.3x(t)Q(t) - (Q(t))^2, \quad Q(5) = -50, \tag{14}$$

Upon substituting the control (13) into (7), the controlled system takes the form

$$\dot{x}(t) = 0.1x^2(t) + Q(t)x(t), \quad x(0) = 1. \tag{15}$$

Note that the comparison of the sliding mode optimal regulator (9)–(11) to the best linear regulator (13)–(15) with respect to the criterion (8) is conducted for illustration purposes, since the sliding mode optimal regulator (9)–(11) should theoretically yield a better result, as follows from Theorem 1.

The results of applying the regulator (13)–(15) to the system (7) are shown in Fig. 2, which presents the graphs of the gain matrix (14) $Q(t)$, the control (13) $u(t)$, the state (15) $x(t)$, and the criterion (8) $J_1(t)$ in the interval $[0,5]$. The value of criterion (8) at the final moment $T = 5$ is $J_1(5) = 1.0693 + 6.6 * 10^{-3} = 1.0759$. To provide better comparison, Figure 3 presents the graphs of the state trajectories $x(t)$ (11) and (15) in detail in the interval $[4.995,5]$.

It can be observed that the optimal sliding mode control (9) yields a certainly better value of the criterion (8) in comparison to the polynomial-quadratic regulator (13). Note again that the linear feedback control fails to provide a causal optimal control for the criterion (8) (see also Remark 3).

For verification purposes, both, the sliding mode and polynomial-quadratic regulators, are applied to minimizing the criterion

$$J = \frac{1}{2}[\int_0^T u^2(t)dt + \int_0^T x^2(t)dt], \tag{16}$$

which coincides with the criteria (8) and (12), if the non-integral term is absent. In this case, the optimal sliding mode regulator

$$u^*(t) = Q^*(t)sign[x(t)],$$
$$\dot{Q}^*(t) = | x(t) | - 0.2x(t)Q^*(t), \quad Q(5) = 0, \tag{17}$$
$$\dot{x}(t) = 0.1x^2(t) + Q^*(t)sign[x(t)], \quad x(0) = 1.$$

and the optimal polynomial-quadratic regulator

$$u(t) = Q(t)x(t),$$
$$\dot{Q}(t) = 1 - 0.3x(t)Q(t) - (Q(t))^2, \quad Q(5) = 0. \tag{18}$$
$$\dot{x}(t) = 0.1x^2(t) + Q(t)x(t), \quad x(0) = 1.$$

yield the same control $u^*(t) = u(t)$ and, accordingly, the same optimal trajectory $x(t)$ and the same final criterion value $J(5) = 1.0691$, although the gain matrices $Q^*(t)$ and $Q(t)$ are different. The graphs of the gain matrices $Q^*(t)$ and $Q(t)$ are shown in Fig. 4.

V. Appendix

Proof of Theorem 1. Necessity. Define the Hamiltonian function [17] for the optimal control problem (1),(3) as

$$H(x,u,q,t) = \frac{1}{2}(u^T R(t)u + x^T L(t)x) + q^T \dot{x}(t) =$$
$$= \frac{1}{2}(u^T R(t)u + x^T L(t)x) + q^T [f(x,t) + B(t)u]. \tag{19}$$

Applying the maximum principle condition $\partial H/\partial u = 0$ to this specific Hamiltonian function (19) yields

$$\partial H/\partial u = 0 \Rightarrow R(t)u(t) + B^T(t)q(t) = 0.$$

Accordingly, the optimal control law is obtained as

$$u^*(t) = -R^{-1}(t)B^T(t)q(t).$$

Let us seek $q(t)$ as a signum function of $x(t)$ multiplied by a gain matrix

$$q(t) = -Q(t)Sign[x(t)], \tag{20}$$

where $Q(t)$ is a square symmetric matrix of dimension $n \times n$. This yields the complete form of the optimal control

$$u^*(t) = R^{-1}(t)B^T(t)Q(t)Sign[x(t)]. \tag{21}$$

Using the co-state equation $dq(t)/dt = -\partial H/\partial x$, which gives

$$-dq(t)/dt = L(t)x(t) + [\partial f(x,t)/\partial x]^T q(t), \tag{22}$$

and substituting (21) into (22), we obtain

$$\dot{Q}(t)Sign[x(t)] + Q(t)d(Sign[x(t)])/dt =$$
$$= L(t)x(t) - [\partial f(x,t)/\partial x]^T Q(t)Sign[x(t)].$$

Substituting the polynomial representation for $f(x,t)$ and taking into account the expression for $\partial f(x,t)/\partial x$

$$\partial f(x,t)/\partial x = a_1(t) + 2a_2(t)x + 3a_3(t)xx^T +$$
$$\ldots + pa_p(t)x \ldots_{p-1 \ times} \ldots x,$$

the following equation for $Q(t)$ is obtained

$$\dot{Q}(t)Sign[x(t)] + Q(t)d(Sign[x(t)])/dt = \tag{23}$$
$$= L(t)x(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \ldots$$
$$+ pa_p(t)x(t) \ldots_{p-1 \ times} \ldots x(t)]^T Q(t)Sign[x(t)].$$

Taking into account that $d(Sign[x(t)])/dx = 0$ almost everywhere outside the sliding manifold $x(t) = 0$, the following equation is obtained

$$\dot{Q}(t)Sign[x(t)] = L(t)x(t) - [a_1(t) + 2a_2(t)x(t) + \tag{24}$$
$$3a_3(t)x(t)x^T(t) + \ldots + pa_p(t)x(t) \ldots_{p-1 \ times} \ldots x(t)]^T \times$$
$$Q(t)Sign[x(t)].$$

Note that if $x(t) = 0$, then $u(t) = 0$; therefore, the value of $Q(t)$ is no longer needed. The equation (24) is satisfied, if $Q(t)$ is assigned as a solution of the equation (5).

Note that if the state $x(t)$ does not reach the sliding manifold $x(t) = 0$ in an interior point of the interval $[0,T]$, the transversality condition [17] for $q(T)$ implies that

$$q(T) = -Q(T)Sign[x(t)] = \partial J/\partial x(T) = \psi Sign[x(t)]$$

and, therefore,

$$Q(T) = -\psi. \tag{25}$$

However, if $x(t)$ reaches the sliding manifold $x(t) = 0$ before the final moment $t = T$, since the problem becomes a two fixed point problem where the terminal point is fixed at an *a priori* unknown time moment when $x(t)$ reaches the sliding manifold $x(t) = 0$. Given that the final state value $x(T)$ remains equal to zero, if the state enters the sliding mode before the final moment $t = T$, only the integral part of the criterion should be minimized over all control laws providing that $x(t)$ reaches the sliding manifold $x(t) = 0$ within the interval $[0,T]$. Since the minimal value of the integral part of the criterion (3) over all possible controls is provided by the linear feedback control solving the optimal polynomial-quadratic regulator problem (see also Remark 1 in Section 3), which may lead to a nonzero final state value $x(T) \neq 0$ (see [17], [18], [16]), the minimal value of the integral part of the criterion (3) over all control laws providing that $x(t)$ reaches the sliding manifold $x(t) = 0$ within the interval $[0,T]$ is given by the control law, which yields the state to the sliding manifold $x(t) = 0$ exactly at the final moment $t = T$. This control law corresponds to the gain matrix $Q_0(t)$ in view of its definition in the paragraph after (5). Thus, the terminal conditions for the equation (5) are correctly defined by Theorem 1. The necessity part is proved.

Sufficiency. The optimality of the optimal control law $u^*(t)$ given in Theorem 1 and by the formula (21) is proved in a standard way (see details, for example, in [21]): composing the Hamilton-Jacobi-Bellman (HJB) equation, corresponding to the Hamiltonian (19), and demonstrating that it is satisfied with the Bellman function $V(x,t) = -x^T Q(t)Sign[x] = -\sum_{i,j=1}^{n} Q_{ij}(t)sign[x_j]x_i$, where $Q_{ij}(t)$ are the entries of the matrix $Q(t)$ solving the equation (5). The demonstration mostly repeats the formulas (22)–(25) in the necessity part. Finally, minimizing the right-hand side of the HJB equation over u yields the optimal control $u^*(t)$ in the form (21). The theorem is proved. ■

Proof of Proposition in Remark 1. Consider the optimal control problem for a linear system (1) with respect to the Bolza-Meyer criterion without a non-integral term

$$J = \frac{1}{2}\int_{t_0}^{T} (u^T(s)R(s)u(s) + x^T(s)L(s)x(s))ds. \quad (26)$$

As follows from the optimal polynomial-quadratic regulator theory [16], the optimal control law can be represented as

$$u^*(t) = (R(t))^{-1}B^T(t)Q(t)x(t), \quad (27)$$

where $Q(t)$ satisfies the equation

$$\dot{Q}(t)x(t) = L(t)x(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \quad (28)$$
$$\ldots + pa_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)]^T Q(t)x(t) - Q(t)a_0(t) -$$
$$Q(t)[a_1(t) + a_2(t)x(t) + a_3(t)x(t)x^T(t) + \ldots$$
$$+ a_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)]x(t) -$$
$$Q(t)B(t)R^{-1}(t)B^T(t)Q(t)x(t),$$

with the terminal condition $Q(T) = 0$, which reflects the presence of the non-homogeneous term $a_0(t)$ in the state equation (1). The optimally controlled system takes the form

$$\dot{x}(t) = A(t)x(t) + B(t)R^{-1}(t)B^T(t)Q(t)x(t), \quad x(t_0) = x_0. \quad (29)$$

Let us show that the optimal polynomial-quadratic regulator (27)-(29) coincides with the optimal sliding mode regulator given by Theorem 1. Indeed, upon introducing the new gain matrix $Q^*(t) = Q(t) * \mid x(t) \mid$, the control law (27) turns to the sliding mode control (4) and the equation (29) coincides with (6). Furthermore, in view of (28) and (29), the newly introduced gain matrix $Q^*(t)$ satisfies the equation

$$\dot{Q}^*(t) = \frac{d(Q(t) * \mid x(t) \mid)}{dt} =$$
$$\frac{dQ(t)}{dt} * \mid x(t) \mid + Q(t) * \frac{d(\mid x(t) \mid)}{dt} =$$
$$(L(t) - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \ldots$$
$$+ pa_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)]^T Q(t)) * \mid x(t) \mid -$$
$$Q(t) * (a_0(t) * Sign[x(t)]) -$$
$$(Q(t)[a_1(t) + a_2(t)x(t) + a_3(t)x(t)x^T(t) + \ldots$$
$$+ a_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)] -$$
$$Q(t)B(t)R^{-1}(t)B^T(t)Q(t)) * \mid x(t) \mid +$$
$$Q(t) * (a_0(t) * Sign[x(t)]) +$$
$$Q(t)([a_1(t) + a_2(t)x(t) + a_3(t)x(t)x^T(t) + \ldots$$
$$+ a_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)] * \mid x(t) \mid +$$
$$R^{-1}(t)B^T(t)Q(t) * \mid x(t) \mid) =$$
$$L(t) * \mid x(t) \mid - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \ldots$$
$$+ pa_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)]^T Q(t) * \mid x(t) \mid =$$
$$L(t) * \mid x(t) \mid - [a_1(t) + 2a_2(t)x(t) + 3a_3(t)x(t)x^T(t) + \ldots$$
$$+ pa_p(t)x(t)\ldots_{p-1\ times}\ldots x(t)]^T Q^*(t),$$

with the terminal condition $Q^*(T) = 0$, where $a * b$ denotes the component-wise product between vectors $a = [a_1, \ldots, a_n] \in R^n$ and $b = [b_1, \ldots, b_n] \in R^n$, which results in the vector $[(a*b)_1, \ldots, (a*b)_n] = [a_1b_1, \ldots, a_nb_n] \in R^n$. The obtained equation for $Q^*(t)$ coincides with (5). The proposition is proved. ■

VI. CONCLUSIONS

This paper presents an optimal control problem for nonlinear polynomial systems, whose solution is given by a sliding mode control. The optimal control problem is considered for a nonlinear polynomial system with a Bolza-Meyer criterion, where the integral control and state energy terms are quadratic and the non-integral term is of the first degree. That type of criteria would be useful in the joint control and parameter identification problems where the objective should be reached for a finite time. It is shown that optimal solution is given by a causal sliding mode control, whereas the conventional polynomial-quadratic regulator fails to provide a feasible solution. It is also verified that both sliding mode and polynomial-quadratic regulators yield the same optimal trajectory, being applied to the optimal control problem with respect to the quadratic Bolza-Meyer criterion without the non-integral term, whose solution is known from the optimal polynomial-quadratic regulator theory. The proposed approach based on a sliding mode control is expected to be applicable to other optimal control problems with non-quadratic criteria, where the conventional linear feedback control would not work.

REFERENCES

[1] V. I. Utkin, *Sliding Modes in Control and Optimization*, Springer, Berlin, 1992.

[2] Y. B. Shtessel, A. S. I. Zinober, and I. Shkolnikov, "Sliding mode control for nonlinear systems with output delay via method of stable system center," *ASME Trans. J. Dyn. Syst. Measur. Contr.*, Vol. 125, 2003, pp. 253–257.

[3] S. Suzuki, Y. Pan, K. Furuta, and S. Hatakeyama, "VS-control with time-varying sliding sector: Design and application to pendulum," *Asian Journal of Control*, Vol. 6, 2004, pp. 307–316.

[4] F. Castaños and L. Fridman, "Analysis and design of integral sliding manifolds for systems with unmatched perturbations," *IEEE Trans. Automatic Control*, Vol. 51, 2006, pp. 853–858.

[5] A. Azemi and E. Yaz, "Sliding mode adaptive observer approach to chaotic synchronization," *ASME Transactions. J. Dynamic Systems, Measurements and Control*, Vol. 122, 2000, pp. 758–765.

[6] J. Davila, L. Fridman, and A. Poznyak, "Observation and identification of mechanical systems via second order sliding modes," *International Journal of Control*, Vol. 79, 2006, pp. 1251–1262.

[7] I. Boiko, L Fridman, A Pisano, and E Usai, "Analysis of chattering in systems with second order sliding modes," *IEEE Trans. Automatic Control*, Vol. 52, 2007, pp. 2085-2102.

[8] V. I. Utkin and L. Shi, "Integral sliding mode in systems operating under uncertainty conditions," *Proc. 35th Conference on Decision and Control*, Kobe, Japan, 1996, pp. 4591–4596.

[9] Y. Xia and Y. Jia, "Robust sliding mode control for uncertain stochastic time-delay systems," *IEEE Trans. Automatic Control*, Vol. 48, 2003, pp. 1086–1092.

[10] Y. Niu, D. W. C. Ho and J. Lam, "Robust integral sliding mode control for uncertain stochastic systems with time-varying delay", *Automatica*, Vol. 41, 2005, pp. 873-880.

[11] P. Shi, Y. Xia, G. P. Liu, and D. Rees, "On designing of sliding mode control for stochastic jump systems," *IEEE Trans. Automatic Control*, Vol. 51, 2006, pp. 97–103.

[12] Y. Niu and D. W. C. Ho, "Robust observer design for Ito stochastic time-delay systems via sliding mode control", *Systems and Control Letters*, Vol. 55, 2006, pp. 781–793.

[13] Y. Niu and D. W. C. Ho, "Robust H_∞ control for nonlinear stochastic systems: A sliding mode approach", *IEEE Trans. Automatic Control*, Vol. 53, 2008, pp. 1695-1701.

[14] M. V. Basin *et al*., "Optimal and robust sliding mode filter for systems with continuous and delayed measurements," *Proc. 41st Conference on Decision and Control*, Las Vegas, NV, 2002, pp. 2594–2599.

[15] M. V. Basin *et al*., "Integral sliding mode design for robust filtering and control of linear stochastic time-delay systems," *Int. J. Robust Nonlinear Control*, Vol. 15, 2005, pp. 407–421.

[16] M. V. Basin and D. Calderon-Alvarez, "Optimal controller for stochastic polynomial systems," *Proc. 47th IEEE Conference on Decision and Control*, Cancun, Mexico, 2008, pp. 4755-4760.

[17] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

[18] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, 1975.

[19] V. I. Utkin, J. Guldner, J. Shi, *Sliding Mode Control in Electromechanical Systems*, Taylor and Francis, London, 1999.

[20] M. V. Basin, J. Perez, M. Skliar, Optimal filtering for polynomial system states with polynomial multiplicative noise, *International J. Robust and Nonlinear Control*, Vol. 16, 2006, pp. 287–298.

[21] M. V. Basin and J. G. Rodriguez-Gonzalez, Optimal control for linear systems with multiple time delays in control input, *IEEE Trans. Automatic Control*, Vol. 51, 2006, pp. 91–96.

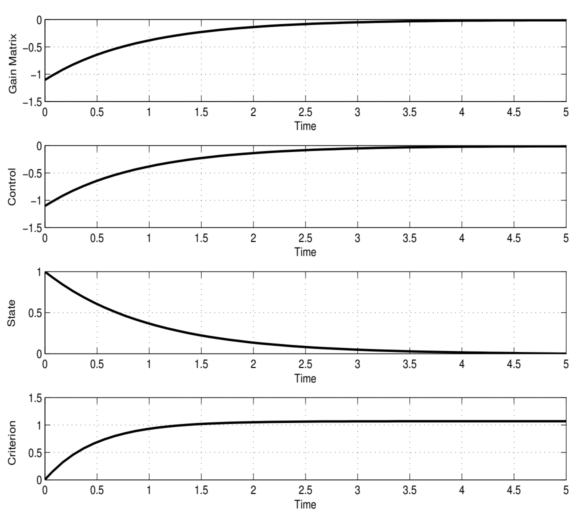

Fig. 1. Sliding mode regulator optimal with respect to criterion J_1. Graphs of the gain matrix (10) $Q^*(t)$, the control (9) $u^*(t)$, the state (11) $x(t)$, and the criterion (8) $J_1(t)$ in the interval $[0,5]$.

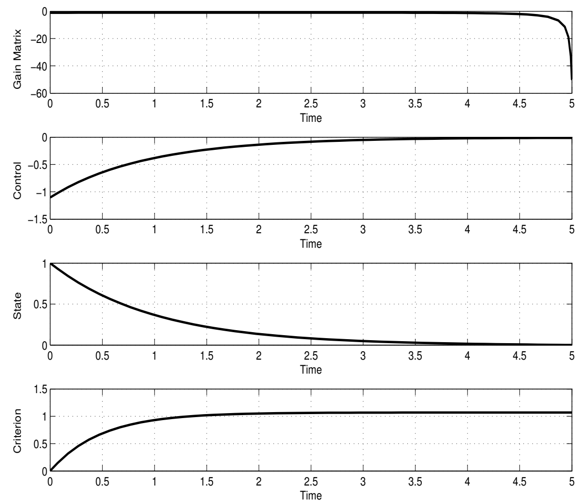

Fig. 2. Polynomial-quadratic regulator. Graphs of the gain matrix (14) $Q(t)$, the control (13) $u(t)$, the state (15) $x(t)$, and the criterion (8) $J_1(t)$ in the interval $[0,5]$.

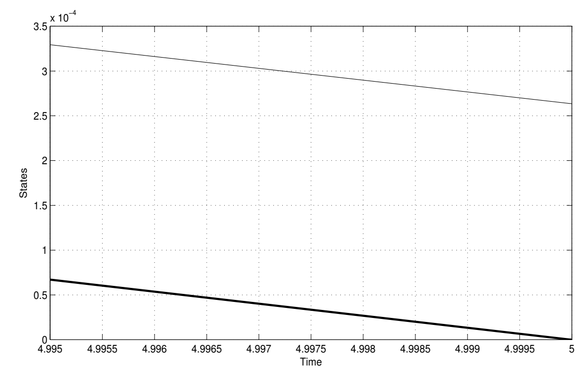

Fig. 3. Graphs of the state trajectories $x(t)$ (11) (thick) and (15) (thin) in detail in the interval $[4.995,5]$.

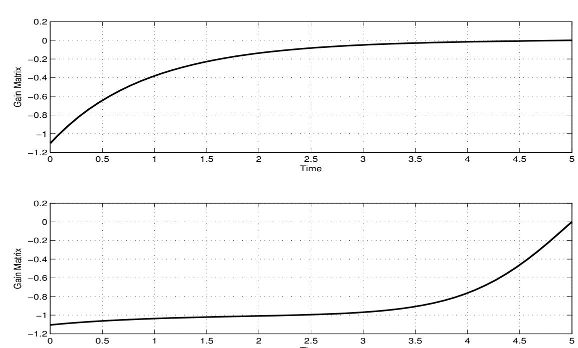

Fig. 4. Graphs of the gain matrices $Q^*(t)$ (above) and $Q(t)$ (below) for criterion J_1 without non-integral term.