

Model Independent Electromagnetic corrections in hadronic τ decays

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Abstract

The long distance correction to the total decay width of the $\tau^\pm \rightarrow K^0 \pi^\pm \nu$ decay is calculated in a model independent approach where a discrimination of photons in the Bremsstrahlung process is assumed. This correction is completely free of UV and IR singularities and moreover, it satisfies electromagnetic gauge invariance. The result of this work can be applied on the tau decays: $\tau^\pm \rightarrow \pi^\pm \pi^0 \nu, K^\pm \pi^0 \nu$.

1 Introduction

It is well known that hadronic τ decays are an ideal laboratory to obtain information about the fundamental parameters within the Standard Model and also some properties of QCD at low energies [1][2]. In particular the $\tau \rightarrow K\pi\nu$ decay has been studied in the past by ALEPH [3] and OPAL [4] and recently at B factories [5][6] where the high statistic measurements provide excellent information about the structure of the spectral functions, parameters of the intermediate states and the total hadronic spectral function. It is also possible to determine the product $|V_{us}|F_+^{K^0\pi^+}(0)$ from this decay although the best determination comes from semileptonic kaon decays [7].

On the theoretical side this sort of processes have a nice feature: the decay amplitude can be factorized into a pure leptonic part and a hadronic spectral function [8] in such a way that the differential decay distribution reads

$$\begin{aligned} \frac{d\Gamma_{K\pi}}{d\sqrt{s}} &= \frac{G_F^2 (|V_{us}|F_+^{K\pi}(0))^2 m_\tau^3}{32\pi^3 s} S_{EW} \left(1 - \frac{s}{m_\tau^2}\right)^2 \\ &\times \left[\left(1 + \frac{2s}{m_\tau^2}\right) q_{K\pi}^3 |\tilde{F}_+^{K\pi}(s)|^2 + \frac{3\Delta_{K\pi}^2 q_{K\pi} |\tilde{F}_0^{K\pi}(s)|^2}{4s} \right], \end{aligned} \quad (1)$$

where a sum over the two posibles decays $\tau^+ \rightarrow K^+\pi^0\nu$ and $\tau^+ \rightarrow K^0\pi^+\nu$ has been done, isospin symmetry is assumed and the reduced vector and scalar form factors have been normalized to one at the origin

$$\tilde{F}_+^{K\pi}(s) = \frac{F_+^{K\pi}(s)}{F_+^{K\pi}(0)}, \quad \tilde{F}_0^{K\pi}(s) = \frac{F_0^{K\pi}(s)}{F_+^{K\pi}(0)}. \quad (2)$$

In this expression $\Delta_{K\pi} \equiv m_K^2 - m_\pi^2$ and the kaon momentum in the rest frame of the hadronic system reads

$$q_{K\pi} = \frac{1}{2\sqrt{s}} \sqrt{(s - (m_K + m_\pi)^2)(s - (m_K - m_\pi)^2)} \times \theta(s - (m_K + m_\pi)^2). \quad (3)$$

A theoretical description[9] of the vector and scalar form factors has been done in the Chiral Theory with Resonances(R χ T) framework[10], providing a successful representation of the data. It is worth to mention that eq.(1) includes the short distance correction S_{EW} [11][12] however the long distance correction is omitted. In the B factories[5][6] an improved experimental precision can be achieved in the future which makes mandatory to have a theoretical analysis of the long distance corrections effects.

It is well known that in all decays with a charged particle the emission of photons is always present altering the dynamics of the decay. The approximative next-to-leading order algorithms [13] are used to simulate the correction due to soft photons where the virtual corrections (one loop) are reconstructed numerically up to the leading logarithms from the real photon corrections. An improved algorithm [14] can be applied if a phenomenological model is used to describe the behaviour of the invariant amplitude.

On the other hand, the first attempt to describe the lepton-hadron interaction used a simple effective interaction approach, nonetheless once the electromagnetic correction is computed an unfortunate feature appears: it depends on the cutoff energy [15] that controls the UV singularity. Nevertheless in the χ PT framework [16] it is possible to describe the interactions between the lightest multiplet of pseudoscalar mesons and the lightest leptons at low energy (below the $\rho(770)$ resonance region) including also real and virtual photons [17][18] and due to the character of χ PT, the UV singularity is cancelled by adding a finite number of appropriated counterterms. At the end one does not have to deal with an UV cutoff but with the finite pieces of the couplings constant of the effective theory. A nice example of this type of electromagnetic correction treatment was done in K_{l3} [19].

Another alternative for the analysis of electromagnetic corrections is given by a Model Independent (MI) approach [20]. In this method the invariant amplitude of the radiative correction is separated into external and internal contribution. The external radiative correction is obtained by considering the emission and absorption of real or virtual photons in external lines. The internal radiative correction corresponds to diagrams where a photon is attached to an internal line and clearly depends on the precise details of the process, in other words this part is model dependent. The aim of this technique is to procure an electromagnetic correction, evaluated once and for all which is gauge invariant, free of UV singularities and contains the dominant logarithms that come from the infrared singularity cancellation.

The MI technique has been applied in the analysis of the electromagnetic corrections to the Dalitz Plot of semileptonic decays of hyperons [21], in the calculation of radiative correction to leptonic decays of pseudoscalar meson[22] and also recently a radiative corrections analysis to the Dalitz plot of the K_{l3} decay[23]. It is worth to mention that authors in [23] have quoted that their result in K_{l3} has been compared with the universal electromagnetic correction given in [19] with a very small quantitative difference. That comparison also help us

to identify the model dependent part in the MI scheme with the corresponding form factor computed within χ P.T. In the effective approach the form factors can be separated into two parts, one of them contains the universal electromagnetic correction that is linked with the model independent electromagnetic factor (MI) and the second one can be seen as the corresponding model dependent form factor (MD), which in the context of χ P.T is made of hadronic loop contributions, finite local terms and electromagnetic pieces that are left aside once the universal correction is defined¹.

In the hadronic τ decays the energy transferred \sqrt{s} to the hadronic system goes from a threshold $(m_1 + m_2)$ up to m_τ , which means that the expansion parameter of χ P.T is no longer valid in all the region. In this respect we can not apply consistently the work done in k_{l3} in computing the electromagnetic corrections in hadronic tau decays.

In this paper we are not committed in to give a description of the form factors, our aim is to present the MI electromagnetic corrections to the $\tau \rightarrow K\pi\nu$ following the techniques given in [20]. In Section 2 we describe briefly the basis of the hadronic tau decay and general effects of electromagnetic correction. In Section 3 the MI electromagnetic corrections for virtual and real photons are computed. In Section 4 we present the MI corrections to the $\tau^+ \rightarrow K^0\pi^+\nu$ decay width and a discussion about our result.

2 τ decay and the Electromagnetic Interaction

2.1 Basic elements on τ decays

Here we do a brief review of the very well known elements of the $\tau^\pm \rightarrow P^\pm P^0 \nu_\tau$ decay where P is a pseudoscalar meson. We start with the invariant amplitude without electromagnetic effects that is expressed [8] as follows

$$\mathcal{M}_\tau^{(0)} = C_{CG} G_F V_{CKM} l_\mu h^\mu, \quad (4)$$

where G_F is the Fermi constant, V_{CKM} is the CKM matrix element, the Clebsch-Gordon coefficient $C_{CG} = (-\frac{\sqrt{2}}{2}, \frac{1}{2}, 1)$ depends on the hadronic final state ($\pi^+ K^0, \pi^0 K^+, \pi^+ \pi^0$) and the leptonic current is given by²

$$l_\mu = \bar{u}_{\nu_\tau}(q) \gamma_\mu \gamma^7 u_\tau(p_\tau), \quad (5)$$

meanwhile the hadronic part h^μ that contains the form factors $F_+(s), F_-(s)$ is written as follows

$$h^\mu = F_+(s)(p^+ - p^0)^\mu + F_-(s)(p^+ + p^0)^\mu, \quad (6)$$

where $p_+(p_0)$ is the 4-momenta of the charged (neutral) pseudo-scalar meson and $s = (p_+ + p_0)^2$.

The density function at tree level is given by

$$\rho^{(0)}(s, u) = |F_+(s)|^2 \mathcal{D}(s, u) + 2F_+(s)F_-^\dagger(s) \mathcal{D}_3(s, u) + |F_-(s)|^2 \mathcal{D}_2(s), \quad (7)$$

¹It is worth to mention that the definition of the universal e.m. correction is not unique meanwhile the e.m. correction computed in the MI approach is very well defined.

²We use the notation $\gamma^7 = 1 - \gamma^5$.

and the \mathcal{D} -functions are given as follows

$$\begin{aligned}
D(s, u) &= 2u^2 + 2u(s - m_\tau^2 - m_+^2 - m_0^2) + \frac{m_\tau^2}{2}(m_\tau^2 - s) + 2m_+^2 m_0^2, \\
D_2(s) &= \frac{m_\tau^2}{2}(m_\tau^2 - s), \\
D_3(s, u) &= \frac{m_\tau^2}{2}(2m_0^2 + m_\tau^2 - s - 2u).
\end{aligned} \tag{8}$$

The differential decay width of the process $\tau^+ \rightarrow P^+ P^0 \nu_\tau$ can be written in a simple form

$$\frac{d\Gamma_{P^+ P^0}^{(0)}}{dsdu} = \frac{|C_{CG} G_F V_{CKM}|^2}{(4\pi)^3 m_\tau^3} \rho^{(0)}(s, u) \tag{9}$$

where $u = (p_\tau - p^+)^2$. After doing the u -integration it is straightforward to get³

$$\begin{aligned}
\frac{d\Gamma_{P^+ P^0}^{(0)}}{ds} &= \frac{|C_{CG} G_F V_{CKM}|^2 m_\tau^3 q_{+0} (1 - \frac{s}{m_\tau^2})^2}{3(4\pi)^3 \sqrt{s}} \left\{ 3|F_-(s)|^2 \right. \\
&\quad \left. + |F_+(s)|^2 \left[\left(1 + \frac{2s}{m_\tau^2}\right) \frac{4q_{+0}^2}{s} + \frac{3\Delta_+^2}{s^2} \right] + 6F_+(s)F_-^\dagger(s) \frac{\Delta_+}{s} \right\}, \tag{10}
\end{aligned}$$

where the momentum q_{+0} in the rest frame of the hadronic system reads

$$q_{+0} = \frac{1}{2\sqrt{s}} \sqrt{(s - (m_+ + m_0)^2)(s - (m_+ - m_0)^2)} \times \theta(s - (m_+ + m_0)^2). \tag{11}$$

It is more familiar and elegant to write eq.(10) in terms of the scalar form factor $F_0(s)$ defined by

$$F_-(s) = \frac{\Delta_+}{s} [F_0(s) - F_+(s)]. \tag{12}$$

With the help of eq.(12), the differential decay width reads

$$\begin{aligned}
\frac{d\Gamma_{P^+ P^0}^{(0)}}{d\sqrt{s}} &= \frac{|G_F V_{CKM}|^2 m_\tau^3}{32\pi^3 s} \left[1 - \frac{s}{m_\tau^2}\right]^2 \left[\frac{3\Delta_+^2 q_{+0}}{4s} |F_0(s)|^2 \right. \\
&\quad \left. + |F_+(s)|^2 \left(1 + \frac{2s}{m_\tau^2}\right) q_{+0}^3 \right] A_{p^+ p^{(0)}}, \tag{13}
\end{aligned}$$

where $A_{p^+ p^{(0)}} = \frac{4}{3} C_{CG}^2$ and its value is equal to $\frac{1}{3}(\frac{2}{3})$ for the hadronic final state $K^+ \pi^0, (K^0 \pi^+)$. It is easy to see that summing both final $K\pi$ states we get eq.(1). In the case of the $\pi^+ \pi^0$ -final state, $A_{\pi\pi} = 4/3$ and the scalar form factor contribution vanishes⁴ due to isospin symmetry.

The energy-momenta conservation defines a set of allowed values for the cinematic variables (u, s) that can be shown in a Dalitz plot which borders are given by

$$(m_+ + m_0)^2 \leq s \leq m_\tau^2, \quad u_-(s) \leq u \leq u_+(s), \tag{14}$$

³ $\Delta_+ = m_+^2 - m_0^2$ where $m_+(m_0^2)$ is the mass of the charged(neutral) meson.

⁴Even considering the physical pion masses, the scalar contribution is negligible.

where

$$\begin{aligned}
u_+(s) &= \frac{1}{2s} \left\{ 2s(m_\tau^2 + m_0^2 - s) - (m_\tau^2 - s)(s + m_+^2 - m_0^2) \right. \\
&\quad \left. + (m_\tau^2 - s)\sqrt{(s - (m_+^2 + m_0^2))^2 - 4m_+^2 m_0^2} \right\} , \\
u_-(s) &= \frac{1}{2s} \left\{ 2s(m_\tau^2 + m_0^2 - s) - (m_\tau^2 - s)(s + m_+^2 - m_0^2) \right. \\
&\quad \left. - (m_\tau^2 - s)\sqrt{(s - (m_+^2 + m_0^2))^2 - 4m_+^2 m_0^2} \right\} . \tag{15}
\end{aligned}$$

Considering the hadronic final state $K^0\pi^+$, the integrated decay width is found to be

$$\Gamma_{K\pi}^{(0)} = |F_+(0)V_{CKM}|^2 G_F^2 C_{CG}^2 \mathcal{N}_i I_{K\pi}^{(0)} \tag{16}$$

where $\mathcal{N}_i = \left[\frac{m_\tau^5}{48\pi^3} \right]$ and the integrated density $I_{K\pi}^{(0)}$ is defined as follows

$$\begin{aligned}
I_{K\pi}^{(0)} &= \int \frac{ds}{m_\tau^2 s \sqrt{s}} \left[1 - \frac{s}{m_\tau^2} \right]^2 \left[|\tilde{F}_+^{K\pi}(s)|^2 \left(1 + \frac{2s}{m_\tau^2} \right) q_{K\pi}^3 \right. \\
&\quad \left. + \frac{3\Delta_+^2 q_{K\pi}}{4s} |\tilde{F}_0^{K\pi}(s)|^2 \right] . \tag{17}
\end{aligned}$$

In order to compute $I_{K\pi}^{(0)}$ one needs a theoretical description of the normalized vector and scalar form factors $\tilde{F}_+^{K\pi}(s)$ and $\tilde{F}_0^{K\pi}(s)$. This task is achieved by a fit to the measured distributions to $\tau^\pm \rightarrow K_s \pi^\pm \nu$. There is a strong effort in computing this integral [24] with a dispersive representation of the form factors but for illustrative purpose we consider the parameterized vector and scalar form factors as given by the Belle Collaboration [6] as follows

$$F_V(s) = F_+(s) = \frac{1}{1 + \beta} [BW_{K^*(892)}(s) + BW_{K^*(1410)}(s)] , \tag{18}$$

where β is the fraction of the $K^*(1410)$ resonance contribution and $BW(s)_R$ is a relativistic Breit-Wigner function

$$BW_R(s) = \frac{M_R^2}{M_R^2 - s - iM_R\Gamma_R(s)} , \tag{19}$$

and $\Gamma_R(s)$ is the s-dependent total width of the resonance,

$$\begin{aligned}
\Gamma_R(s) &= \tilde{\Gamma}_{0,R} \frac{s}{M_R^2} \left(\frac{\sigma_{K\pi}(s)}{\sigma_{K\pi}(m_R^2)} \right)^3 , \\
\sigma_{K\pi}(s) &= \frac{2q_{K\pi}(s)}{\sqrt{s}} . \tag{20}
\end{aligned}$$

For the scalar form factor $F_0(s)$ we take the description that includes only the $K_0^*(800)$ resonance

$$F_0(s) = \kappa \frac{s}{M_{K_0^*(800)}^2} BW_{K_0^*(800)}(s) . \tag{21}$$

where κ is a complex constant and represents the fraction of the scalar resonance contribution.

The mass and width of $K^*(892)$, $K^*(1410)$ are fixed from [25] meanwhile the parameters of $K_0^*(800)$ are taken from [26]. With the values of Table 3 given in [6], it is found that $I_{K_S\pi^\pm}^{(0)} = 0.384221$. Assuming that

$$\begin{aligned} \mathcal{B}(\tau^\pm \rightarrow K^0\pi^\pm\nu) &= \mathcal{B}(\tau^\pm \rightarrow K_S\pi^\pm\nu) + \mathcal{B}(\tau^\pm \rightarrow K_L\pi^\pm\nu) \\ &= 2\mathcal{B}(\tau^\pm \rightarrow K_S\pi^\pm\nu) , \end{aligned} \quad (22)$$

then we get

$$I_{K^0\pi^\pm}^{(0)} = 2I_{K_S\pi^\pm}^{(0)} \sim 0.768 . \quad (23)$$

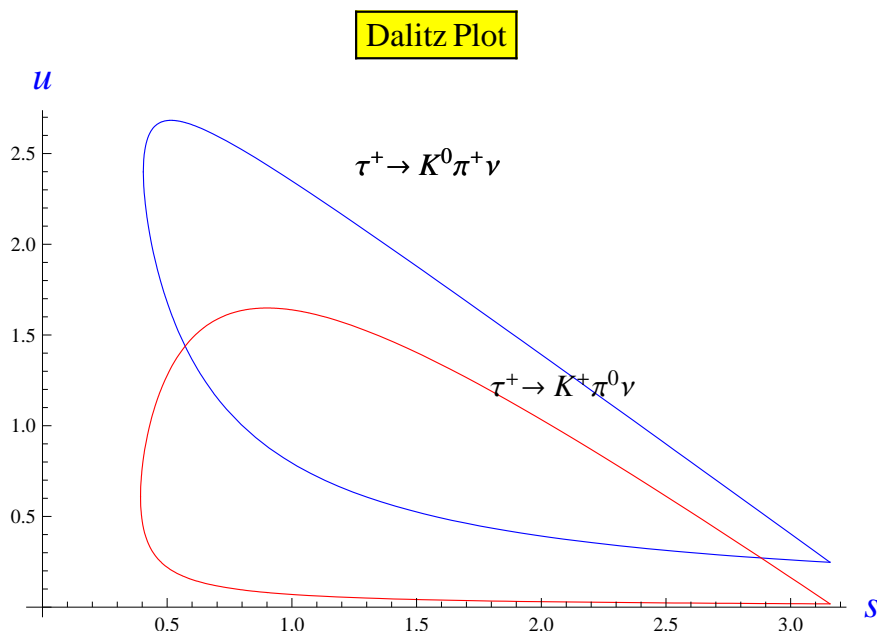


Figure 1: The Dalitz plot without electromagnetic corrections for the tau decay into the $K\pi$ hadronic final state, the Mandelstam variables are defined in the text and masses are given in GeV.

2.2 An overview on electromagnetic corrections

Here we address some general effects of the electromagnetic corrections. First we assume the simplest and easiest scenario: there is just one form factor that is not affected by the one loop integration, in other words its dependence on s is negligible and we denote this by writing $F_+(s) = F_V$. In this case the tree level amplitude for the $\tau^+ \rightarrow P^+ P^0 \nu$ decay reads

$$\mathcal{M}^{(0)} = G_F V_{CKM} C_{CG} F_V l_\mu (p_+ - p_0)^\mu . \quad (24)$$

The amplitude for the one loop electromagnetic correction with pointlike meson-photon interaction is found to be

$$\mathcal{M}^v = G_F V_{CKM} C_{CG} F_V l_\mu \frac{\alpha}{4\pi} [f_+^{e.m.}(u)t_-^\mu + f_-^{e.m.}(u)t_+^\mu] , \quad (25)$$

where $t_{\pm}^{\mu} = (p^{\pm} \pm p^0)^{\mu}$. It is well known that the function $f_{+}^{e.m.}(u)$ encloses ultraviolet (UV) and infrared (IR) singularities meanwhile $f_{-}^{e.m.}(u)$ contains just an UV singularity. The IR divergence is cancelled by taking into account the real photon emission meanwhile the UV problem can be solved with a cut-off in the one-loop integration that replaces the singularity⁵. In short this method consists in making the next replacement

$$\frac{2}{4-D} - \gamma_E + \ln[4\pi] + \ln\left[\frac{m^2}{\mu^2}\right] \rightarrow \ln\left[\frac{\Lambda^2}{m^2}\right]. \quad (26)$$

where the cut-off Λ can be some resonance mass and m is a characteristic mass of the process. The replacement makes sense because represents that something is hidden in the effective 4-body interaction encoded in eq.(24).

In order to make an analogy we recall the one loop corrections (at short distances) computed [27] within the Standard Model for the process $\tau \rightarrow \nu_{\tau} d \bar{u}$ where the pure QED correction in the Fermi model is UV divergent but if all the electroweak corrections are included (Z and W virtual exchange) then the ultraviolet cut-off ($\ln[\frac{\Lambda^2}{m_{\tau}^2}]$) is replaced naturally by a large logarithm namely $\ln[m_Z/m_{\tau}]$ [11]. In this case the hidden objects in the Fermi model are Z and W bosons.

On the other hand, when the electromagnetic corrections are computed in the MI approach, the discussion about the value of the cut-off Λ is left aside. For instance consider we compute the e.m. corrections to eq.(24) following the work of [20], in this case the one loop invariant matrix can be written as follows

$$\mathcal{M}^v = \mathcal{M}^{(0)} \frac{\alpha}{4\pi} f_{m.i.}^{e.m.}(u, \lambda^2) + \mathcal{M}_{+} \frac{\alpha}{4\pi} f_{m.i.}^r(u) + \mathcal{M}^{(0)} \frac{\alpha}{4\pi} f_{+}^{m.d.}(u), \quad (27)$$

where

$$\mathcal{M}_{+} = G_F V_{CKM} C_{CG} F_V \bar{u}_{\nu_{\tau}}(q) \gamma_{\mu} \gamma^7 \not{p}^{+} u_{\tau}(p_{\tau}) t_{-}^{\mu}. \quad (28)$$

The first and second pieces of eq.(28) are the e.m. corrections independent of structure effects, come from the convection and spin term, are gauge invariant, free of UV divergences and all the IR singularity is located in this term. The last piece gathers in the function $f_{+}^{m.d.}(u)$, the e.m. effects on the vector form factor, details of strong interactions and the model dependent assumptions to cancel the UV divergences. Notice that in assuming that the vector form factor is the dominant we are taking into account only the effects at $\mathcal{O}(\alpha)$ to this form factor. Adding eqs.(24,27) we get

$$\mathcal{M} = \mathcal{M}^{(0)'} \left[1 + \frac{\alpha}{4\pi} f_{m.i.}^{e.m.}(u, \lambda^2) \right] + \mathcal{M}'_{+} \frac{\alpha}{4\pi} f_{m.i.}^r(u), \quad (29)$$

where the vector form factor has been redefined at order α as follows,

$$F'_{V}(u) = F_V + \frac{\alpha}{4\pi} f_{+}^{m.d.}(u). \quad (30)$$

This fact is indicated by a prime on the amplitudes of eq.(29). As a consequence of eq.(30), it is assumed that $F'_{V}(u)$ is extracted from the experiment and hence

⁵ $D = 4 - \epsilon$ assuming the one loop integrals are done in the Dimensional Regularization Method, however others prescriptions are equivalents.

gives information about the structure dependence and also helps to select the best model or theory that describes this form factor and its complications. The same procedure is applied when the tree level amplitude depends on two form factors, in this case the model dependent amplitude can be written as follows

$$\mathcal{M}_{md}^v = G_F V_{CKM} C_{CG} l_\mu \frac{\alpha}{4\pi} [F_+^{m.d.}(s, u) t_-^\mu + F_-^{m.d.}(s, u) t_+^\mu] . \quad (31)$$

In this general case the form factors $F_+^{m.d.}(s, u)$ and $F_-^{m.d.}(s, u)$ that represent the mixing of hadronic and e.m. effects are redefined as follows

$$\begin{aligned} F'_+(s, u) &= F_+(s) + \frac{\alpha}{4\pi} F_+^{m.d.}(s, u) , \\ F'_-(s, u) &= F_-(s) + \frac{\alpha}{4\pi} F_-^{m.d.}(s, u) . \end{aligned} \quad (32)$$

3 Model Independent Radiative Corrections

In this section the one loop and real photon corrections to $\tau^+ \rightarrow P^+ P \nu$ are computed following the techniques explained in [20]. A characteristic of this correction is that it is specified completely by the momenta and spin states of the initial and final particles.

3.1 Virtual photons.

The MI one loop electromagnetic corrections are obtained from the diagrams shown in fig.(2) with the tree level amplitude given in eq.(4). The amplitude of

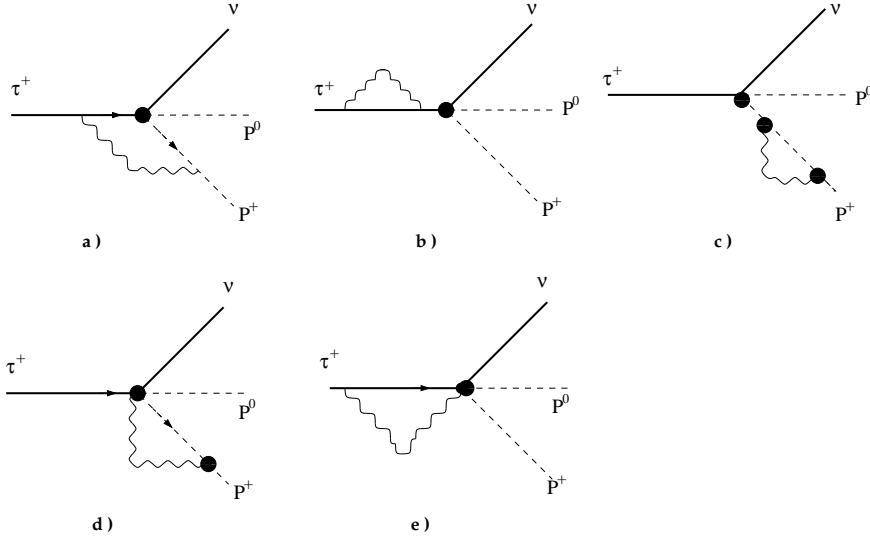


Figure 2: The so-called external diagrams, where a photon line connects external charged legs correspond to a), b) and c), where $P^+(P^0)$ denotes the charged(neutral) scalar meson.

the first external diagram (fig.2a) reads

$$\begin{aligned} \mathcal{M}_a &= e^2 \bar{u}_\nu(q) G \frac{-i}{(2\pi)^4} \int dk^4 [F_+(s')(\not{p}^+ - \not{p}^0 + \not{k}) \\ &\quad + F_-(s')(\not{p}^+ + \not{p}^0 + \not{k})] \gamma^7 \frac{\not{p}_\tau + \not{k} - m_\tau}{k^2[(p_\tau + k)^2 - m_\tau^2]} \frac{[2\not{p}^+ + \not{k}]}{(p^+ + k)^2 - m_+^2} u_\tau(p_\tau) \end{aligned} \quad (33)$$

where now $s' = (p^0 + p^+ + k)^2$ and here we define $G = G_F V_{CKM} C_{CG}$ in order to avoid long expressions. According to our aim and following the result given in [20], the amplitude is separated in one piece that depends only on general QED properties and a second piece that depends on the description of the structure. This assumption allows us to write the amplitude \mathcal{M}_a in the following form

$$\begin{aligned} \mathcal{M}_a &= \mathcal{M}_\tau^{(0)} \frac{-ie^2}{(2\pi)^4} \int \frac{(2p_\tau + k) \cdot (2p^+ + k) dk^4}{k^2[(p_\tau + k)^2 - m_\tau^2][(p^+ + k)^2 - m_+^2]} \\ &\quad + G \bar{u}_\nu(q) h_\nu \gamma^\nu \gamma^7 \frac{ie^2}{(2\pi)^4} \int \frac{\frac{1}{2}[\gamma^\alpha, \not{k}](2p^+ + k)_\alpha dk^4}{k^2[(p_\tau + k)^2 - m_\tau^2][(p^+ + k)^2 - m_+^2]} u_\tau(p_\tau) \\ &\quad + \mathcal{M}_a^{r,md} \end{aligned} \quad (34)$$

The infrared singularity, some UV divergences and the major finite correction come from the first line that corresponds to the convection term meanwhile the second line corresponds to the spin term which is UV and IR finite [20]. The rest of the amplitude $\mathcal{M}_a^{r,md}$ can be written as indicated in eq.(31) and contains all details of strong interactions. From here we assume that the model dependent part is included in the redefinition of the form factors as given in eq.(32). The convection amplitude, first line in eq.(34), is written⁶ in terms of the tree level amplitude times the electromagnetic correction as follows,

$$\begin{aligned} \mathcal{M}_a^{cc} &= \mathcal{M}_\tau^{(0)} \frac{\alpha}{4\pi} [B0[m_\tau^2, 0, m_\tau^2] - B0[u, m_+^2, m_\tau^2] + B0[m_+^2, 0, m_\tau^2] \\ &\quad + 4p_\tau \cdot p^+ C0[m_+^2, u, m_\tau^2, \lambda^2]] . \end{aligned} \quad (35)$$

The convection contribution of the scalar meson self-energy⁷, fig.(1c), is given by

$$\mathcal{M}_{SE}^{cc} = \mathcal{M}_\tau^{(0)} \frac{1}{2} \frac{4i\pi\alpha}{(2\pi)^4} \int \frac{dk^4 (2p^+ + k) \cdot (2p^+ + k)}{k^2[(p^+ + k)^2 - m_+^2]^2} . \quad (36)$$

The very well know lepton self-energy, fig.(1b), reads⁸

$$\mathcal{M}_{SE}^\tau = \mathcal{M}_\tau^{(0)} \frac{\alpha}{4\pi} \frac{1}{2} \left[-\Delta_{UV} + \ln \left[\frac{m_\tau^2}{\mu^2} \right] - 4 - 2 \ln \left[\frac{\lambda^2}{m_\tau^2} \right] \right] , \quad (37)$$

where λ is the fictitious mass of the photon used in order to control the IR divergence and μ is the well known mass parameter introduced in the dimensional

⁶The integration is done using the FeynCal[30] package for Mathematica and the Passarino-Veltman functions can be evaluated with LoopTools. In order to see the IR singularity cancellation we use analytical expression given in the Appendix.

⁷See the ref.[20]

⁸In dimensional regularization $\Delta_{UV} = \frac{2}{4-D} - \gamma_E + \ln[4\pi]$.

regularization method. The total convection amplitude is the sum of (35,36,37) and reads

$$\begin{aligned}
\mathcal{M}_{m.i.}^{cc} &= \mathcal{M}_\tau^{(0)} \frac{\alpha}{4\pi} f_{m.i.}^v \\
&= \mathcal{M}_\tau^{(0)} \frac{\alpha}{4\pi} \left[2 \ln \left[\frac{m_+ m_\tau}{\lambda^2} \right] + \frac{m_+ m_\tau}{u} \left[\frac{1}{x_t} - x_t \right] \ln[x_t] \right. \\
&\quad \left. + 2m_+^2 y_t C0[m_+^2, u, m_\tau^2, \lambda^2] - \frac{m_+^2}{2u} (1 - r_t) \ln[r_t] - 1 \right] \quad (38)
\end{aligned}$$

The notation is the same that in [31] and the expression of $C0[m_+^2, u, m_\tau^2, \lambda^2]$ is obtained from the general form given in [32]. The total convection amplitude is free of UV singularities and is also electromagnetic gauge invariant which can be easily checked by adding to the photon propagator the term $\xi k^\mu k^\nu / k^2$, where ξ is an arbitrary parameter and then seeing that ξ -dependent contributions from the lepton self energy and convection meson self energy cancel the respective terms coming from eq.(35).

The second line of eq.(34) is found to be

$$\mathcal{M}_2 = \mathcal{M}_\tau^{(0)} \frac{\alpha}{4\pi} (2p_\tau \cdot p^+) f^r + G \bar{u}_{\nu_\tau}(q) h_\nu \gamma^\nu \gamma^7 \not{p}^+ u(p_\tau) \frac{\alpha}{4\pi} [2m_\tau f^r] \quad (39)$$

This piece is gauge invariant by itself and free of IR and UV singularities and the function f^r reads

$$\begin{aligned}
f^r &= \frac{1}{m_+^2 (y_\tau^2 - 4r_\tau)} \left[\frac{m_+^2 (2 - y_\tau)}{u} \left[\frac{2 - y_\tau}{2} \ln[r_\tau] - \frac{x_\tau}{1 - x_\tau^2} \frac{\ln[x_\tau]}{\sqrt{r_\tau}} (y_\tau^2 - 4r_\tau) \right] \right. \\
&\quad \left. - \ln[r_\tau^2] \right] . \quad (40)
\end{aligned}$$

In order to get a complete gauge invariance result, the total MI amplitude must contain the sum of eqs.(38,39), thus the correction to the differential tau decay width is written as follows

$$\begin{aligned}
\frac{d\Gamma_{P^+ P^0}}{dsdu} &= \frac{|G|^2}{(4\pi)^3 m_\tau^3} \left[\rho^{(0)}(s, u) \left[1 + \frac{\alpha}{2\pi} (f_{m.i.}^v + 2p_\tau \cdot p^+ f^r) \right] \right. \\
&\quad + \frac{\alpha}{2\pi} [m_\tau^2 f^r] \left\{ |F_+(s)|^2 \mathcal{G}(s, u) + 2F_+^\dagger(s) F_-(s) \mathcal{H}(s, u) \right. \\
&\quad \left. \left. + |F_-(s)|^2 \mathcal{E}(s, u) \right\} \right] \quad (41)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}(s, u) &= s(m_+^2 - u) + m_+^2 (4u - 4m_0^2) + m_\tau^2 (m_0^2 - m_+^2) , \\
\mathcal{E}(s, u) &= s(m_+^2 - u) + m_\tau^2 m_0^2 - m_+^2 m_\tau^2 , \\
\mathcal{H}(s, u) &= s(m_+^2 + u) - m_\tau^2 m_0^2 - m_+^2 m_\tau^2 . \quad (42)
\end{aligned}$$

It is worth to mention that in eq.(41), the function $f_{m.i.}^v$ contains all the IR singularity of the one loop correction which is cancelled after taking into account the real-photon emission [33].

In the case of two pions in the hadronic final state, where the vector form factor is the dominant, the formula (41) reads

$$\frac{d\Gamma_{\pi\pi}}{dsdu} = \frac{||GF_V(s)||^2}{(4\pi)^3 m_\tau^3} \left[\left[1 + \frac{\alpha}{2\pi} (f_{m.i.}^v + 2p_\tau \cdot p^+ f^r) \right] \mathcal{D}(s, u) + \frac{\alpha}{2\pi} (m_\tau^2 f^r) \mathcal{G}(s, u) \right] \quad (43)$$

3.2 Real Photon Emission

In order to be consistent with the work done in the one loop correction, the model independent part of the radiative process must be defined, a task that is achieved by using the low-energy theorems [34][35].

According to the Low Theorem [34], the radiative invariant amplitude denoted as \mathcal{M}^γ can be expanded in powers of the photon energy k for small k as follows

$$\mathcal{M}^\gamma = \frac{\mathcal{M}}{k} + \mathcal{M}_1 k^0 + k\mathcal{M}_2 + \dots \quad (44)$$

where the dots symbolize terms with powers of order ≥ 2 in k . The first piece \mathcal{M} (Low-term) and \mathcal{M}_1 can be calculated completely from the non-radiative invariant amplitude meanwhile \mathcal{M}_2 and the next elements on the serie depend on the theoretical model that describes the details of the photon emission from either hadronic external lines or an internal hadronic vertex. This means that eq.(44) establishes the definition of model independent and model dependent term in the radiative amplitude.

On the other hand, it was shown in [35] that the unpolarized and squared amplitude of the radiative process can be splitted into two parts, one element of order $1/k^2$ that comes entirely from the Low-term and the rest that contains contributions of order k^0, k^1, \dots as it is indicated in the following equation

$$\begin{aligned} \sum_{spins} |\mathcal{M}^\gamma|^2 &= \sum_{spins} |\mathcal{M}^{(0)}|^2 e^2 \left[\frac{p^+ \cdot \epsilon(k)}{p^+ \cdot k} - \frac{p_\tau \cdot \epsilon(k)}{p_\tau \cdot k} \right]^2 \\ &+ \sum_{spins} |\mathcal{M}'|^2 k^0 + \sum_{spins} |\mathcal{M}''|^2 k^1 + \dots, \end{aligned} \quad (45)$$

where $\epsilon(k)$ is the photon polarization vector and \sum_{spins} indicates an average over initial spin states and a sum over final spin states, except over the photon degrees of freedom. The first piece is the Low-term which is precisely the convection term in the MI scheme and encloses the appropriated terms that cancels the IR singularity in eq.(38). In this work we considere only the Low-term for the radiative amplitude⁹.

According to the previous lines our gauge invariant MI radiative amplitude for the $\tau^+ \rightarrow P^+ P^0 \nu \gamma$ decay reads

$$\mathcal{M}_\tau^\gamma = \mathcal{M}_\tau^{(0)'} e \left[\frac{p^+ \cdot \epsilon(k)}{p^+ \cdot k} - \frac{p_\tau \cdot \epsilon(k)}{p_\tau \cdot k} \right]. \quad (46)$$

The soft photon approximation[36] was computed in a previous work [37] with a careful handling of the infrared singularity [38] and it was shown that the radiative correction depends on a cutoff energy ω_0 ¹⁰. However in this work we considere an alternative procedure [15],[39],[40] that proposses a separation of the Dalitz-Plot region in such a way that the uncomfortable dependence is avoided.

In computing the well known invariant integrals for the real photon correction,

⁹The term of order k^0 includes contributions from the interference between the model independent and the model dependent term, therefore it is model dependent.

¹⁰An asumption of the soft photon approximation is that in an experiment, there is a minimal energy ω_0 for detecting a real photon

a novel approach was done in the work of A. Martínez *et al* [23], obtaining the same result that Ginsberg [39], however we follow the technique of the later. In the radiative process $\tau \rightarrow K\pi\nu\gamma$ a new variable arises, known as the invariant mass of the undetected particles and denoted by $x = (q + k)^2$ with

$$x_-(s, u) \leq x \leq x_+(s, u) \quad (47)$$

where $k = (k_0, \vec{k})$ are the energy and the momentum of the photon and the maximal and minimal value of x are given in the Appendix. In order to compute the real photon contribution some assumptions have to be considered, in this respect we adopt those given in [23] which means the following:

- The allowed kinematical region of all values for (u, s) that satisfy the relation of the three-body space phase, given by eqs.(14,15).
- The values of the Lorentz invariant x consistent with the first point.

In other words we consider the set of values of u, s and x that defines a region whose borders are given as follows

$$D_{III} = \{u_-(s) \leq u \leq u_+(s), s_{min} \leq s \leq s_{max}, \lambda^2 \leq x \leq x_+(s, u)\} \quad (48)$$

It is important to point out that the IR divergence is precisely within this region. Here the minimal value of x is written in terms of λ , the fictitious mass of the photon, the same regulator used in handling the IR singularity in eq.(38). As a consequence of these remarks, it is assumed a discrimination of real photons that could be done in an experimental setup by means of an analysis of the 4-body radiative Dalitz Plot(See fig.3)[41]. The discriminated photons are inside the complementary region D_c , only accesible to the radiative process, where the energy of the photon is never zero and as a consequence, the radiative amplitude is free of IR singularity. This region is defined for the following set of values,

$$D_c = \left\{ u_+ \leq u \leq (m_\tau - m_\pi)^2, x_- \leq x \leq x_+, \right. \\ \left. s_{min} \leq s \leq \frac{m_\tau(m_\tau m_\pi + m_K^2 - m_\pi^2)}{m_\tau - m_\pi} \right\} \quad (49)$$

Once it has been clarified our assumptions, we can compute the contribution of the Low term eq.(46) with the corresponding phase space described in eq.(48). Notice that we are computing the eq.(18) given in [39] but for the $\tau \rightarrow K\pi\nu\gamma$ process.

The differential radiative decay width reads

$$\frac{d\Gamma_{K\pi}^\gamma}{dsdu} = \frac{G_F^2 |V_{us} F_+^{K\pi}(0)|^2 C_{CG}^2 \rho(s, u)}{m_\tau^3 (4\pi)^3} \frac{\alpha}{\pi} (I_{1,1} + I_{2,0} + I_{0,2}) . \quad (50)$$

The very well know invariant integrals are given in [39], details of the computation and the notation are presented in the appendix, here we just write the

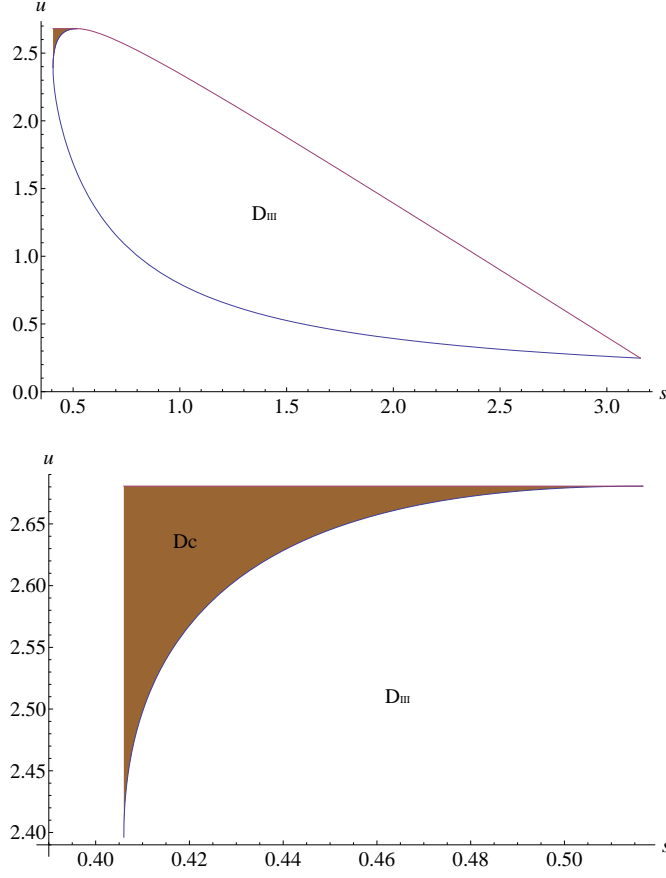


Figure 3:

The upper plot shows the projection of the 4-body radiative Dalitz Plot onto the u - s plane where can be seen that the 3-body Dalitz Plot is inside this region. The lower plot shows an amplification of the projected complementary region D_c (see text) accessible only to the radiative process.

result in our case as follows,

$$\begin{aligned}
I_{2,0} &= \ln \left[\frac{m_\tau^2 - s}{x_+(s, u)} \right] + \ln \left[\frac{\lambda}{m_\tau} \right], \\
I_{0,2} &= \ln \left[\frac{m_\tau^2 - s - u + m_{K^0}^2}{x_+(s, u)} \right] + \ln \left[\frac{\lambda}{m_{\pi^+}} \right], \\
I_{1,1} &= \frac{x_\tau y_\tau}{\sqrt{r_\tau} (1 - x_\tau^2)} \left\{ Li_2 \left[\frac{-a^2}{4r_+} \right] - Li_2 \left[\frac{-4r_-}{a^2} \right] - \ln[x_\tau] \ln \left[\frac{y_\tau^2 - 4r_\tau}{\sqrt{r_\tau}} \right] \right. \\
&\quad \left. + \ln[x_\tau] \ln \left[\frac{\lambda^2}{m_\tau m_{\pi^+}} \right] - \ln[x_\tau] \ln \left[\frac{x_\tau x_+^2(s, u)}{4r_+} \right] \right\}. \tag{51}
\end{aligned}$$

4 Correction to the $\tau^+ \rightarrow K^0 \pi^+ \nu$ decay

The results of the previous sections are applicable to the hadronic final modes ($K^0 \pi^+$, $K^+ \pi^0$, $\pi^0 \pi^+$), however we are interested only in the first mode. Consequently we present here the MI electromagnetic corrections to the differential width decay, due to virtual photons eq.(41) and real photons eq.(50) computed within the region eq.(48),

$$\frac{d\Gamma_{K^0 \pi^+}}{dsdu} = \frac{G_F^2 |V_{us} F_+^{K\pi}(0)|^2 C_{CG}^2}{(4\pi)^3 m_\tau^3} \left\{ |\tilde{F}_+(s)|^2 \tilde{\mathcal{D}}(s, u) + 2\tilde{F}_-(s) \tilde{F}_+(s) \tilde{\mathcal{D}}_3(s, u) + |\tilde{F}_-(s)|^2 \tilde{\mathcal{D}}_2(s, u) \right\}, \quad (52)$$

where $C_{CG}^2 = 1/2$ and the $\tilde{\mathcal{D}}$ -functions corrected are given by

$$\begin{aligned} \tilde{\mathcal{D}}(s, u) &= \mathcal{D}(s, u) + \frac{\alpha}{2\pi} \left\{ \mathcal{D}(s, u) f_I^{m.i.} + \mathcal{G}(s, u) f_{II}^{m.i.} \right\} \\ \tilde{\mathcal{D}}_2(s, u) &= \mathcal{D}_2(s) + \frac{\alpha}{2\pi} \left\{ \mathcal{D}_2(s) f_I^{m.i.} + \mathcal{E}(s, u) f_{II}^{m.i.} \right\} \\ \tilde{\mathcal{D}}_3(s, u) &= \mathcal{D}_3(s, u) + \frac{\alpha}{2\pi} \left\{ \mathcal{D}_3(s, u) f_I^{m.i.} + \mathcal{H}(s, u) f_{II}^{m.i.} \right\} \end{aligned} \quad (53)$$

where

$$\begin{aligned} f_I^{m.i.} &= 2[I_{2,0} + I_{0,2} + I_{1,1}] + f_{m.i.}^v + m_{\pi^+}^2 y_\tau f^r, \\ f_{II}^{m.i.} &= m_\tau^2 f^r. \end{aligned} \quad (54)$$

The simple form of eq.(52) gives a hand to identify the effect of the e.m. corrections

- The density function eq.(7) is corrected at $\mathcal{O}(\alpha)$ by the M.I. corrections.
- The form factors includes, for definition in eq.(32), the model dependent effects at $\mathcal{O}(\alpha)$.

Integrating eq.(52) we obtain the total decay width corrected by the model independent e.m. corrections

$$\Gamma_{K^0 \pi^+} = \Gamma_{K^0 \pi^+}^{(0)} [1 + \delta_{EM}^{m.i.}] \quad (55)$$

where the e.m. correction function reads

$$\begin{aligned} \delta_{EM}^{m.i.} &= \frac{[\alpha/2\pi]}{m_\tau^8 I_{K\pi}^{(0)}} \frac{3}{4} \int \left\{ |\tilde{F}_+(s)|^2 [\mathcal{D}(s, u) f_I^{m.i.} + \mathcal{G}(s, u) f_{II}^{m.i.}] \right. \\ &\quad + |\tilde{F}_-(s)|^2 [\mathcal{D}_2(s) f_I^{m.i.} + \mathcal{E}(s, u) f_{II}^{m.i.}] \\ &\quad \left. + 2\tilde{F}_-(s) \tilde{F}_+(s) [\mathcal{D}_3(s, u) f_I^{m.i.} + \mathcal{H}(s, u) f_{II}^{m.i.}] \right\} dsdu \end{aligned} \quad (56)$$

In order to estimate the electromagnetic correction we follow the theoretical description for the form factors as given in section(2.1), hence we obtain $\delta_{EM}^{m.i.} = -0.127\%$. The aim of this work has been achieved by writing eq.(56), however the Dalitz plot corrections can be obtained straightforward by using eq.(53).

On the other hand, Antonelli *et al*[42] have given an estimate of the long distance electromagnetic corrections to $\tau^+ \rightarrow K^0\pi^+\nu$, where the structure dependent effects have been neglected and their approach relies in the analysis already done in Kl_3 and $\tau \rightarrow \pi\pi\nu$. They found $\delta_{em}^{K^0\pi^+} = (-0.15 \pm 0.2)\%$ where 0.2 is the uncertainty they assigned to the unknown structure dependent effects.

5 Conclusions

Summarizing the work done, we provide the MI electromagnetic corrections to the $\tau^+ \rightarrow K^0\pi^+\nu$ decay following the procedure described in [20] considering both virtual and real photon within the 3-body phase space region eq.(48). As it has been pointed out this correction is electromagnetic gauge invariant, free of IR singularities, free of UV singularities and most important it does not have an UV cutoff. This approach considers that all structure dependence is included inside the form factors $\tilde{F}_\pm(s)$ after an appropriated redefinition. In this respect our work is not focused in the theoretical description of the form factors and the corresponding solution to the remaining UV problem.

On the other hand, it is important to accentuate that eq.(56) can be used for the three hadronic final states ($\pi^\pm\pi^0, \pi^\pm K^0, \pi^0 K^\pm$) with the corresponding changes. Concerning the τ decay into the 2π final state, the universal electromagnetic correction given in [31] is slightly different from the MI correction, we stressed this to be more qualitative than quantitative.

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Appendix.

Kinematics.

In the radiative process a new invariant Mandelstam variable is introduced, $x = (q + k)^2$ whose maximum and minimum value are given as follows

$$x_\pm(s, u) = \frac{1}{2m_\pi^2} \left[u(m_+^2 - m_0^2) - m_+^2(m_+^2 - m_0^2) + m_\tau^2(m_+^2 + m_0^2) - s(m_\tau^2 - m_+^2 - u) \pm \lambda^{1/2}(u, m_+^2, m_\tau^2)\lambda^{1/2}(s, m_+^2, m_0^2) \right] \quad (57)$$

where the well known Källén function reads

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc) . \quad (58)$$

Invariant amplitudes.

According to the definition given in [39], the radiative function of the Low term is written as follows

$$g^\gamma = I_{11} + I_{20} + I_{02} \quad (59)$$

These type of integrals have been already studied in [39], here we present brief details about computing them for our situation. We start with the easiest one which we write it as follows

$$I_{2,0} = - \lim_{\lambda \rightarrow 0} \int_{\lambda^2}^{x_+(t,u)} p_\tau^2 i_{2,0} dx ,$$

$$i_{2,0} = \frac{1}{2\pi} \int \frac{d^3 \vec{k} d^3 \vec{q}}{k_0 q_0} \frac{\delta(\vec{P} - \vec{k} - \vec{q}) \delta(E_\nu - k_0 - q_0)}{[(p_\tau - k)^2 - m_\tau^2]^2} . \quad (60)$$

where $P = p_\tau - p_+ - p_0$ and $E_\nu = E_\tau - E_+ - E_0$. The regulator for the IR singularity is chosen to be the same than the one used in the one loop corrections which means that $k^2 = \lambda^2$ for a real photon with a small fictitious mass. A frame is chosen in such a way that $\vec{P} = 0$, hence

$$i_{2,0} = \frac{1}{2\pi} \int \frac{d^3 \vec{k} d^3 \vec{q}}{k_0 q_0} \frac{\delta(-\vec{k} - \vec{q}) \delta(E_\nu - k_0 - q_0)}{4[p_\tau \cdot k - \frac{\lambda^2}{2}]^2} . \quad (61)$$

Integration over the photon variable allow us to eliminate one delta function, then eq.(61) reads

$$i_{2,0} = \frac{1}{8\pi} \int \frac{d^3 \vec{q}/q_0}{\sqrt{q_0^2 + \lambda^2}} \frac{\delta(E_\nu - \sqrt{q_0^2 + \lambda^2} - q_0)}{[E_\tau \sqrt{q_0^2 + \lambda^2} + \vec{p}_\tau \cdot \vec{q} - \lambda^2/2]^2} . \quad (62)$$

The next integration is computed in spherical coordinates, first over the azimuth angle ϕ and then over q_0 , then last equation reads

$$i_{2,0} = \int_0^\pi \frac{(E_\nu^2 - \lambda^2) \sin \theta d\theta}{2[E_\tau(E_\nu^2 + \lambda^2) + |\vec{p}_\tau|(E_\nu^2 - \lambda^2) \cos \theta - \lambda^2 E_\nu]^2} \quad (63)$$

Throwing away contributions of equal or greater order than $\mathcal{O}(\lambda^4)$, the previous expression is found to be

$$i_{2,0} = \frac{(x - \lambda^2)}{x^2 m_\tau^2 + 4\lambda^2 (P \cdot p_\tau)^2 - 2\lambda^2 p_\tau^2 x} . \quad (64)$$

Finally the x -integration in eq.(60) is done straightforward

$$I_{2,0} = - \lim_{\lambda \rightarrow 0} \int_{\lambda^2}^{x_+(t,u)} p_\tau^2 i_{2,0} dx = \ln \left[\frac{m_\tau^2 - s}{x_+(s,u)} \right] + \ln \left[\frac{\lambda}{m_\tau} \right] . \quad (65)$$

The computation of $I_{0,2}$ is quite similar, here we just present the result

$$I_{0,2} = - \lim_{\lambda \rightarrow 0} \int_{\lambda^2}^{x_+(s,u)} p_+^2 i_{0,2} dx = \ln \left[\frac{m_\tau^2 - u - s + m_0^2}{x_+(s,u)} \right] + \ln \left[\frac{\lambda}{m_+} \right] . \quad (66)$$

The third integral is computed following the same procedure, first we write it as follows

$$I_{1,1} = \lim_{\lambda \rightarrow 0} \int_{\lambda^2}^{x_+(s,u)} 2p_\tau \cdot p_+ i_{1,1} dx ,$$

$$i_{1,1} = \frac{1}{2\pi} \int \frac{d^3 \vec{q} d^3 \vec{k} \delta(\vec{k} + \vec{q}) \delta(E_\nu - k_0 - q_0)}{4q_0 k_0 [E_\tau k_0 - \vec{p}_\tau \cdot \vec{k} - \lambda^2/2] [E_+ k_0 - \vec{p}_+ \cdot \vec{k} + \lambda^2/2]} . \quad (67)$$

After integrating the delta function $\delta(\vec{k} + \vec{q})$ we get

$$i_{1,1} = \frac{1}{2\pi} \int \frac{d^3 \vec{q} \delta(E_\nu - k_0 - q_0)}{4q_0 k_0 [E_\tau k_0 + \vec{p}_\tau \cdot \vec{q} - \lambda^2/2] [E_+ k_0 + \vec{p}_+ \cdot \vec{q} + \lambda^2/2]} , \quad (68)$$

where $k_0 = \sqrt{q_0^2 + \lambda^2}$. The Feynman trick allows us to combine propagators as follows

$$\frac{1}{[E_\tau k_0 + \vec{p}_\tau \cdot \vec{q} - \lambda^2/2] [E_+ k_0 + \vec{p}_+ \cdot \vec{q} + \lambda^2/2]} = \int_0^1 \frac{dz}{[c + \vec{q} \cdot \vec{p}]^2} \quad (69)$$

where now

$$c = E_+ k_0 + z((E_\tau - E_+)k_0 - \lambda^2) + \lambda^2/2 ,$$

$$\vec{p} = \vec{p}_\tau z + \vec{p}_+(1-z) . \quad (70)$$

After using eq.(69) and the properties of the delta function, the Eq.(68) reads

$$i_{1,1} = \frac{1}{2\pi} \frac{E_\nu^2 - \lambda^2}{2E_\nu^2} \int_0^1 \int_0^{2\pi} \int_0^\pi \frac{\sin \theta d\theta d\phi dz}{4[\bar{c} + \frac{E_\nu^2 - \lambda^2}{2E_\nu} |\vec{p}| \cos \theta]^2}$$

$$= \frac{E_\nu^2 - \lambda^2}{4E_\nu^2} \int_0^1 \frac{dz}{\bar{c}^2 - (|\vec{p}| \frac{E_\nu^2 - \lambda^2}{2E_\nu})^2} , \quad (71)$$

where

$$\bar{c} = E_+ \frac{E_\nu^2 + \lambda^2}{2E_\nu} + z((E_\tau - E_+) \frac{E_\nu^2 + \lambda^2}{2E_\nu} - \lambda^2) + \lambda^2/2 . \quad (72)$$

In order to do the z-integration, the denominator is written as a polynomial function on z,

$$\bar{c}^2 - (|\vec{p}|q_0)^2 = \frac{1}{4E_\nu^2} \left\{ z^2 \left[((E_\tau - E_+)(E_\nu^2 + \lambda^2) - 2E_\nu \lambda^2)^2 - \vec{p}_\tau^2 (E_\nu^2 - \lambda^2)^2 \right. \right.$$

$$+ 2\vec{p}_\tau \cdot \vec{p}_+ (E_\nu^2 - \lambda^2)^2 - \vec{p}_+^2 (E_\nu^2 - \lambda^2)^2 \left. \right]$$

$$+ 2z \left[E_+ (E_\nu^2 + \lambda^2) \left[(E_\tau - E_+) (E_\nu^2 + \lambda^2) - 2\lambda^2 E_\nu \right] \right.$$

$$+ (-\vec{p}_\tau \cdot \vec{p}_+ + \vec{p}_+^2) (E_\nu^2 - \lambda^2)^2$$

$$+ \lambda^2 E_\nu (E_\tau - E_+) (E_\nu^2 + \lambda^2) - 2\lambda^4 E_\nu^2$$

$$+ \left[E_+^2 (E_\nu^2 + \lambda^2)^2 + \lambda^4 E_\nu^2 \right.$$

$$\left. \left. + 2\lambda^2 p_+ \cdot P(E_\nu^2 + \lambda^2) - \vec{p}_+^2 (E_\nu^2 - \lambda^2)^2 \right] \right\}$$

$$= \frac{1}{4E_\nu^2} [\alpha z^2 + 2\beta z + \gamma] , \quad (73)$$

The coefficients, in terms of Lorentz invariant products, are written as follows

$$\begin{aligned}
\alpha &= (x - \lambda^2)^2 (m_\tau^2 + m_+^2 - 2(p_\tau \cdot p_+)) + 4\lambda^4 x \\
&\quad + 4\lambda^2 (p_\tau \cdot P - p_+ \cdot P) ((p_\tau \cdot P - p_+ \cdot P) - (x + \lambda^2)) \\
\beta &= (x - \lambda^2)^2 (p_\tau \cdot p_+ - m_+^2) + 2\lambda^4 \{P \cdot p_\tau - 3P \cdot p_+\} \\
&\quad + 4\lambda^2 \{P \cdot p_\tau P \cdot p_+ - (P \cdot p_+)^2\} - 2\lambda^4 x \\
&\quad + (x - \lambda^2)\lambda^2 \{P \cdot p_\tau - 3P \cdot p_+\} \\
\gamma &= (x - \lambda^2)^2 m_+^2 + \lambda^2 (2P \cdot p_+) (2P \cdot p_+ + x + \lambda^2) + \lambda^4 x
\end{aligned} \tag{74}$$

It is straightforward to find that eq.(71) reads

$$i_{1,1} = \int_0^1 \frac{(x - \lambda^2) dz}{\alpha z^2 + 2\beta z + \gamma} = \frac{(x - \lambda^2)}{2\sqrt{\beta^2 - \alpha\gamma}} \ln \left[\frac{\beta + \gamma + \sqrt{\beta^2 - \alpha\gamma}}{\beta + \gamma - \sqrt{\beta^2 - \alpha\gamma}} \right]. \tag{75}$$

The coefficients given in eq.(74), can be written in terms of the invariant variables (s, u) with the help of the following scalar products

$$\begin{aligned}
P \cdot p_\tau &= \frac{1}{2}(x + m_\tau^2 - s) \\
P \cdot p_+ &= \frac{1}{2}(m_\tau^2 + m_0^2 - s - u) \\
p_\tau \cdot p_+ &= \frac{1}{2}(m_\tau^2 + m_+^2 - u)
\end{aligned} \tag{76}$$

Then with eq.(75) and eq.(76) we write eq.(67) in the conventional form given by Ginsberg¹¹,

$$I_{1,1} = \lim_{\lambda \rightarrow 0} \int_{\lambda^2}^{x_+} dx \frac{E_\tau}{\delta} \ln \left[\frac{x^2 E_\tau + 2\lambda^2 \epsilon_\nu (s - m_\tau^2) + (x - \lambda^2) \bar{\delta}}{x^2 E_\tau + 2\lambda^2 \epsilon_\nu (s - m_\tau^2) - (x - \lambda^2) \bar{\delta}} \right] \tag{77}$$

where

$$\begin{aligned}
\bar{\delta} &= \sqrt{x^2 |\vec{p}_\tau|^2 + \lambda^2 a^2}, & E_\tau &= \frac{m_\tau^2 + m_+^2 - u}{2m_+} \\
a^2 &= -x_+(s, u)x_-(s, u), & |\vec{p}_\tau| &= \frac{\sqrt{\lambda(u, m_\tau^2, m_+^2)}}{2m_+} \\
\epsilon_\nu &= \frac{u + s - m_\tau^2 - m_0^2}{2m_+}
\end{aligned} \tag{78}$$

Recalling the definition given by Ginsber for r_\pm ,

$$\begin{aligned}
r_\pm &= \frac{x_\tau}{m_\tau} \left\{ \Omega_+ \pm \sqrt{\Omega_+^2 - \frac{1}{16} m_\tau^2 a^4} \right\} \\
\Omega_+ &= \epsilon_\nu |\vec{p}_\tau|^2 (s - m_\tau^2) - \frac{1}{4} a^2 E_\tau.
\end{aligned} \tag{79}$$

¹¹Eq.(25) in [39]

and using the master formula¹², we get the final result

$$I_{1,1} = \frac{x_\tau y_\tau}{\sqrt{r_\tau}(1-x_\tau^2)} \left\{ Li_2 \left[\frac{-a^2}{4r_+} \right] - Li_2 \left[\frac{-4r_-}{a^2} \right] + \ln[x_\tau] \ln \left[\frac{\lambda^2}{m_\tau m_+} \right] \right. \\ \left. - \ln[x_\tau] \ln \left[\frac{y_\tau^2 - 4r_\tau}{\sqrt{r_\tau}} \right] - \ln[x_\tau] \ln \left[\frac{x_\tau x_+^2(s, u)}{4r_+} \right] \right\} \quad (80)$$

where

$$r_\tau = \frac{m_\tau^2}{m_+^2}, \quad y_\tau = 1 + r_\tau - \frac{u}{m_+^2} \\ x_\tau = \frac{1}{2\sqrt{r_\tau}} \left[y_\tau - \sqrt{y_\tau^2 - 4r_\tau} \right] \quad (81)$$

Notice this result is useful also for the $\tau^+ \rightarrow K^+ \pi^0 \nu \gamma$ decay, where we must put $m_+ = m_{K^+}$ and $m_0 = m_{\pi^0}$.

Scalar function C0.

In the one loop calculation, it is found the 3-point scalar function C0 that can be evaluated by LoopTools. However, in order to see the cancellation of the IR singularity, the C0 function is written in terms of logarithms and dilogarithms with the help of the general form [32],

$$C0[m_+^2, u, m_\tau^2, \lambda^2] = C0[m_+^2, u, m_\tau^2, \lambda^2, m_+^2, m_\tau^2] \\ = \frac{x_t}{m_+ m_\tau (1-x_t^2)} \left[-\frac{1}{2} \ln^2[x_t] - \frac{\pi^2}{6} + 2 \ln[x_t] \ln[1-x_t^2] \right] \\ + \frac{1}{8} \ln^2[r_t] + Li_2[x_t^2] + Li_2 \left[1 - \frac{x_t}{\sqrt{r_t}} \right] + Li_2[1 - x_t \sqrt{r_t}] \\ - \ln[x_t] \ln \left[\frac{\lambda^2}{m_+ m_\tau} \right] \quad (82)$$

The B0 function is very well known and all the specific cases can be obtained easily from its definition[43].

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