UNIVERSIDAD AUTÓNOMA DE NUEVO LEÓN
FACULTAD DE CIENCIAS FÍSICO-MATEMÁTICAS

Hypothesis test for changes in variance using a statistic based in P-values in a time series of normal independent observations.

Por

Samuel Uriel Armendáriz Hernández

Como requisito parcial para obtener el grado de:

Maestría en Ciencias con Orientación en Matemáticas

JUNIO 2015

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Content

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Abstract

The present work proposes a Hypothesis Test to detect a shift in the variance of a series of independent normal observations using a statistic based on the p-values of the F distribution. Since the probability distribution function of this statistic is intractable, critical values were we estimated numerically through extensive simulation. A regression approach was used to simplify the quantile evaluation and extrapolation. The power of the test was simulated using Monte Carlo simulation, and the results were compared with the Chen test (1997) to prove its efficiency. Time series analysts might find the test useful to address homoscedasticity studies were at most one change might be involved.

CHAPTER 1. - INTRODUCTION

1.1. Introduction

There are not ideal processes, every process presents variability. Quality engineers define quality is defined as the reduction of variability. Due to this, it is necessary to monitor the variability of a process. In some industries, the shift in variance is very important. For instance, a pharmaceutical process cannot exceed some critical amount of an active substance and reduce drastically in the next batch. Maybe the mean will be under control but variability might trigger fatal or ineffective cases. Nevertheless, most of the efforts to detect change points have been developed to monitor change points for a shift in mean. In the literature reviewed in Chapter two, only 7 articles regarding change-points in variance were found, 5 of them were written after 2012.

In summary:

- There is a need in some industries to keep their processes with constant variance.
- There are few tools available today to attach these requirements from a change-point approach.
- The number of researchers developing tools to detect shifts in variance is increasing in the last years. This shows the fact that the statistic community is interested in this field.
- Cordero (2012) and Tercero (2012) have used p-values of chi-squared and F distributions respectively to create control-charts for this purpose.
- Concerning a retrospective analysis, hypothesis tests based on p-values of a shift in variance have not been developed yet.

Due to this, in this research a statistic based on the p-values of iterative F-tests for detecting shifts in variance is proposed.

Statistical Control Process (SPC) studies how to monitor the process and determine whether it is under statistical control to make predictions and strategic decisions. It consists of 2 phases: Phase 1: estimation and Phase 2: detection. Shewhart is considered the father of SPC. In 1931 he published his book called "Economic Control of Quality of Manufactured" where he established the philosophy and designed the first basic statistical tool to monitor the process, the first control charts, where he included a dispersion graph displaying the data, the mean and the upper and lower limit controls. This control chart is currently used by most of the practitioners in the manufacturing as can be read in Western Electric- A Brief history:

"By the turn of the century, Western Electric had trained individuals as inspectors to assure specification and quality standards, in order to avoid sending bad products to the customer. In the 1920's, Western Electric's Dr. Walter Shewhart took manufacturing quality to the next level--employing statistical techniques to control processes to minimize defective output. When Dr. Shewhart joined the Inspection Engineering Department at Hawthorne in 1918, industrial quality was limited to inspecting finished products and removing defective items. That all changed in May 1924. Dr. Shewhart's boss, George Edwards, recalled: "Dr. Shewhart prepared a little memorandum only about a page in length. About a third of that page was given over to a simple diagram which we would all recognize today as a schematic control chart. That diagram, and the short text which preceded and followed it, set forth all of the essential principles and considerations which are involved in what we know today as process quality control." Mr. Edwards had observed the birth of the modem scientific study of process control. That same year, Dr. Shewhart created the first statistical control charts of manufacturing processes, which involved statistical sampling procedures. Shewhart published his findings in a 1931 book, Economic Control of Quality of Manufactured Product".

The Change-point analysis is a branch of SPC which deals with the detection and estimation of changes in series of observations; this is, given a sequence of observations of the variables $(X_1, \dots, X_T)$ with distribution

$$X_i \sim \begin{cases} f(\theta_1), i = 1,2, \dots, \tau \\ f(\theta_2), i = \tau + 1, \dots, T \end{cases}$$

Find τ; the initial moment when a change occurs. In this research, we propose a test for detecting changes in variance in a sequence of independent normal observations.

1.2. History and background of change point analysis for shift in variance

In 1994 Carla Inclan and George Tiao used cumulative sums of squares to search for change points systematically at different pieces of the series. This approach was based on a centered version of the cumulative sum of squares following an algorithm to find multiple change points in an iterative way.

In 1997 Jie Chen and Arjun K. Gupta designed a hypothesis test for shift in variance using Schwarz Information criterion and Maximum Likelihood function of the model. They presented it again in 2001. This test will be deeply studied in Chapter 5.

In 2005 Hawkins and Zamba extended the work of Hawkins et al. (2003) to develop a control chart for shift in variance with the change point methodology and based in the Generalized Likelihood Ratio Test with the Bartlett's correction adapted for its sequential use.

Tercero et al. (2012) developed a generalized LR control chart capable of detecting shifts in variance and estimate the initial moment of this change at the same time using the p-value function of the F statistic. These p-values are used to construct the Hypothesis Test of the present work.

Cordero et al. (2012) presented a generalized likelihood control chart by using the p-value of the Chi-Squared statistic capable of detecting shifts in variance and estimate the initial moment of the change at the same time.

Garza (2013) developed estimators for shift in mean, variance or both and MLE for time series with multiple change points using a construction Heuristic and Genetic Algorithm to assess the problem of finding MLES, which is a NP optimization problem.

Finally, Villanueva (2013) created a non-parametric control chart based in the model proposed by Hawkins and Zamba (2005).

1.3. Problem Statement

Let $X_1, X_2, \ldots X_T$ be a set of independent observations following the next distribution:

$$X_i \sim \begin{cases} N(\mu_1, \sigma_1^2), t = 1,2, \ldots, \tau \\ N(\mu_1, \sigma_2^2),, t = \tau + 1, \ldots, T \end{cases}$$

In this research a hypothesis based on F test is designed to determine if there is a shift in the variance will be developed. The null hypothesis for this test is $H_0 : \tau = T$ and the alternative hypothesis is $H_1 : \tau \neq T$.

Chapter 1 presents the proposal of this research; chapter 2 presents the literature review in change point and SPC. Chapter 3 contains the theoretical framework necessary to develop this research. Readers with a strong statistical background can omit it. Chapter 4 presents the proposed test, an evaluation of the power of this test and a comparison with Chen's test (1997). After this, a Chapter with conclusions and ideas for future work is added. Finally, the appendix shows the Matlab code, a complete table of all the critical values estimated and all the stages simulated for power of the test analysis for proposed test and Chen's test (1997).

1.4. Research questions

As was mentioned in section 1.1, in this research a test to detect changes in variance in a normal sequence of observations based on the P-values of the F test is proposed. To analyze the efficiency of this test the following steps are required:

- Calculate the power of the test proposed with simulated data to validate if really detects a shift in variance when it exists and doesn't detect a shift in variance when it doesn't exist.
- If the test works correctly, it must be compared with another test under the same conditions to decide in the first place the test with the biggest accuracy and secondly some advantages and disadvantages of both tests.

Based on this, the following questions are posed:

Question 1. Is the proposed test an unbiased test?

Question 2. How is the power of the proposed test compared to the power of another similar test?

1.5. Research hypotheses

According to questions posed in section 1.4, the correspondent hypotheses are.

Hypothesis 1. The proposed test is an unbiased test, this is, and the probability of committing a type I error is less than the significance level and the probability of committing a type 2 error is at least that of the significance level.

Hypothesis 2. The power of this test is bigger than at least, the power of another test in literature, this is; the proposed test rejects the null hypothesis more times when it is false while both tests are performed under the same conditions.

1.6. Objective

The main objective of the present research is to develop a new unbiased test for changes in variance based in the p-values of F distribution. To achieve this goal, series of normally distributed data with constant mean and variance must be simulated. Let T be the size of one series and $X_1, X_2, \dots X_T$ be a set of independent normal observations. For each $k;\ 2 \leq k \leq T-2$ we calculate the p value of F test to detect if there is change in variance before and after X_k. We take the smallest p-value, which indicates the point where the probability to have a decrement in variance is maximum. Analogously the biggest p-value indicates the point with maximum probability to have an increment in variance. After replicating this process several times, we will choose specific quantiles for each p-value representing significance level desired in order to design the test statistic.

1.7 Justification

Applied mathematicians solve real life problems that generate value to society using math. Therefore a strict analysis of the potential impact and contributions to science and industry generated with this research is necessary. The present section contains such analysis.

1.7.1 Scientific justification

This research aims to cover the following requirements in science:

- The problem of detecting a shift in variance has received less attention than the detection of changes in the mean. This could be because the power of this kind of tests is very poor.
- The development of this test can be used in a control chart for an online monitoring of the variance of the process.
- Some of the methods that detect shifts in mean assume equality in variances. Therefore, this hypothesis test can be used as a previous step for those methods.

1.7.2 Practical justification

Montgomery (2008), states that quality is inversely proportional to the variability. This is why industry must maintain variance in the least possible level. The first step to attach this goal is to identify the steps of the process presenting uncontrolled variance. After this, decisions must be taken and strategies must be designed according to the situation.

This research will provide a new tool that will help industry to automatize the first step using the state of the art in Change Point Analysis.

Examples of industries whose requirements of variance must be extremely exigent are:

- Pharmaceutical industry.
- Biometrical security systems.
- Jewelry creation.
- Fire alarms.
- Automobile parts.

1.8 Scope and limits

This model only works supposing independent normally distributed data. However, this technique can be generalized to other distributions using different statistics based on p-values. Moreover equality in means $\mu_1 = \mu_2$ is assumed.

CHAPTER 2. LITERATURE REVIEW

In this section a complete literature review is presented. In order to introduce the reader gradually, this section begins presenting contributions to Change Point analysis in general and finalizing in those works closely related with the hypothesis test developed here.

Thus the subsections of this chapter are:

- A general overview of change point analysis (2.1)
- Tests for shift in variance (2.2)
- GLR control charts (2.3)
- P value based tests and charts (2.4)

Table 1 presents all the sources chronologically with a brief description to ease the search of a specific paper. This table does not include textbooks consulted for section 3. These books are listed in that section.

Author, Year	Contribution
(Page, 1954)	Schemes for detecting shift in mean.
(Page, 1955)	Hypothesis test for change in mean.
(Page, 1957)	Hypothesis test for change in mean of a normal distribution.
(Hinkley, 1970)	Detecting change point in mean for normal observations using MLE.
(Sen & Srivastava, 1975)	Bayesian statistics and statistics depending of estimations of change point for hypothesis test for change in variance are shown. Then they find distribution function for some of the Bayesian statistics.
(Hawkins, 1977)	Hypothesis test for change in mean.
(Pettitt, 1979)	Hypothesis test using non parametric techniques for change in distribution.
(Bhattacharya, 1994)	Hypothesis test for change in mean.
(Inclan & Tiao, 1994)	Iterated cumulative sums of squares algorithm to detect shift in variance.
(Chen & Gupta, 1997)	Hypothesis test for change in variance. Is the test chosen to compare with Proposed Power test.

(Samuel, Pigniatello, & Calvin, 1998)	Control chart for detecting change in mean.
(Chen & Gupta, 2001)	Some hypothesis tests, among them there is one of shift in variance.
(Hawkins, Qiu, & Wook, 2003)	Methodology for detecting and diagnosing step changes based on imperfect process knowledge for change point in mean/
(Hawkins & Zamba, 2005)	Developed a control chart for detecting shifts in variance.
(Amiri & Allahyari, 2011)	A complete literature review.
(Fotopoulos, Jandhyala, & Kahpalova, 2010)	Derived exact computable expressions for the asymptotic distribution of the change point MLE for change in mean.
(Tercero, Martinez, & Ramirez, 2012)	Development a generalized LR control chart capable of detecting shifts in variance and estimate the initial moment of this change at the same time using the p value function of the F statistic.
(Cordero & Gonzalez, 2012)	Generalized likelihood control chart designed by using the P Value of the Chi Squared statistic capable of detecting shifts in variance and estimate the initial moment of the change at the same time.
(Tercero, Garza, & Cordero, 2013)	Hypothesis test for change in mean.
(Villanueva, 2013)	Non parametrical control chart for change point in variance.
(Garza, 2013)	Estimators for shift in mean, variance or both and estimation of multiple change points in time series normally distributed using a construction Heuristic and Genetic Algorithm.
(Perez, 2013)	Doctoral thesis formed by 3 researches Using sequential control charts for detecting shift and variances.
(Zhou, Zou, Zhang, & Wang, 2009)	Non parametric control chart based on change point methodology using the Mann-Whitney statistic and adding a EWMA to detect sifts in mean.

Table 1: Literature review

2.1 Change point analysis.

To analyze CPA literature the main focuses must be clearly defined

- **Classical analysis** of Change point problem implies the use of descriptive and inferential statistical tools to realize change point estimations. This approach considers known the data distribution although its parameters maybe unknown.
- **Non parametric analysis** supposes that function of distribution cannot be defined a prior. This adds complexity to the problem of detect if the process is under statistical control because parametric statistical inference methods are not valid here.

2.1.1 Classical Analysis

One of the pioneers in this area is Page. In 1954 he presented a control chart based in the cumulated sum of observations (CUSUM), reporting bigger sensibility to small and sustained changes than Shewhart control Charts. In the next years he continued publishing in this area.

In 1970 Hinkley founds the basis for using MLE to estimate the change point, which implies the use of derivatives to determine maximum and minimum values.

In 1998 Samuel, Pignatiello and Calvin presented an alternative for change point detection from the use of MLE for shift in mean once the control chart detects that process is out of statistical control.

2.1.2 Non parametric analysis.

Again Page is a pioneer in this field proposing a test based in the redefinition of a variable and binomial distribution in 1955.

Bhattacharya began to develop non parametric procedures in the way known today using Wilcoxon's signal test. Pettit (1979) worked with this approach for Bernoulli and Binomial distribution functions.

2.2 Tests for shift in variance.

In 1997 Jie Chen and Arjun K. Gupta designed a hypothesis test for shift in variance using Schwarz Information criterion and Maximum Likelihood function of the model. Performance of this test is very similar to CUSUM based tests.

Perez (2013) published used a self-start control chart and MLE for Sequential detection and estimation of sustained shifts in mean and variance of time series for normally distributed observations.

On the other hand Garza (2013) used a construction Heuristic and Genetic Algorithm for designing estimators for shift in mean, variance or both and estimation of multiple change points in normally distributed time series in order to assess the problem of finding MLES, a NP optimization problem.

2.3 GLR control charts

Hawkins and Zamba (2005) developed a GLR test based on Bartlett's test for mean and variance of normally distributed data using only one test for all cases: change only in mean, only in variance or both parameters for sequential use.

Zou, Zou, Zhang & Wang (2009) developed a non-parametric control chart based in Hawkins (2003) change point methodology using the Mann-Whitney statistic and adding a EWMA to detect sifts in mean.

Villanueva (2013) used Hawkins methodology (2005) with a non-parametric statistic to create a control chart for shit in variance and finding correspondent control limits.

2.4 P-value based tests and charts.

Cordero (2012) et al. developed a generalized control chart based in the P value of chi squared test with the objective of detecting and estimating shifts in variance in normal process in phase II based on the GLR procedure. This GLR chart for variance in a normal process takes into account the knowledge of the process resulting in a chart with better performance than similar charts.

Meanwhile Tercero et al. (2012) created a Generalized Likelihood Ratio chart capable of detecting shifts in variance and estimate the initial moment of this change at the same time. They used a moving window instead an ever growing set of observations and the P value for F test was taken to build an estimator and a statistic to be controlled. This method works in Phase I of SPC.

After observing the evolution of CPA and SPC the next step is to design a Hypothesis test based on P-values of F distribution. If the performance of this test is the expected, many other tests using P-values of other distributions can be designed.

CHAPTER 3. THEORETICAL FRAMEWORK

The Hypothesis test proposed in the present work is based on the P-values of the F distribution. In order to make this contribution accessible to most of the readers the present Chapter is included. This chapter covers the following themes:

- **Hypothesis tests** (3.1). This concept is the heart of the research because we are creating a new hypothesis test.
- **P-values** (3.2). The hypothesis test is based on P-values.
- **The F distribution** (3.3). The P-values used in this research come from F distribution. Future researches probably will need P-values from other distributions.
- **Regression Analysis** (3.4). Regression was need to fit a curve over obtained quantiles.

Readers with strong statistical basis can omit this section without problem.

3.1 Hypothesis tests

A **statistical hypothesis** is a conjecture about one or more random variables. For instance, if a smartphone manufacturing company wants to choose if the time of discharge of the battery when the smartphone is not used is at least 24 hours, they have to prove the hypothesis $\theta \geq 24$ where θ is the parameter of the exponential distribution.

To achieve this we need to define a rules set to decide if we accept or reject our hypothesis. This set of rules is called the **Hypothesis Test**. We will call H_0 to the hypothesis to be proved and H_1 to the alternate hypothesis (when H_0 is not accepted).

When testing the hypothesis we can make 2 types of errors. The rejection of H_0 when is true is called the **type 1 error**. The probability of committing a type 1 error is denoted with α. On the other hand if we accept H_0 when is false we are making a **type 2 error**. The probability of doing this kind of error is denoted by β.

The rejection region of H_0 is called the critical region of the test. The probability of obtain a value of the statistic inside this region when $H_0$is true is called the size of the critical region, this is α. This probability is also called the **significance level**.

A good hypothesis test has small α and β. If we maintain the size of the sample constant, the type 1 error decreases when type 2 error increases and vice versa. The only way to reduce both errors is increasing the size of the sample.

The **power of the test** is defined by the probability of rejecting H_0 inside and outside the critical region, this is:

$$\pi(\theta) = \begin{cases} \alpha(\theta), & \theta \in H_0 \\ 1 - \beta(\theta), & \theta \in H_1 \end{cases}$$

This function is useful to compare two tests. A test is better than other wen it has the smallest probability of committing a type 1 error β.

3.2 P-values

The p value is a function of the observed sample results (a statistic) that is used for testing a statistical hypothesis. If the P value is equal or smaller than the significance level, it suggests that the observed data are inconsistent with the assumption that the null hypothesis is true and thus that hypothesis must be rejected. When p value is calculated correctly, such a test is guaranteed to control the type 1 error rate to be no greater than α.

To summarize, the p- value is the extreme value where we can accept the null hypothesis. If the statistic obtains a value less than the P-value, is very probable that you are outside of the acceptance region. In terms of this research, getting a statistic's value less than the P-value is very probable that there is a change in variance. If we take the least p value, we are obtaining the point in which a shift in variance is the most probable.

3.3 The F distribution

3.3.1 Definition

The F distribution plays a very important role in the sampling of normal data. This distribution is named in honor of Sir Ronald A. Fisher, one of the most prominent statisticians of the past century. F distribution is usually defined in terms of χ^2 distribution.

If U and V are random chi- squared variables with independent distributions and ν_1, ν_2 degrees of freedom respectively then

$$F = \frac{U/\nu_1}{V/\nu_2}$$

Is a random variable with F Distribution, this is, a random variable whose density is given by:

$$g(f) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}(\frac{\nu_1}{\nu_2})^{\frac{\nu_1}{2}} \cdot f^{\frac{\nu_1}{2}-1}(1+\frac{\nu_1}{\nu_2}f)^{-\frac{1}{2}(\nu_1+\nu_2)}$$

For $f > 0$ and $g(f) = 0$ in any other case.

This result is applied in situations where we are interested in comparing the variance of 2 populations. For instance the problem of estimating $\frac{\sigma_1^2}{\sigma_2^2}$ or validate if $\sigma_1^2 = \sigma_2^2$.

3.3.2 F test

An F Test is a statistical test in which the test statistic has an F distribution under the null hypothesis.

Given random samples of normally distributed independent variables of sizes n_1 and n_2 and variances σ_1^2, σ_2^2 the critical region to prove the null hypothesis $\sigma_1^2 = \sigma_2^2$ versus the alternative $\sigma_1^2 \neq \sigma_2^2$ are:

- $\frac{S_1^2}{S_2^2} \geq f_{\alpha,n_1-1,n_2-1}$ If $S_1^2 \geq S_2^2$.

- $\frac{S_2^2}{S_1^2} \geq f_{\alpha,n_2-1,n_1-1}$ If $S_1^2 \leq S_2^2$.

3.4 Regression Analysis

The distribution of the test statistic is unknown yet. However, quantiles can be estimated through simulations, where curves can be fitted as a function of the sample size. To fit a curve is useful for two reasons: first, is more practical to work with a simple equation than using a huge table. In the other hand, an equation allows users to calculate values of the test statistic for sizes whose computation time make them unpractical to calculate in production line.

The chosen tool for this purpose is regression analysis; a statistical process for estimating relationship among variables.

In the present work we will study the behavior of a dependent variable and an independent variable.

3.4.1 Linear Simple Regression

The simplest case is the Simple Linear Regression. Here we have a relationship of the form

$$Y_i = \alpha + \beta x_i + \epsilon_i$$

Where Y_i is a random variable and x_i is just another observable variable.

The difference between $\widehat{Y_i} = \alpha + \beta x_i$ and the actual observation Y_i is called the error ϵ_i. Our objective when fitting a curve is to minimize the sum of errors. But when we have both positive and negative errors they can eliminate between them and small sum or mean of errors is not always synonym of the best curve fitted. Instead of this, Statisticians minimize the sum of square of the errors (SSE), since this is always a nonzero quantity, offering the same advantages that variance add to mean. Due to this the method for calculating slope and interception parameters is called the Method of Minimum Squares.

3.4.2 Regression with transformed variables.

A function that relates y with x is intrinsically linear if after transforming one or both variables they have a linear relationship. The next table resume the most useful intrinsically linear functions. When appears “log” you can use logarithm to the base 10 or e.

Function	Transformation(s).	Linear form
a) Exponential $y = \alpha e^{\beta x}$	$y' = \ln(y)$	$y' = \ln(\alpha) + \beta x$
b) Power $y = \alpha x^{\beta}$	$y' = \log(y)$ $x' = \log(x)$	$y' = \log(\alpha) + \beta x'$
c) Logarithmical $y = \alpha + \beta \log(x)$	$x' = \log(x)$	$y = \alpha + \beta x'$
d) Reciprocal $y = \alpha + \frac{\beta}{x}$	$x' = \frac{1}{x}$	$y = \alpha + \beta x'$

Table 2: the most useful intrinsically linear functions

The next image is taken from Devore’s book and will help us to select the most appropriate transformation for each model.

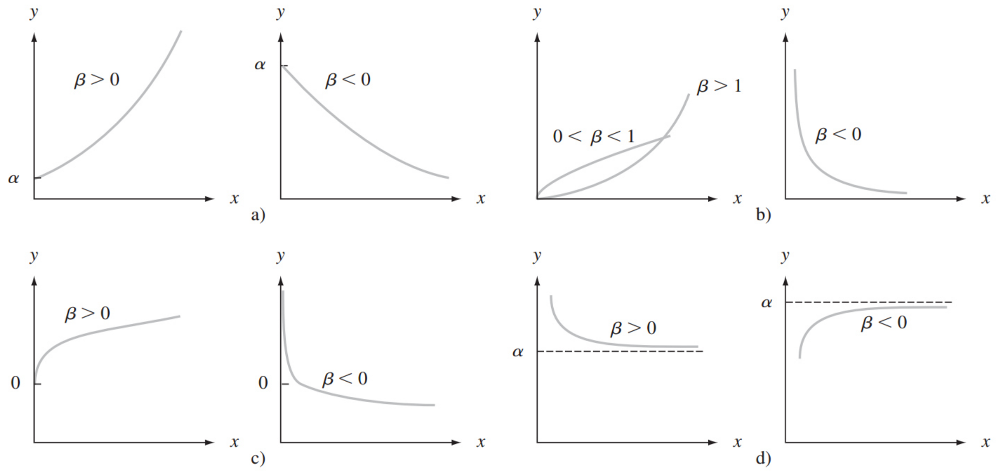


Figure 1: Graphs of intrinsically linear functions in Table 2

In this section the statistical background necessary to develop this research was presented. The information provided in this section is an introductory description of key topics. For further information the sources consulted are listed below.

- **“Estadística matemática con aplicaciones”** (Freund & Miller, 1999).
- **“Probabilidad y estadística para ingeniería y ciencias”** (Devore, 2008).
- **“Statistical inference”** (Casella & Berger, 2002).

CHAPTER 4. PROPOSED MODEL

This section describes the proposed test for changes in variance. To attach this goal with the proper order this research is structured in the following way.

- **Model (4.1).** The model is formally enunciated, the methodology is extensively described and the S statistic is defined.
- **Quantile estimation (4.2).** Most important quantiles are numerically calculated for sample sizes of at most 500.
- **Regression (4.3).** A curve fitting method is selected and calculated in order to predict values of S statistic for sample sizes bigger than 500.
- **Power of the test (4.4).** The performance of the test is calculated under some stages and results are described.
- **Comparison with Chen's test (4.5).** Finally, the performance of this test is compared to the performance pf the test presented by Chen & Gupta (1997).

In order to avoid making this work too extensive, in this section are presented only the most representative results. To consult the complete results of the simulations and the MATLAB ® codes please consult the appendix.

4.1 Model

Let $X_1, X_2, \dots\ X_T$ be a set of independent observations following the next distribution:

$$X_i \sim \begin{cases} N(\mu_1, \sigma_1^2), t = 1,2, \dots, \tau \\ N(\mu_1, \sigma_2^2),, t = \tau + 1, \dots, T \end{cases}$$

Let be $H_0: \sigma_1^2 = \sigma_2^2$ and $H_1: \sigma_1^2 \neq \sigma_2^2$and suppose that τ, the change-point is unknown.

For each $X_k; 3 \leq k \leq T - 2$ we estimate the mean and variance prior and after the change point:

$$\hat{\mu}_{0,T,k} = \frac{1}{k} \sum_{i=1}^{k} X_i$$

$$\hat{\mu}_{1,T,k} = \frac{1}{T-k}\sum_{i=k+1}^{T} X_i$$

$$\hat{\sigma}^2{}_{0,T,k} = \frac{1}{k-1}\sum_{i=1}^{k}(X_i - \hat{\mu}_{0,n,k})^2$$

$$\hat{\sigma}^2{}_{1,T,k} = \frac{1}{T-k-1}\sum_{i=k+1}^{T}(X_i - \hat{\mu}_{1,n,k})^2$$

And the statistic is computed as follows for each $X_k; 3 \leq k \leq T-2$.

$$r_{k.T} = \frac{\hat{\sigma}^2{}_{1,T,k}}{\hat{\sigma}^2{}_{0,T,k}}$$

This statistic follows a F_{ν_1,ν_2} distribution where $\nu_1 = T-1-k$ and $\nu_2 = k-1$.

The P-value $p_{T,k}$ for this test is obtained by:

$$p_{T,k} = P(F_{V_1,V_2} > r_{k.T})$$

We reject H_0 with type 1 error equal to α if

$\min(p_{T,k}) < S_{T,\alpha}$ For left tail.

$\max(p_{T,k}) > S_{T,\alpha}$ For right tail.

Where $S_{T,\alpha}$ is the quantile of the p-value distribution for size= T and significance level α.

To guarantee this type 1 error we must find the values of $S_{T,\alpha}$ under H_0 such that

$\mathrm{P}(\min(p_{T,k}) < S_{T,\alpha}) = \alpha$ For left tail.

$P(\max(p_{T,k}) > S_{T,\alpha}) = \alpha$ For right tail.

Given that the distribution of $\min(p_{T,k})$ and $\max(p_{T,k})$ are intractable, it is impossible to calculate the values of $S_{T,\alpha}$. Therefore thes values will be obtained in the next section through a simulation.

4.2 Quantile estimation

The exact distribution of the statistic based on the p-values is unknown yet, and then is needed to calculate some quantiles of this distribution numerically in order to use them as critical values of the statistic.

For each $5 \leq T \leq 500$ we calculate the statistic as shown in the section 4.1 and save the results in the vector p. Then this vector is sorted in increasing order. We take the biggest and the smallest element of p. This process is repeated 1000 times and saved in two vectors: $t(1000)$ for the biggest p values and $b(1000)$ with the smallest. These vectors are sorted again. Finally we take the elements $b(10)$, $b(50)$, $b(100)$, $b(200)$, $t(801)$, $t(901)$, $t(951)$ and $t(991)$, which represent the Quantiles $S_{.01}$, $S_{.05}$, $S_{.1}$, $S_{.2}$, $S_{.8}$, $S_{.9}$, $S_{.95}$ and $S_{.99}$. These elements were saved in an excel file. The time of execution of this program was of over 2 weeks. Matlab code used to implement this algorithm can be consulted in the Appendix 1 and the complete table of all Quantiles can be consulted in the Appendix 2.

Figure 2 and figure 3 display plots of the data horizontal axis represent the size of the sample and vertical axis represents the P-value.

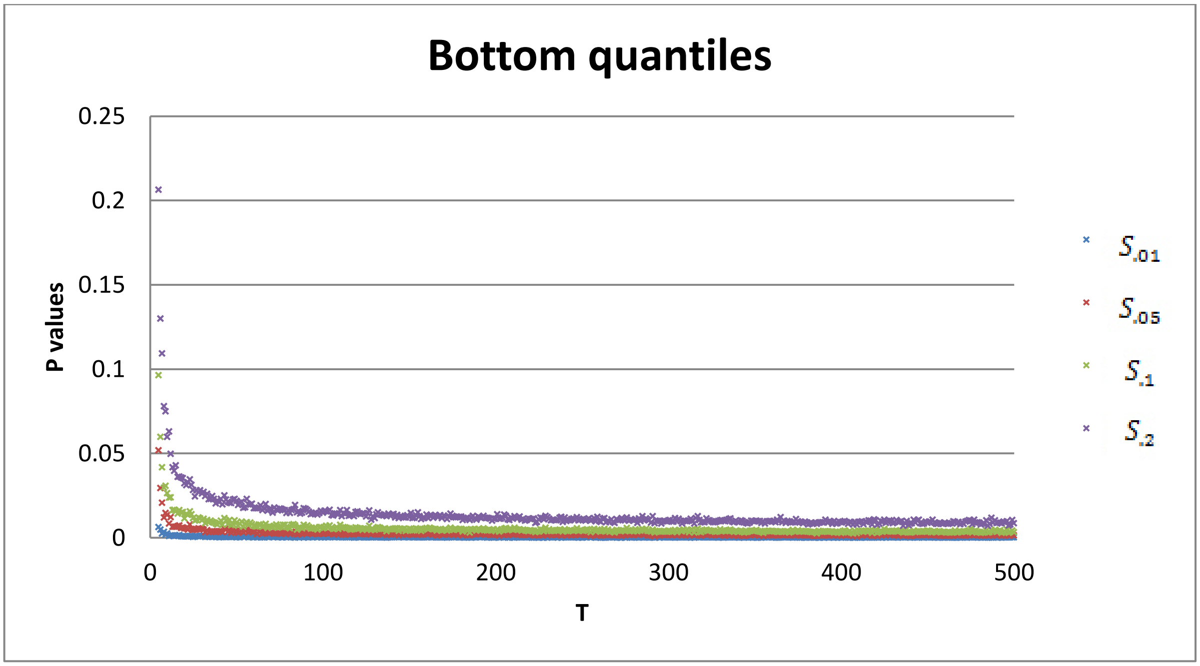


Figure 2: Bottom quantiles of statistic based on p-values

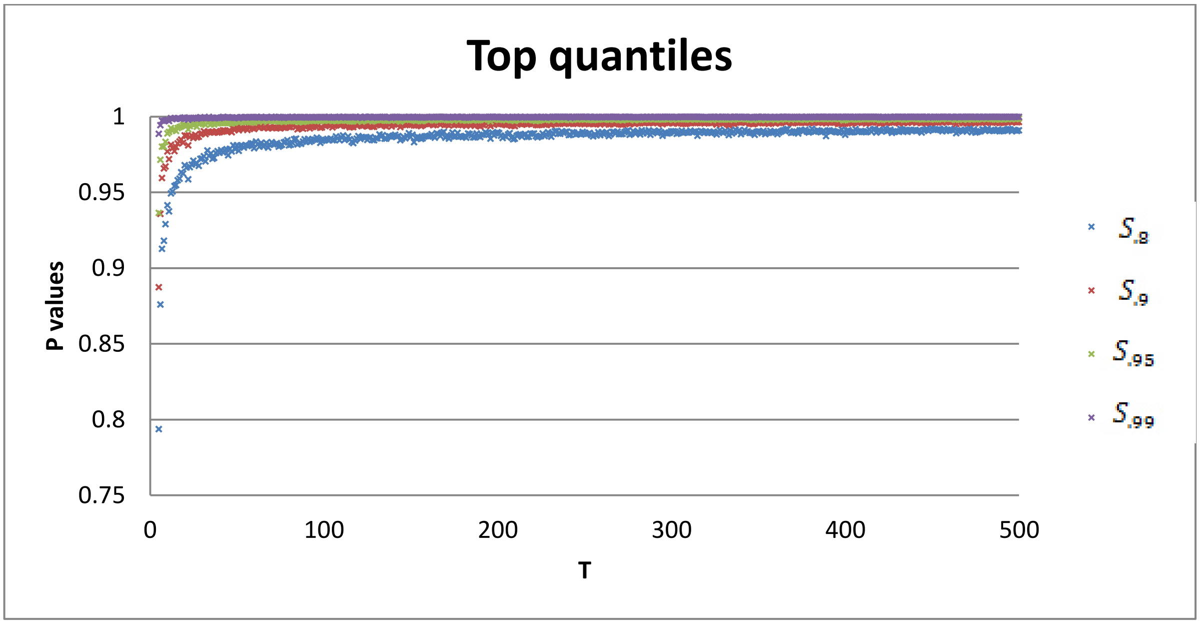


Figure 3: Top quantiles of statistic based on p-values

The exact distribution of the quantiles is unknown yet. As was mentioned in section 3.4, is necessary to fit a curve for two reasons: first, is more practical to work with a simple equation than using a huge table and having an equation allows users to calculate values of the test statistic for sizes whose computation time make them unpractical to calculate in production line.

According to the data visualization in figure 3 and 4 and comparing with figure 1 a reciprocal transformation is suggested.

4.3 Reciprocal regression.

The transformation consists in make $T' = \frac{1}{T}$ and $p' = p$. If this transformation is the most appropriate then n' and p' will be linearly related and both regressions will have the same parameters, this is, if $p' = aT' + b \implies p = \frac{a}{T} + b$. After doing this transformation we obtain the following correlation coefficients (Table 3):

Quantile 10/1000	0.917876215
Quantile 50/1000	0.90538046
Quantile 100/1000	0.938721886

Quantile 200/1000	0.958885348
Quantile 801/1000	-0.960948301
Quantile 901/1000	-0.925966965
Quantile 951/1000	-0.884818859
Quantile 991/1000	-0.860428257

Table 3: Correlation coefficients after reciprocal transformation.

Figure 4 displays the data after the transformation.

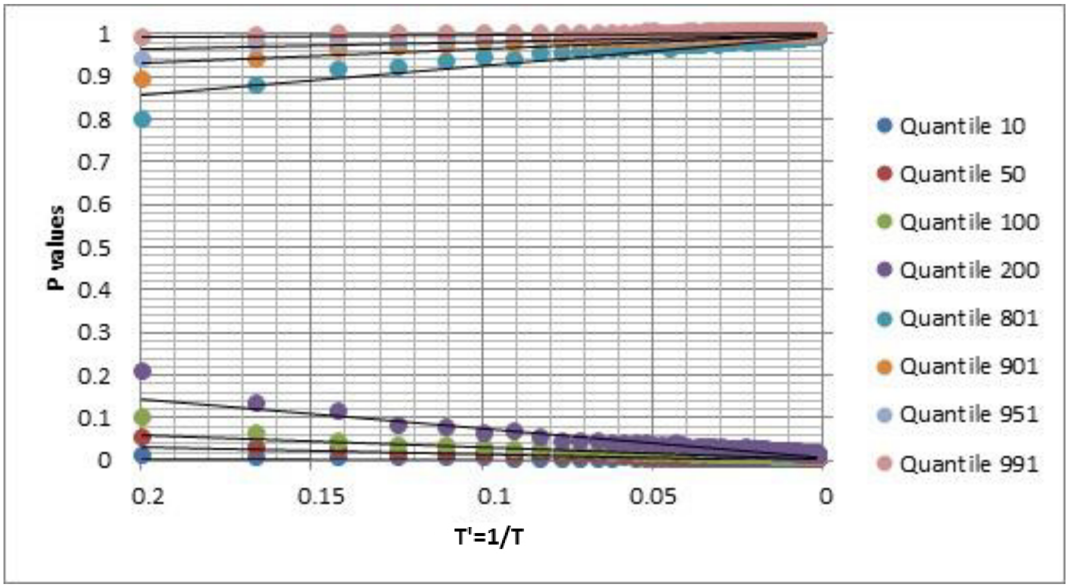


Figure 4: Quantiles after reciprocal transformation

An exhaustive analysis was required to ensure the accuracy of this regression technique. For this purpose we compared this method with the proposed by Hawkins (2005) and used by Villanueva (2005).

1. Give initial values to the next equation $p(T) = a + b\ln.05 + \frac{c+d\ln.05}{\sqrt{T-e}}$ where 0.05 is the significance level.
2. Calculate the sum of the square errors.
3. Using the SOLVER included in the MS Excel software we obtain the minimum value of the sum of the square errors by varying the values of the parameters of the equation.

The reciprocal regression had a better performance with smaller sum of squares of errors, smaller maximum absolute error and the errors didn't show tendency as in the Hawkins equation. For $S_{.8}$ proposed test obtained sum of square errors of 0.006536435 against 0.17181223 obtained by Hawkins regression. Maximum absolute error for proposed test was 0.004073487 and for Hawkins regression was 0.011019286. Figure 5 represents the errors of the regression and figure 6 contains the plots of both curves. Observe that reciprocal equation gets stable after T=50.

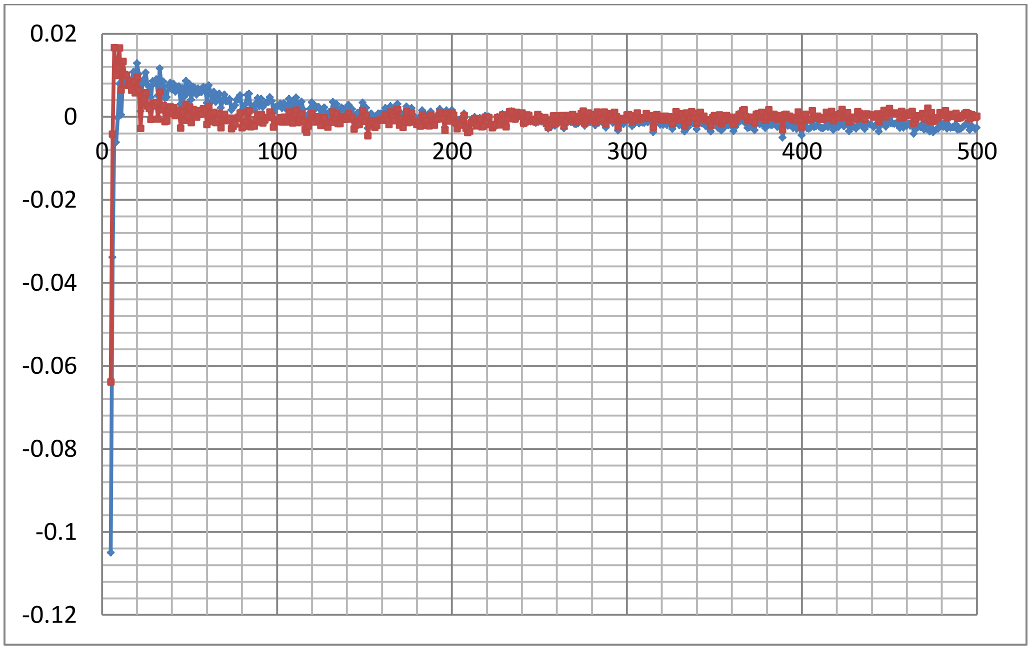


Figure 5: Hawkins (blue) versus Reciprocal (red) regression error.

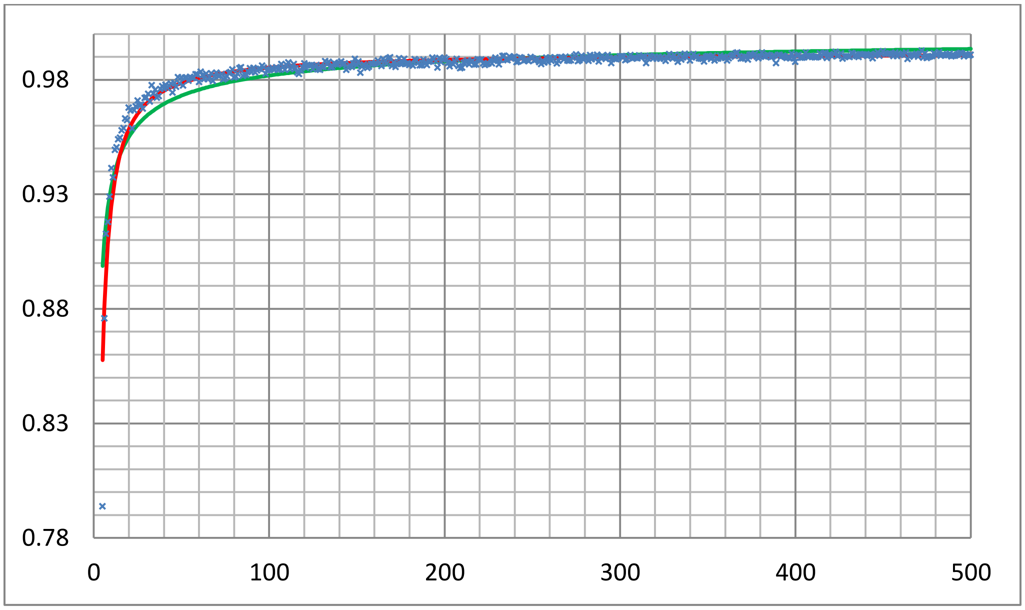


Figure 6: Hawkins fitted curve (green) vs. Reciprocal fitted curve (red).

Once we proved the efficiency of this regression, we eliminate the first 50 data for all the Quantiles. Then we repeated the regression process obtaining very accurate equations for $T \geq 51$. Table 4 contains the equation, sum of square errors and maximum absolute error for each quantile.

Quantile	P value equation	Sum of square errors	Maximum absolute error
$S_{.01}$	$y = 0.0215/T + 0.0002$	5.28509×10^{-6}	1.53856×10^{-7}
$S_{.05}$	$y = 0.1259/T + 0.0013$	5.13829×10^{-5}	1.56216×10^{-6}
$S_{.1}$	$y = 0.2773/T + 0.0032$	0.000153936	3.83051×10^{-6}
$S_{.2}$	$y = 0.6848/T + 0.0079$	0.000481205	1.10268×10^{-5}
$S_{.8}$	$y = -0.7059/T + 0.9918$	0.000571248	1.67328×10^{-5}
$S_{.9}$	$y = -0.2788/T + 0.9967$	0.000162006	3.60606×10^{-6}
$S_{.95}$	$y = -0.1172/T + 0.9986$	5.27951×10^{-5}	1.52534×10^{-6}
$S_{.99}$	$y = -0.0184/T + 0.9998$	5.50215×10^{-6}	2.44966×10^{-7}

Table 4: Quantile equations.

And for $T \leq 50$ the exact quantiles are presented in table 5.

T	$S_{.01}$	$S_{.05}$	$S_{.1}$	$S_{.2}$	$S_{.8}$	$S_{.9}$	$S_{.91}$	$S_{.99}$
5	0.006281654	0.051898567	0.096447892	0.206450389	0.793816129	0.887433142	0.93650131	0.988702562
6	0.00491284	0.02956886	0.059877712	0.129978907	0.875907015	0.93576931	0.971488867	0.994266561
7	0.002617857	0.02076724	0.041913994	0.109310511	0.912708253	0.959506561	0.980235353	0.997326592
8	0.003527766	0.012120241	0.029634693	0.07805619	0.917976273	0.965748537	0.980092467	0.997168952
9	0.001864831	0.014705964	0.030908449	0.07507764	0.929010629	0.96701141	0.983398764	0.997547166
10	0.00262071	0.013665	0.026528795	0.059773781	0.941394135	0.976895856	0.988920762	0.997946414
11	0.001081597	0.008486449	0.023856217	0.063039043	0.93744109	0.971876594	0.989971893	0.997107917
12	0.001444736	0.012250571	0.023979854	0.049865043	0.94934864	0.981511799	0.992404698	0.998986848
13	0.001499019	0.006616104	0.016403075	0.041864665	0.95053208	0.979062224	0.990728806	0.998429037
14	0.001123607	0.006646856	0.016699654	0.039742909	0.954068979	0.977275643	0.991779828	0.998916939
15	0.001310778	0.006593983	0.015941447	0.042956757	0.954761443	0.979986274	0.990568345	0.998497937
16	0.001195564	0.006848465	0.014358527	0.036191175	0.95786007	0.982176686	0.992472446	0.998963121
17	0.001656906	0.007259888	0.016415925	0.036268004	0.95893534	0.983489889	0.993102043	0.998617629
18	0.000903789	0.005887336	0.014974443	0.035831466	0.963154232	0.984916326	0.993437959	0.998521705
19	0.000782177	0.006177839	0.015346783	0.03534597	0.962448541	0.982778637	0.994373668	0.999109327
20	0.000859101	0.005585629	0.011574396	0.03228666	0.967915639	0.98763408	0.994482116	0.999329755
21	0.000956109	0.005664029	0.014158796	0.031213293	0.966523406	0.987304452	0.994054712	0.999059505
22	0.000823876	0.004862915	0.013559189	0.032825859	0.958699899	0.981092629	0.99156066	0.997859694
23	0.001057962	0.007899986	0.015645745	0.034549283	0.966700154	0.986571085	0.993683444	0.998766683
24	0.001034028	0.005263389	0.01396847	0.031418554	0.968405768	0.987591285	0.995292944	0.998958552
25	0.000529317	0.004629828	0.011682942	0.028380448	0.970947664	0.985921173	0.994427718	0.999110206
26	0.000695347	0.004849328	0.010172928	0.024651011	0.968830476	0.9880692	0.995258377	0.999139719
27	0.001026647	0.004964639	0.010676741	0.028177858	0.969129611	0.986366926	0.99335473	0.99864106
28	0.000831316	0.005570962	0.012152271	0.027057955	0.967437045	0.986739241	0.995080043	0.998988153
29	0.001150428	0.005266861	0.011415616	0.028258208	0.972054137	0.988476346	0.995536049	0.999537906
30	0.001003513	0.005510761	0.010328261	0.026273203	0.972264895	0.988820264	0.995745907	0.999242244
31	0.001275103	0.004763208	0.01131335	0.027210968	0.973842773	0.990222848	0.995983628	0.999279086
32	0.000525106	0.003253471	0.008603691	0.024796291	0.970492529	0.988422713	0.994513464	0.9990019
33	0.000687567	0.004480272	0.010519031	0.026138531	0.977638756	0.989624002	0.995922962	0.99911675
34	0.000723594	0.003390081	0.00922889	0.022814853	0.973932587	0.989851813	0.996396042	0.999802381
35	0.00083763	0.003766593	0.010192387	0.02308011	0.975620429	0.989612591	0.99492796	0.999024042
36	0.000510157	0.003845853	0.009348099	0.024744708	0.972184613	0.990063453	0.99560023	0.999127342
37	0.000630365	0.003802335	0.009006464	0.023009365	0.972837663	0.989391566	0.994735229	0.998894711
38	0.00051203	0.003329733	0.008538765	0.020390194	0.975966291	0.990257631	0.995697764	0.999267393
39	0.000740834	0.003677159	0.007954698	0.022120167	0.977219352	0.989482758	0.995499706	0.999282649
40	0.000381906	0.003897783	0.008909007	0.02112644	0.976045976	0.98961907	0.995967404	0.999048987
41	0.000505817	0.003968294	0.010104511	0.022988032	0.977804994	0.990407413	0.99589944	0.999383911
42	0.000402911	0.003310108	0.00771358	0.019878728	0.976346536	0.990882712	0.995811993	0.999722945
43	0.000671136	0.005381887	0.011670857	0.025134899	0.977555129	0.990658728	0.995812004	0.999306576
44	0.00042473	0.004940848	0.010346605	0.022613644	0.978370224	0.990138391	0.995848153	0.999443281

45	0.000553331	0.003456211	0.00804661	0.021297296	0.97442729	0.990354526	0.995806498	0.999534078
46	0.000490521	0.005386224	0.010489919	0.020703091	0.978664269	0.990757144	0.995817819	0.999496304
47	0.000424423	0.002913309	0.00750276	0.020551616	0.976884812	0.989733613	0.995391889	0.999325336
48	0.000769527	0.003847857	0.00813313	0.022625716	0.981062925	0.991866308	0.996121989	0.999216434
49	0.000922083	0.004553121	0.010393097	0.023091529	0.97844255	0.991522113	0.996163899	0.99927986
50	0.000350961	0.004021736	0.009062285	0.021870338	0.980962703	0.993302279	0.996933472	0.999431223

Table 5: Quantile table for T<51

4.4 Power of the test

Power of the test allows us to analyze the performance of the test, this is, verify is the test rejects the null hypothesis when alternative hypothesis is correct. To accomplish this proposed test must be executed under stages when variances change (alternative hypothesis). Thus the power of the test was calculated with all the combination of the next stages.

- Size of the series (T) of 20,30,100,300
- Change point localization (τ/T) of 0.1, 0.3 and 0.5
- Shifts in variance (σ_0^2 / σ_1^2) of $1.25^{-3}, 1.25^{-2}, 1.25^{-1}, 1$ and $1, 1.25, 1.25^2, 1.25^3$

Each stage is replicated 1000 times and the probability $\pi(\theta)$ is calculated by the times that null hypothesis is rejected divided by 1000.

To achieve this we used a Matlab® script. The results were saved as excel tables and jpg pictures. The time of execution for this program was 7783.116291 seconds.

Table 6 and 7 show the results for specific stages for two tails. Horizontal axis represents shifts in variance and vertical axis represents the probability of rejecting the null hypothesis.

The complete list of figures can be consulted in Appendix 3.1.

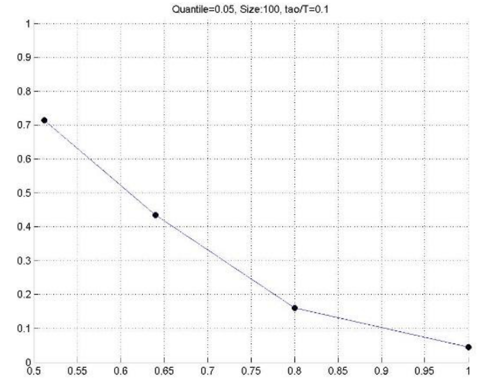
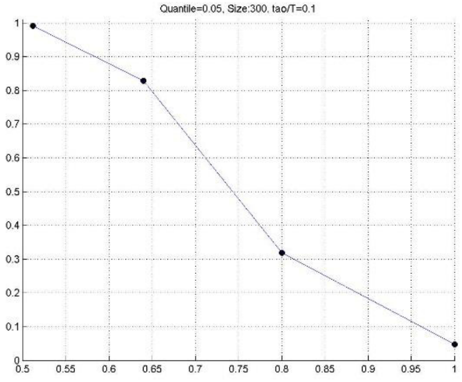
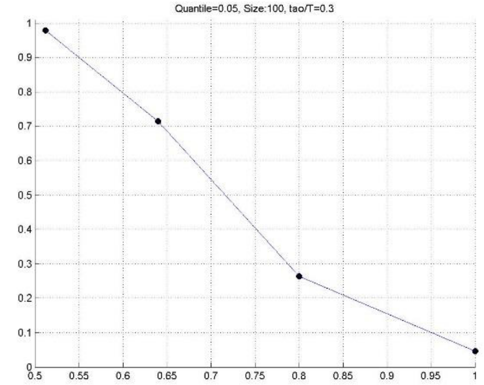
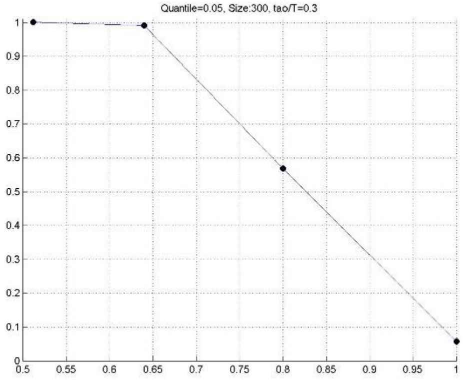
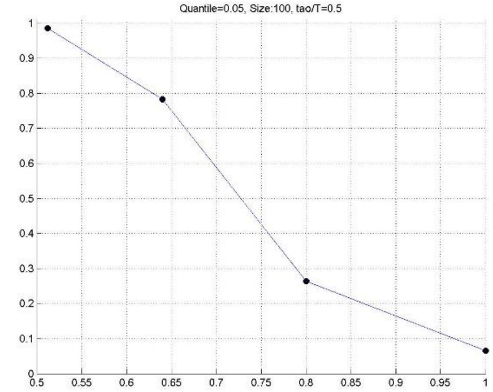
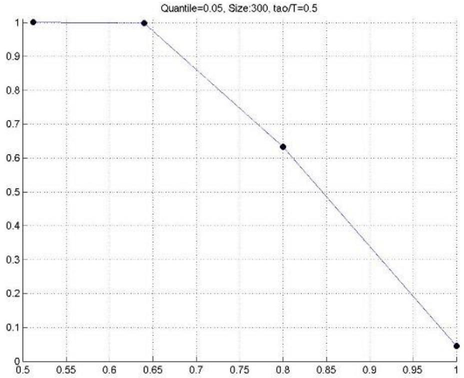
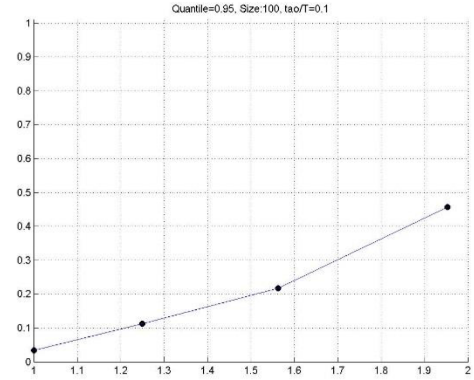
$\alpha = 0.05$	$T = 100$	$T = 300$
$\frac{\tau}{T} = .01$		
$\frac{\tau}{T} = .03$		
$\frac{\tau}{T} = .05$		

Table 6: Power of proposed test for α=0.05

<table>
<tr><th>$\alpha = 0.95$</th><th>$T = 100$</th><th>$T = 300$</th></tr>
<tr><td>$\frac{\tau}{T} = .01$</td><td>


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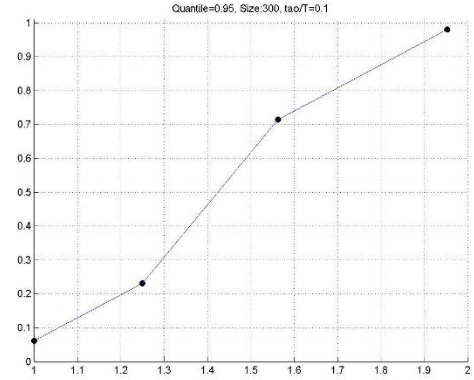

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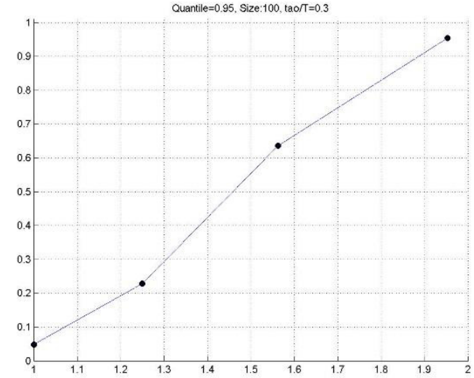
<tr><td>$\frac{\tau}{T} = .03$</td><td>


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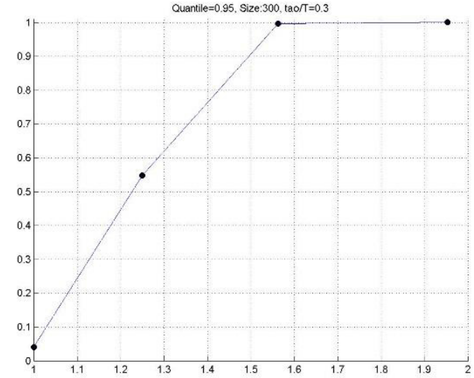

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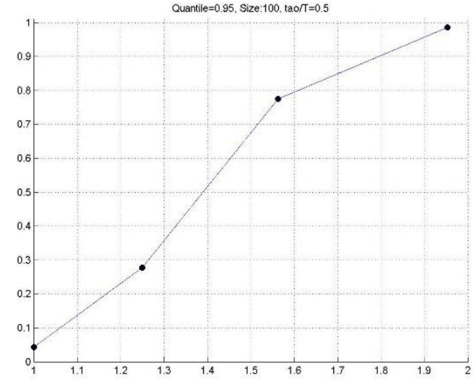
<tr><td>$\frac{\tau}{T} = .05$</td><td>


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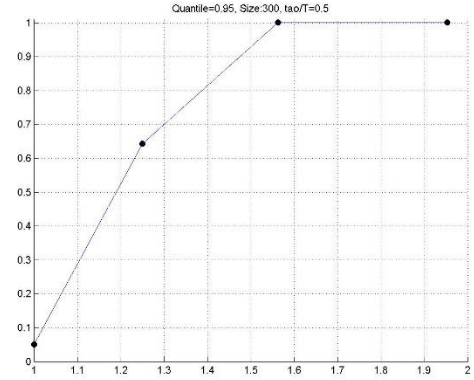


</td></tr>
</table>

Table 7: Power of Proposed test for α=0.95

Note that when variance is equal to 1 the probability of rejecting the null hypothesis is almost equal to the significance level. This indicates that simulated quantiles are closely near to the real ones. We can also observe that the power of the test grows faster when the shift in variance is located in the half of the series. In the same way power increases when size increases.

4.5 Comparison with Chen's Test (1997).

In 1997 Jie Chen and K. Gupta wrote an article proposing a Hypothesis Test for shift in variance. They used the Schwarz Information Criterion (SIC) defined as $-2\log\left(L(\hat{\theta})\right) + p\log(T)$ where $L(\hat{\theta})$ is the maximum likelihood function for the model, p is the number of free parameters in the model and T is the sample size.

They accept H_0 if $SIC(T) < \min_k SIC(k) + c_\alpha$ and reject when $SIC(T) > \min_k SIC(k) + c_\alpha$ where:

$$SIC(T) = T\log(2\pi) + T\log(\sigma^2) + T + \log(T)$$

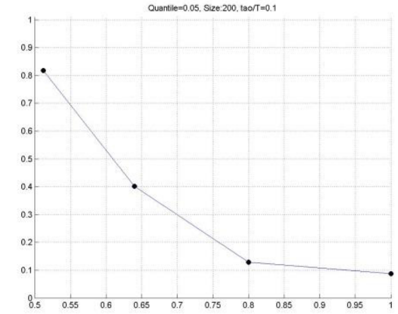
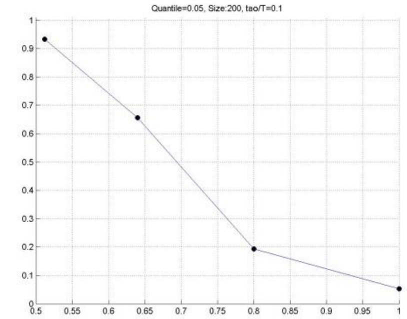
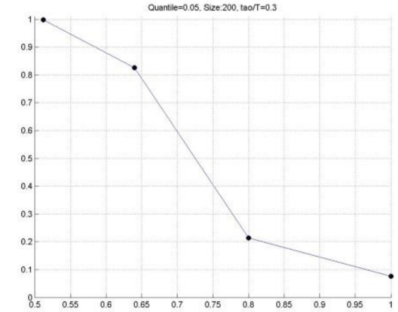
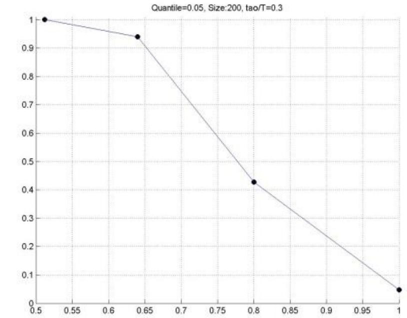
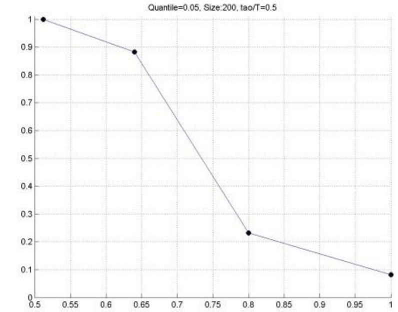
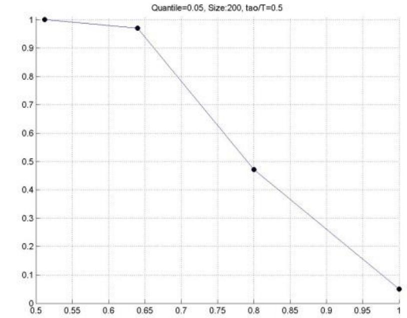
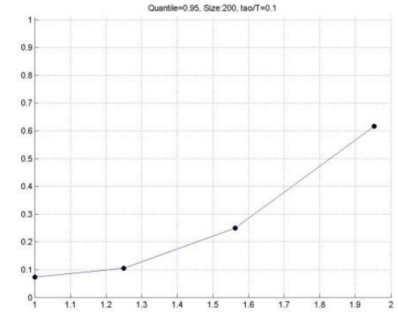
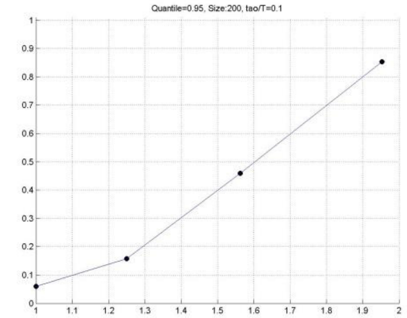
$$SIC(k) = T\log(2\pi) + k\log(\sigma_1^2) + (T-k)\log(\sigma_1^2) + T + 2\log(T)$$

The number $\min_{2\leq k\leq T-2} SIC(k)$ is the change point and c_α is obtained from table 8.

Size	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
20	19.106	9.691	6.698
50	17.284	9.171	6.208
100	16.280	8.626	5.799
200	15.416	8.088	5.359

Table 8: Values of c_α

The power of this test was calculated for these stages, power of proposed test was calculated for T=200 and results were compared. Table 9 shows the results of the comparison for size 200 and significance level of .05. Note that the power of proposed test is bigger than the power of Chen's test.

Parameters (Quantile, size, $\frac{\tau}{T}$)	Chen	Proposed
(0.05, 200, .1)		
(0.05, 200, .3)		
(0.05, 200, .5)		
(0.95, 200, .1)		

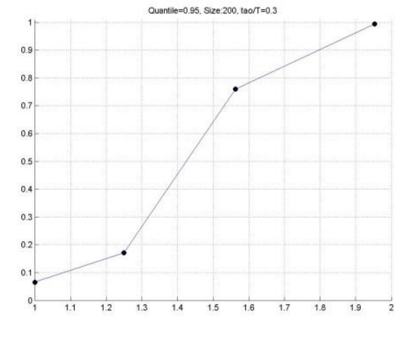
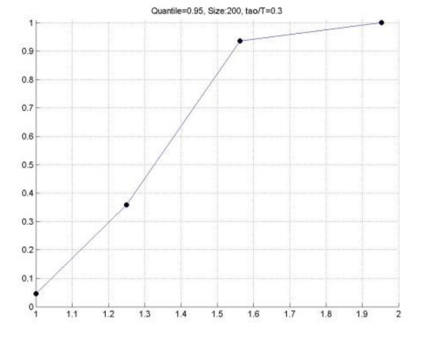
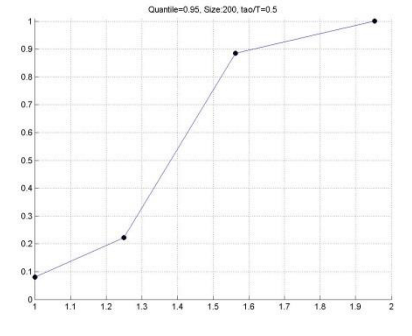
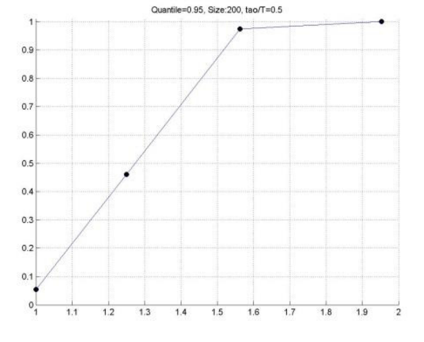
(0.95, 200, .3)		
(0.95, 200, .5)		

Table 9: Comparison between Chen's test and Proposed's test.

Observe that the test developed in this research accepts the null hypothesis more times when it is true because the height lowest point (where there is no change in variance) is more near to the significance level. In the other hand, this test rejects more times the null hypothesis when it is false, indicated by the bigger height of the other 3 points.

Based on these observations we can ensure than proposed test has a better performance than Chen's test.

4.6. – Numerical Example

The S&P 500 (the Standard and Poor's 500), is one of the most used American stock index.

Figure 5 shows data between July 2004 and July 2009. In this figure, the reader can graphically observe a decrease in the variance. The proposed test was applied with significance level of .05. Null hypothesis was rejected and the change-point (red) was estimated in the 23rd observation. This result is very similar to the obtained by Villanueva (2013). The execution time of the Matlab® script was 0.002799 seconds.

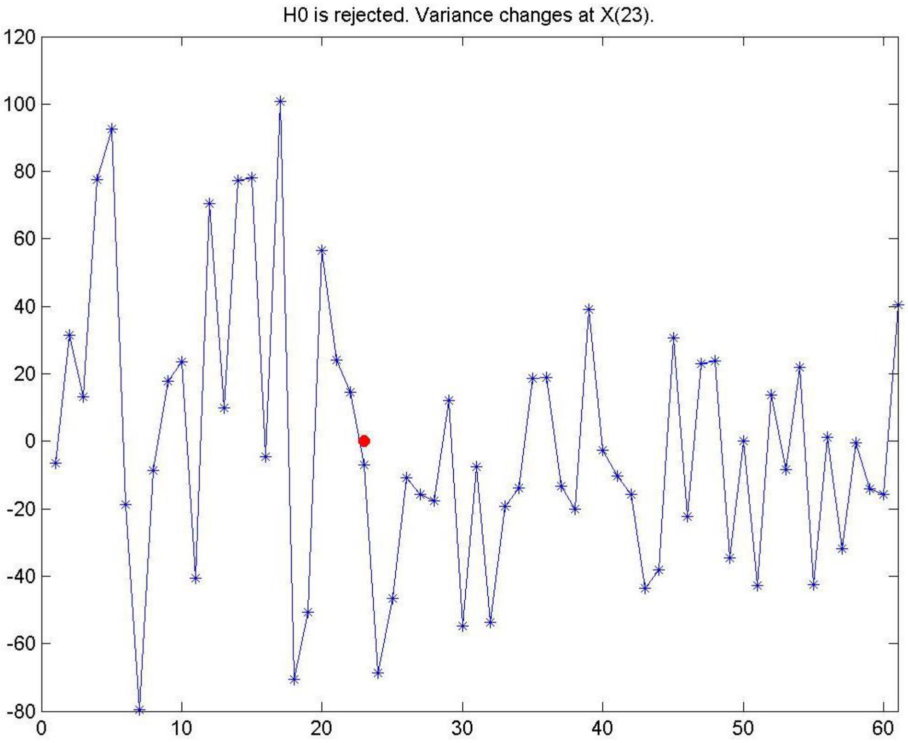


Figure 7: Data from S&P index between July 2004 and July 2009

CONCLUSIONS AND FUTURE WORK

Conclusions

There are not ideal processes, this is, every process presents variability. In the other hand quality is defined as the reduction of variability. Therefore is necessary to monitor the variability of a process in order to design strategies to reduce it. In spite of this Change Point Analysis researchers have dedicated most of the efforts to detect change points in mean. Due to this, a hypothesis test based on p values of F distribution was designed to detect shifts in variances

The first conclusion is that this hypothesis test has proved its efficiency for detecting changes in variances for simulated data as we can see in sections 4.4. Moreover the test reported better results than Chen's test, which efficiency is very similar to CUMSUM as we can see in section 4.5. With these two results the research hypotheses stated in section 1.5 are proved.

A remarkable fact is that the regression curves are the best ones found in the literature, reporting maximum absolute errors near of a ten millionth for sample sizes bigger than 50.

This method offers the following advantages if it is compared with the most common existent techniques.

- Computing time of Control Charts is very high. The execution time of the numerical example was of 0.002799 seconds making this tool able to use in production lines.
- GLR based tools like Chen test (1997) are not very accurate. As was shown in section 4.5 this method offered better results.
- Statistic test like proposed by Hawkins (2005) assume data with the same distribution. This test uses F distribution for different parameters in each step.

The main disadvantage of this tool is that it works only with normal series of independent observations with no change in mean.

Future work

To continue enriching the present work the following two lines are suggested.

First, as was mentioned in subsection 4.2, the exact distribution of the proposed statistic is unknown yet. Theoretical asymptotical properties of this statistic can be obtained.

Second, this methodology can be extended to use P-values for other distributions to design hypothesis tests for detecting shifts not only in variance but also in mean. This is, taking the general model.

$$X_i \sim \begin{cases} f(\theta_1), i = 1,2, \dots, \tau \\ f(\theta_2), i = \tau + 1, \dots, T \end{cases}$$

With the Hypotheses $H_0: \tau = T$ and $H_1: \tau \neq T$

Applying the Likelihood Ratio Test for all $k = 2,3 \dots, T - 2$ we can obtain $\min(p_{T,k})$ and $\max(p_{T,k})$ and their respective quantiles in the same way proposed in this work.

A particular case is to use the p-values of the chi squared distribution in the next way.

Let $X_1, X_2, \dots X_T$ be a set of independent observations following the next distribution:

$$X_i \sim \begin{cases} N(\mu_1, \sigma_1^2), t = 1,2, \dots, \tau \\ N(\mu_2, \sigma_2^2),, t = \tau + 1, \dots, T \end{cases}$$

Parameters μ_1, μ_2 and σ_1^2 are known and σ_2^2 is unknown. Hypotheses are $H_0: \sigma_1^2 = \sigma_2^2$ and $H_1: \sigma_1^2 \neq \sigma_2^2$.

The likelihood ratio for this test is based on the statistic

$$\chi_{\tau,k}^2 = \frac{(k - \tau)S_{\tau,k}^2}{\sigma_1^2}$$

The P –value is obtained from the next equation

$$P_{\tau,k} = P(\chi_{k-1}^2 \geq \chi_{\tau,k}^2)$$

And in this case the estimator of the change point maximizes the P-value, this is:

$$\widehat{\tau_k} = \max_{\tau}\{P_{\tau,k}\}$$

To summarize, there is opportunity to continue researching the application of P-values to detect and estimate change points in parameters of statistical distributions.

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APPENDIX

A.1 Matlab codes

A.1.1 Quantile estimation

```
clc
clear all
for n=5:500;
for rep=1:1000;
    for i =1:n
        x(i)=randn;
    end
    miu0=zeros(n,1);
    miu1=zeros(n,1);
    sigma0=zeros(n,1);
    sigma1=zeros(n,1);
    for k = 3:n-2
        for i=1:k
            miu0(k)=miu0(k)+x(i);
        end
        miu0(k)=miu0(k)/k;
        for i=1:k
            sigma0(k)=sigma0(k)+(x(i)-miu0(k))^2;
        end
        sigma0(k)=sigma0(k)/(k-1);
        for i=(k+1):n
            miu1(k)=miu1(k)+x(i);
        end
        miu1(k)=miu1(k)/(n-k);
        for i=(k+1):n
            sigma1(k)=sigma1(k)+(x(i)-miu1(k))^2;
        end
        sigma1(k)=sigma1(k)/(n-1-k);
        r(k)=sigma1(k)/sigma0(k);
        v1(k)=n-1-k;
        v2(k)=k-1;
        p(k-2)=fcdf(r(k),v1(k),v2(k));
    end
    po=sort(p);
    b(rep)=po(1);
    t(rep)=po(n-4);
end
b=sort(b);
t=sort(t);
result(n-4,1)=n;
result(n-4,2)=b(10);
result(n-4,3)=b(50);
result(n-4,4)=b(100);
result(n-4,5)=b(200);
result(n-4,6)=t(801);
result(n-4,7)=t(901);
result(n-4,8)=t(951);
result(n-4,9)=t(991);
end
```

```
filename = 'distf500.xlsx';
xlswrite(filename,result)
```

A.1.2 Power of the test

```
tic
clc
clearvars
image=0;
alpha(:,1)=[.01,.05,.1,.2];
alpha(:,2)=[.8,.9,.95,.99];
varshift(:,1)=[1,1/1.25,1/1.25^2,1/1.25^3];
varshift(:,2)=[1,1.25,1.25^2,1.25^3];
twenty(:,1)=[0.000859101 0.005585629 0.011574396 0.0322866];
twenty(:,2)=[0.967915639 0.98763408 0.994482116 0.999329755];
fifty(:,1)=[0.000350961 0.004021736 0.009062285 0.021870338];
fifty(:,2)=[0.980962703 0.993302279 0.996933472 0.999431223];
a1(:,1)=[.0215 .1259 .2773 .6848];
a1(:,2)=[-.7059 -.2788 -.1172 -.0184];
a2(:,1)=[.002 .0013 .0032 .0079];
a2(:,2)=[.9918 .9967 .9986 .9998];
T=[20,50,100,300];
changePos=[.1,.3,.5];
cont=zeros(1,4);
for cola=1:2
    for a=1:4
        for size=1:4
            for tonT=1:3
            change=changePos(tonT)*T(size);
            for rep=1:1000
                if size==1
                    s=twenty(a,cola);
                elseif size==2
                    s=fifty(a,cola);
                else
                    s=a1(a,cola)/T(size) + a2(a,cola);
                end
                x=zeros(1,T(size));
                for var =1:4
                    for i=1:change
                        x(i)=randn;
                    end
                    for i=change+1:T(size)
                        x(i)=varshift(var,cola)*randn;
                    end
                    miu0=zeros(T(size),1);
                    miu1=zeros(T(size),1);
                    sigma0=zeros(T(size),1);
                    sigma1=zeros(T(size),1);
                    r=zeros(T(size)-4,1);
                    v1=zeros(T(size)-4,1);
                    v2=zeros(T(size)-4,1);
                    p=zeros(T(size)-4,1);
                    for k = 3:T(size)-2
                        for i=1:k
                            miu0(k)=miu0(k)+x(i);
```

```
                        end
                        miu0(k)=miu0(k)/k;
                        for i=1:k
                            sigma0(k)=sigma0(k)+(x(i)-miu0(k))^2;
                        end
                        sigma0(k)=sigma0(k)/(k-1);
                        for i=(k+1):T(size)
                            miu1(k)=miu1(k)+x(i);
                        end
                        miu1(k)=miu1(k)/(T(size)-k);
                        for i=(k+1):T(size)
                            sigma1(k)=sigma1(k)+(x(i)-miu1(k))^2;
                        end
                        sigma1(k)=sigma1(k)/(T(size)-1-k);
                        r(k-2)=sigma1(k)/sigma0(k);
                        v1(k-2)=T(size)-1-k;
                        v2(k-2)=k-1;
                        p(k-2)=fcdf(r(k-2),v1(k-2),v2(k-2));
                    end
                    if cola==2
                        pv=max(p);
                        if pv>s
                             cont(var)=cont(var)+1;
                        end
                    else
                        pv=min(p);
                        if pv<s
                             cont(var)=cont(var)+1;
                        end
                    end
                end
            end
            cont=cont/rep;
            image=image+1;
            name=strcat('Quantile= ',num2str(alpha(a,cola)),...
', Size: ',num2str(T(size)),', tao/T= ',num2str(changePos(tonT)));
            xlswrite(num2str(image),cont);
            figure
            scatter(varshift(:,cola),cont,'k','filled')
            title(name);
            hold on
            plot(varshift(:,cola),cont,'b')
            hold on
            ylim([0 1.01])
            grid on
            saveas(gcf,num2str(image),'jpg')
            end
        end
    end
end
toc
```

A.1.3 Comparison for decreasing variance

```
tic
clc
clearvars
```

```
image=0;
alpha(:,1)=[.01,.05,.1];
varshift(:,1)=[1,1/1.25,1/1.25^2,1/1.25^3];
twenty(:,1)=[19.106 9.961 6.698];
fifty(:,1)=[17.284 9.171 6.208];
hundred(:,1)=[16.280 8.626 5.799];
twohundred(:,1)=[15.416 8.088 5.359];
T=[20,50,100,200];
changePos=[.1,.3,.5];
cont=zeros(1,4);
cola=1;
    for a=1:3%3
        for size=1:4%4
            for tonT=1:3%3
            change=changePos(tonT)*T(size);
            for rep=1:1000
                if size==1
                    ca=twenty(a,cola);
                elseif size==2
                    ca=fifty(a,cola);
                elseif size==3
                    ca=hundred(a,cola);
                elseif size==4
                    ca=twohundred(a,cola);
                end
                x=zeros(1,T(size));
                for var =1:4
                    for i=1:change
                        x(i)=randn;
                    end
                    for i=change+1:T(size)
                        x(i)=varshift(var,cola)*randn;
                    end
                    miu0=zeros(T(size),1);
                    miu1=zeros(T(size),1);
                    sigma0=zeros(T(size),1);
                    sigma1=zeros(T(size),1);
                    r=zeros(T(size)-4,1);
                    v1=zeros(T(size)-4,1);
                    v2=zeros(T(size)-4,1);
                    sick=zeros(T(size)-4,1);
                    for k = 3:T(size)-2
                        for i=1:k
                            miu0(k)=miu0(k)+x(i);
                        end
                        miu0(k)=miu0(k)/k;
                        for i=1:k
                            sigma0(k)=sigma0(k)+(x(i)-miu0(k))^2;
                        end
                        sigma0(k)=sigma0(k)/(k-1);
                        for i=(k+1):T(size)
                            miu1(k)=miu1(k)+x(i);
                        end
                        miu1(k)=miu1(k)/(T(size)-k);
                        for i=(k+1):T(size)
                            sigma1(k)=sigma1(k)+(x(i)-miu1(k))^2;
                        end
```

```
                        sigma1(k)=sigma1(k)/(T(size)-1-k);
                        sick(k-2)=T(size)*log(2*pi)+k*log(sigma0(k))+...
(T(size)-k)*log(sigma1(k))+T(size)+2*log(T(size));
                    end
                    miu=0;
                    sigma=0;
                        for i=1:T(size)
                            miu=miu+x(i);
                        end
                        miu=miu/T(size);
                        for i=1:T(size)
                            sigma=sigma+(x(i)-miu)^2;
                        end
                        sigma=sigma/(T(size)-1);

sicn=T(size)*log(2*pi)+T(size)*log(sigma)+T(size)+log(T(size));
                        sic=min(sick);
                        if sicn>(sic+ca)
                             cont(var)=cont(var)+1;
                        end
                end
            end
            cont=cont/rep;
            image=image+1;
            name=strcat('Quantile= ',num2str(alpha(a,cola)),',...
Size: ',num2str(T(size)),', tao/T= ',num2str(changePos(tonT)));
         %   xlswrite(num2str(image),cont);
            figure
            scatter(varshift(:,cola),cont,'k','filled')
            title(name);
            hold on
            plot(varshift(:,cola),cont,'b')
            hold on
            ylim([0 1.01])
            grid on
            saveas(gcf,num2str(image),'jpg')
            end
        end
    end
toc
```

A.1.4 Comparison for increasing variance

```
tic
clc
clearvars
image=0;
alpha(:,1)=[.01,.05,.1];
varshift(:,1)=[1,1.25,1.25^2,1.25^3];
twenty(:,1)=[19.106 9.961 6.698];
fifty(:,1)=[17.284 9.171 6.208];
hundred(:,1)=[16.280 8.626 5.799];
twohundred(:,1)=[15.416 8.088 5.359];
T=[20,50,100,200];
changePos=[.1,.3,.5];
cont=zeros(1,4);
cola=1;
```

```
    for a=1:3%3
        for size=1:4%4
            for tonT=1:3%3
            change=changePos(tonT)*T(size);
            for rep=1:1000
                if size==1
                    ca=twenty(a,cola);
                elseif size==2
                    ca=fifty(a,cola);
                elseif size==3
                    ca=hundred(a,cola);
                elseif size==4
                    ca=twohundred(a,cola);
                end
                x=zeros(1,T(size));
                for var =1:4
                    for i=1:change
                        x(i)=randn;
                    end
                    for i=change+1:T(size)
                        x(i)=varshift(var,cola)*randn;
                    end
                    x=fliplr(x);
                    miu0=zeros(T(size),1);
                    miu1=zeros(T(size),1);
                    sigma0=zeros(T(size),1);
                    sigma1=zeros(T(size),1);
                    r=zeros(T(size)-4,1);
                    v1=zeros(T(size)-4,1);
                    v2=zeros(T(size)-4,1);
                    sick=zeros(T(size)-4,1);
                    for k = 3:T(size)-2
                        for i=1:k
                            miu0(k)=miu0(k)+x(i);
                        end
                        miu0(k)=miu0(k)/k;
                        for i=1:k
                            sigma0(k)=sigma0(k)+(x(i)-miu0(k))^2;
                        end
                        sigma0(k)=sigma0(k)/(k-1);
                        for i=(k+1):T(size)
                            miu1(k)=miu1(k)+x(i);
                        end
                        miu1(k)=miu1(k)/(T(size)-k);
                        for i=(k+1):T(size)
                            sigma1(k)=sigma1(k)+(x(i)-miu1(k))^2;
                        end
                        sigma1(k)=sigma1(k)/(T(size)-1-k);
                        sick(k-2)=T(size)*log(2*pi)+k*log(sigma0(k))...
+(T(size)-k)*log(sigma1(k))+T(size)+2*log(T(size));
                    end
                    miu=0;
                    sigma=0;
                        for i=1:T(size)
                            miu=miu+x(i);
                        end
                        miu=miu/T(size);
```

```
                        for i=1:T(size)
                            sigma=sigma+(x(i)-miu)^2;
                        end
                        sigma=sigma/(T(size)-1);

sicn=T(size)*log(2*pi)+T(size)*log(sigma)+T(size)+log(T(size));
                        sic=min(sick);
                        if sicn>(sic+ca)
                             cont(var)=cont(var)+1;
                        end
                end
            end
            cont=cont/rep;
            image=image+1;
            name=strcat('Quantile= ',num2str(1-alpha(a,cola)),...
', Size: ',num2str(T(size)),', tao/T= ',num2str(changePos(tonT)));
         %   xlswrite(num2str(image),cont);
            figure
            scatter(varshift(:,cola),cont,'k','filled')
            title(name);
            hold on
            plot(varshift(:,cola),cont,'b')
            hold on
            ylim([0 1.01])
            grid on
            saveas(gcf,num2str(image),'jpg')
            end
        end
    end
toc
```

A 1.5 Numerical Example

```
clearvars
clc
tic
x=[-6.36 31.4 13.19 77.65 92.64 -18.84 -79.67 -8.63 17.88 23.47 -40.65
70.46 9.79 77.32 78.2 -4.45 100.79 -70.48 -50.55 56.49 24.12 14.55 -7.03
-68.74 -46.67 -10.91 -15.81 -17.7 12.05 -54.75 -7.49 -53.71 -19.21 -13.81
18.73 18.91 -13.38 -20.15 39.12 -2.52 -10.35 -15.62 -43.63 -38.09 30.65 -
22.33 23.01 23.74 -34.65 0.17 -42.85 13.85 -8.48 21.81 -42.47 1.19 -31.79
-0.58 -14.16 -15.78 40.52];
T=61;
p=zeros(1,T-4);
miu0=0;
sigma0=0;
miu1=0;
sigma1=0;
for k = 3:T-2
        for i=1:k
            miu0=miu0+x(i);
        end
        miu0=miu0/k;
        for i=1:k
            sigma0=sigma0+(x(i)-miu0)^2;
        end
        sigma0=sigma0/(k-1);
```

```
        for i=(k+1):T
            miu1=miu1+x(i);
        end
        miu1=miu1/(T-k);
        for i=(k+1):T
            sigma1=sigma1+(x(i)-miu1)^2;
        end
        sigma1=sigma1/(T-1-k);
        r=sigma1/sigma0;
        v1=T-1-k;
        v2=k-1;
        p(k-2)=fcdf(r,v1,v2);
end
[~,pos]=find(p==min(min(p)));
pv=min(p);
s= -0.1172/T + 0.9986;
if s<pv
    name='H0 is accepted, there is no change in variance';
else
    name=strcat('H0 is rejected. Variance changes at...
X(',num2str(pos),').');
end
toc
figure
day=linspace(1,61,61);
plot(day,x,'b*-')
xlim([0 61])
title(name)
hold on
scatter(pos,pv,'r','filled')
saveas(gcf,'Numerical example','jpg')
```

A.2 All the Quantiles

T	$S_{.01}$	$S_{.05}$	$S_{.1}$	$S_{.2}$	$S_{.8}$	$S_{.9}$	$S_{.91}$	$S_{.99}$
5	0.006281654	0.051898567	0.096447892	0.206450389	0.793816129	0.887433142	0.93650131	0.988702562
6	0.00491284	0.02956886	0.059877712	0.129978907	0.875907015	0.93576931	0.971488867	0.994266561
7	0.002617857	0.02076724	0.041913994	0.109310511	0.912708253	0.959506561	0.980235353	0.997326592
8	0.003527766	0.012120241	0.029634693	0.07805619	0.917976273	0.965748537	0.980092467	0.997168952
9	0.001864831	0.014705964	0.030908449	0.07507764	0.929010629	0.96701141	0.983398764	0.997547166
10	0.00262071	0.013665	0.026528795	0.059773781	0.941394135	0.976895856	0.988920762	0.997946414
11	0.001081597	0.008486449	0.023856217	0.063039043	0.93744109	0.971876594	0.989971893	0.997107917
12	0.001444736	0.012250571	0.023979854	0.049865043	0.94934864	0.981511799	0.992404698	0.998986848
13	0.001499019	0.006616104	0.016403075	0.041864665	0.95053208	0.979062224	0.990728806	0.998429037
14	0.001123607	0.006646856	0.016699654	0.039742909	0.954068979	0.977275643	0.991779828	0.998916939
15	0.001310778	0.006593983	0.015941447	0.042956757	0.954761443	0.979986274	0.990568345	0.998497937
16	0.001195564	0.006848465	0.014358527	0.036191175	0.95786007	0.982176686	0.992472446	0.998963121
17	0.001656906	0.007259888	0.016415925	0.036268004	0.95893534	0.983489889	0.993102043	0.998617629
18	0.000903789	0.005887336	0.014974443	0.035831466	0.963154232	0.984916326	0.993437959	0.998521705
19	0.000782177	0.006177839	0.015346783	0.03534597	0.962448541	0.982778637	0.994373668	0.999109327

20	0.000859101	0.005585629	0.011574396	0.03228666	0.967915639	0.98763408	0.994482116	0.999329755
21	0.000956109	0.005664029	0.014158796	0.031213293	0.966523406	0.987304452	0.994054712	0.999059505
22	0.000823876	0.004862915	0.013559189	0.032825859	0.958699899	0.981092629	0.99156066	0.997859694
23	0.001057962	0.007899986	0.015645745	0.034549283	0.966700154	0.986571085	0.993683444	0.998766683
24	0.001034028	0.005263389	0.01396847	0.031418554	0.968405768	0.987591285	0.995292944	0.998958552
25	0.000529317	0.004629828	0.011682942	0.028380448	0.970947664	0.985921173	0.994427718	0.999110206
26	0.000695347	0.004849328	0.010172928	0.024651011	0.968830476	0.9880692	0.995258377	0.999139719
27	0.001026647	0.004964639	0.010676741	0.028177858	0.969129611	0.986366926	0.99335473	0.99864106
28	0.000831316	0.005570962	0.012152271	0.027057955	0.967437045	0.986739241	0.995080043	0.998988153
29	0.001150428	0.005266861	0.011415616	0.028258208	0.972054137	0.988476346	0.995536049	0.999537906
30	0.001003513	0.005510761	0.010328261	0.026273203	0.972264895	0.988820264	0.995745907	0.999242244
31	0.001275103	0.004763208	0.01131335	0.027210968	0.973842773	0.990222848	0.995983628	0.999279086
32	0.000525106	0.003253471	0.008603691	0.024796291	0.970492529	0.988422713	0.994513464	0.9990019
33	0.000687567	0.004480272	0.010519031	0.026138531	0.977638756	0.989624002	0.995922962	0.99911675
34	0.000723594	0.003390081	0.00922889	0.022814853	0.973932587	0.989851813	0.996396042	0.999802381
35	0.00083763	0.003766593	0.010192387	0.02308011	0.975620429	0.989612591	0.99492796	0.999024042
36	0.000510157	0.003845853	0.009348099	0.024744708	0.972184613	0.990063453	0.99560023	0.999127342
37	0.000630365	0.003802335	0.009006464	0.023009365	0.972837663	0.989391566	0.994735229	0.998894711
38	0.00051203	0.003329733	0.008538765	0.020390194	0.975966291	0.990257631	0.995697764	0.999267393
39	0.000740834	0.003677159	0.007954698	0.022120167	0.977219352	0.989482758	0.995499706	0.999282649
40	0.000381906	0.003897783	0.008909007	0.02112644	0.976045976	0.98961907	0.995967404	0.999048987
41	0.000505817	0.003968294	0.010104511	0.022988032	0.977804994	0.990407413	0.99589944	0.999383911
42	0.000402911	0.003310108	0.00771358	0.019878728	0.976346536	0.990882712	0.995811993	0.999722945
43	0.000671136	0.005381887	0.011670857	0.025134899	0.977555129	0.990658728	0.995812004	0.999306576
44	0.00042473	0.004940848	0.010346605	0.022613644	0.978370224	0.990138391	0.995848153	0.999443281
45	0.000553331	0.003456211	0.00804661	0.021297296	0.97442729	0.990354526	0.995806498	0.999534078
46	0.000490521	0.005386224	0.010489919	0.020703091	0.978664269	0.990757144	0.995817819	0.999496304
47	0.000424423	0.002913309	0.00750276	0.020551616	0.976884812	0.989733613	0.995391889	0.999325336
48	0.000769527	0.003847857	0.00813313	0.022625716	0.981062925	0.991866308	0.996121989	0.999216434
49	0.000922083	0.004553121	0.010393097	0.023091529	0.97844255	0.991522113	0.996163899	0.99927986
50	0.000350961	0.004021736	0.009062285	0.021870338	0.980962703	0.993302279	0.996933472	0.999431223
51	0.000579504	0.002918846	0.007089016	0.01939825	0.977447419	0.990990717	0.995939336	0.99906209
52	0.000388833	0.003567349	0.008333002	0.020758709	0.980888726	0.992274339	0.997214169	0.999678434
53	0.000990333	0.00424673	0.009524814	0.021358928	0.980925856	0.992308915	0.997010168	0.999598036
54	0.000745464	0.003413828	0.006575853	0.017798536	0.979528051	0.991827951	0.997359973	0.999337461
55	0.000650955	0.004051522	0.008179013	0.01819017	0.98128922	0.992764982	0.996257847	0.99968114
56	0.000976173	0.00479808	0.009299999	0.02302193	0.980660716	0.991121442	0.996382547	0.999417475
57	0.000324069	0.003060037	0.007717594	0.019766135	0.980913023	0.993263529	0.997049158	0.999699363
58	0.000567734	0.004263307	0.00911373	0.019641522	0.9823656	0.993497483	0.997173291	0.999711732
59	0.000545908	0.003677082	0.00809961	0.020309341	0.982360858	0.992342588	0.996198645	0.999509781
60	0.000365845	0.003667112	0.00704835	0.017504839	0.979001815	0.991682954	0.996114855	0.999597007

61	0.000689792	0.003967765	0.006830114	0.017577253	0.98351792	0.99400988	0.997109701	0.99953541
62	0.000313339	0.002281894	0.006836995	0.017524722	0.981936992	0.993277458	0.997026773	0.99969533
63	0.000904404	0.003693145	0.007935101	0.017964648	0.980223558	0.992657355	0.997273718	0.999446629
64	0.000394494	0.003794901	0.007659184	0.019248735	0.982600093	0.993686728	0.997942399	0.999724747
65	0.000318726	0.002435258	0.006575726	0.016331487	0.980313089	0.992630911	0.996393061	0.999462213
66	0.000375187	0.00292821	0.006938117	0.017456569	0.981277242	0.992682136	0.997712525	0.999449886
67	0.000727553	0.003506354	0.008323764	0.020051073	0.982742832	0.993898672	0.997926796	0.999538193
68	0.000433414	0.003351611	0.005866264	0.016121588	0.979511549	0.992297841	0.997026322	0.999463298
69	0.000310759	0.002529264	0.007010491	0.016881409	0.981526403	0.99285559	0.997286711	0.99936271
70	0.000627757	0.003161509	0.006503755	0.017122028	0.983075757	0.992342506	0.997034568	0.999771242
71	0.000544019	0.003002335	0.00620242	0.014861813	0.981648985	0.992640069	0.997522696	0.999495668
72	0.000569654	0.002610679	0.006994394	0.017866956	0.981842736	0.993150762	0.997038594	0.999574089
73	0.000638046	0.002589513	0.007829929	0.016765333	0.982534091	0.99245374	0.99736318	0.999394666
74	0.00034642	0.002840786	0.005959803	0.016228133	0.980088027	0.992617769	0.99667609	0.999717446
75	0.000400983	0.002548178	0.006375102	0.017360868	0.980734001	0.992264207	0.99690039	0.999606329
76	0.00038692	0.002327007	0.006739121	0.017093857	0.981521183	0.993352821	0.996854428	0.999384089
77	0.000370485	0.003457914	0.007608807	0.01660426	0.982981165	0.99263458	0.996387586	0.999281899
78	0.000631051	0.003334926	0.007698185	0.01635513	0.983311514	0.992737833	0.997222473	0.999654534
79	0.000356616	0.002352898	0.006321171	0.014797569	0.984420365	0.993862772	0.99765178	0.999709217
80	0.000502305	0.003056446	0.006238688	0.016229443	0.980976176	0.992960953	0.996933373	0.999461373
81	0.000406614	0.002275897	0.005611339	0.014696317	0.981431783	0.993156206	0.996897577	0.999604254
82	0.000447981	0.003460493	0.007748851	0.016353168	0.982658021	0.99425748	0.997643482	0.999736802
83	0.000381928	0.003492117	0.008116897	0.017075156	0.985053285	0.994758951	0.997577414	0.999378591
84	0.000210096	0.002858371	0.007339831	0.019373048	0.985472366	0.993301628	0.997339375	0.999573789
85	0.000505751	0.002048067	0.004830293	0.015130732	0.981721316	0.991521037	0.996560759	0.999660886
86	0.000436791	0.002220457	0.00606658	0.015855056	0.982402659	0.992475065	0.996431023	0.999513288
87	0.000531634	0.002898699	0.006226483	0.014410381	0.982057612	0.992674907	0.996920518	0.999526491
88	0.000548136	0.002189997	0.005606934	0.014891675	0.983629861	0.991927278	0.996450554	0.999563836
89	0.000184982	0.002878334	0.007662779	0.017595701	0.985057077	0.993751624	0.996946653	0.999585792
90	0.000471526	0.00275806	0.006322295	0.017271084	0.983309557	0.993404484	0.996948806	0.999385881
91	0.000430093	0.002944707	0.00726529	0.016450512	0.985184437	0.993988719	0.997778282	0.999685007
92	0.000248366	0.0028169	0.006346455	0.015494314	0.983264699	0.992313299	0.997231941	0.999622701
93	0.000455811	0.002739287	0.005470093	0.013874208	0.98402366	0.99302467	0.997294878	0.999538994
94	0.0002219	0.002136014	0.00553464	0.014277033	0.984718367	0.993356696	0.997094488	0.999439357
95	0.000453822	0.002811002	0.005609928	0.014498613	0.984832969	0.993874351	0.99740246	0.999645137
96	0.000620747	0.002706014	0.005973543	0.014891779	0.98615097	0.994786377	0.997662457	0.99962961
97	0.000378973	0.00211367	0.004910753	0.013776588	0.985333165	0.994459765	0.997416432	0.999698908
98	0.000664008	0.002448541	0.006356738	0.014659312	0.982676506	0.992571814	0.996763525	0.999502631
99	0.000320686	0.002788665	0.006428374	0.016037875	0.98405483	0.992868121	0.997437585	0.999644454
100	0.00047929	0.002461246	0.005627021	0.014939015	0.984287665	0.994241921	0.997323013	0.999606409
101	0.000290545	0.002529523	0.005967137	0.015975155	0.984570869	0.993702494	0.997400719	0.999691066

102	0.000453248	0.002818098	0.007044416	0.015754074	0.985217069	0.994483391	0.997596748	0.99955039
103	0.00027186	0.002330359	0.006321803	0.015553176	0.984285021	0.994416367	0.997383656	0.999736707
104	0.000632107	0.003016052	0.007252446	0.016587378	0.98549047	0.994374223	0.997696236	0.999492831
105	0.000539611	0.002922393	0.00591466	0.013559958	0.984396887	0.992910502	0.997023829	0.999375192
106	0.000490942	0.002202271	0.005230119	0.013126348	0.984622277	0.99390515	0.997201325	0.999702759
107	0.000335089	0.002218317	0.005472845	0.014595242	0.986870531	0.994032736	0.99704056	0.999518278
108	0.000640527	0.002903952	0.006353125	0.014267391	0.985567921	0.994830161	0.997361042	0.999505157
109	0.000419115	0.002059703	0.005151706	0.014050673	0.984369287	0.994553554	0.99801757	0.999747981
110	0.000500627	0.002927985	0.007678078	0.016375922	0.986442877	0.99458683	0.997557789	0.999356141
111	0.00032635	0.002252227	0.005870778	0.012878501	0.987610524	0.995422775	0.997434738	0.999720681
112	0.000279227	0.002099669	0.006443877	0.014701475	0.98466914	0.994004857	0.99732139	0.99968671
113	0.000328436	0.001862034	0.006129553	0.014186914	0.985408908	0.993769773	0.997324256	0.99963007
114	0.000437613	0.002645999	0.005826095	0.016491658	0.98685879	0.994799974	0.997607775	0.999564074
115	0.000436004	0.002684718	0.005556541	0.012943936	0.985081347	0.992914414	0.996782928	0.999521644
116	0.000505428	0.002131067	0.004873423	0.013296149	0.983237584	0.99336499	0.99690673	0.999568135
117	0.000245904	0.002269833	0.005056942	0.013731751	0.982566946	0.992951639	0.996924709	0.999543791
118	0.000165569	0.001922435	0.005156268	0.012750976	0.984117448	0.994869998	0.9976686	0.999813483
119	0.000373368	0.002886197	0.006156638	0.015318076	0.985529915	0.993880853	0.99704709	0.9996098
120	0.000514121	0.002450733	0.005731258	0.01570609	0.987171478	0.994548655	0.997801157	0.999489762
121	0.000304851	0.002487054	0.005333273	0.01428911	0.985247216	0.994944979	0.997863951	0.999803061
122	0.00023732	0.002507469	0.005559793	0.013874523	0.98649712	0.993731716	0.996653389	0.999282871
123	0.000457228	0.002449647	0.004726421	0.014485857	0.98626438	0.994023009	0.997279855	0.999633453
124	0.00039093	0.002211545	0.005384781	0.014468192	0.986363069	0.994625627	0.997618913	0.999721175
125	0.000247743	0.00198664	0.005196296	0.012605446	0.984578016	0.993401571	0.997632574	0.999520969
126	0.000366151	0.002589109	0.005389636	0.014598624	0.985925435	0.993959781	0.997793176	0.999760831
127	0.000385735	0.002854409	0.007123915	0.016145455	0.986500405	0.994671715	0.997904195	0.999800564
128	0.000232971	0.002289307	0.004707954	0.010718511	0.984694995	0.993171499	0.996903162	0.999445356
129	0.000446742	0.002617593	0.005318052	0.012751466	0.984339202	0.99331586	0.997339766	0.999542407
130	0.000532519	0.002274325	0.004652523	0.011743391	0.985622479	0.994342951	0.997613512	0.999531742
131	0.000326272	0.002073026	0.005661482	0.013961864	0.985591322	0.994437797	0.997602469	0.999605224
132	0.000249771	0.002076933	0.006280569	0.015026182	0.988127773	0.994876561	0.997762639	0.999538518
133	0.000366441	0.002770485	0.006006135	0.013953675	0.98676978	0.994704286	0.997785593	0.999687928
134	0.000312009	0.002496054	0.005264805	0.013508592	0.987814756	0.994961626	0.998047417	0.999789556
135	0.000324964	0.002188357	0.005260892	0.013159175	0.985406842	0.994009747	0.997379071	0.999786282
136	0.000353567	0.002822307	0.00558476	0.013485201	0.98544416	0.993537227	0.997235344	0.999392627
137	0.000162021	0.001578785	0.003969297	0.012782611	0.987216424	0.99514902	0.997571307	0.999661451
138	0.00026275	0.002049376	0.005226977	0.014373098	0.986888588	0.994684509	0.997844215	0.999699286
139	0.000358105	0.002456079	0.005030869	0.013233166	0.986537094	0.993505865	0.997208435	0.999172685
140	0.000286148	0.002146702	0.005004102	0.013772284	0.987414698	0.995499535	0.997862507	0.999685125
141	0.000703088	0.002306293	0.005296073	0.01410506	0.988042112	0.995181608	0.997581982	0.999779595
142	0.000452401	0.002097525	0.005642925	0.012055646	0.98672979	0.994788149	0.997837718	0.999718268

143	0.000500061	0.002719071	0.005621263	0.013507583	0.987183541	0.99481119	0.997593676	0.999627311
144	0.000240369	0.001514403	0.004178022	0.011602976	0.984400114	0.99308097	0.996551065	0.999494214
145	0.000549688	0.002952257	0.005468724	0.013067857	0.985406871	0.994624144	0.997400958	0.999689708
146	0.000403068	0.001987631	0.005037809	0.012495628	0.986468216	0.995141087	0.998090283	0.99959819
147	0.000276158	0.002589385	0.005357002	0.012024888	0.985595748	0.994300327	0.997775525	0.999637793
148	0.000457715	0.001983264	0.004185212	0.012091959	0.987086833	0.994577582	0.997909558	0.999646647
149	0.00046118	0.00191335	0.004748867	0.013383705	0.989064506	0.995476014	0.998253419	0.999752553
150	0.000316551	0.002379168	0.005408986	0.012336029	0.986452849	0.994862058	0.998216734	0.999503099
151	0.000559865	0.002506871	0.005148724	0.014715677	0.98798565	0.99524146	0.997798954	0.999681296
152	0.000390521	0.001818608	0.004225484	0.011622526	0.983065346	0.994051602	0.997585678	0.999605359
153	0.000235549	0.001996364	0.00454985	0.011789527	0.985797969	0.994036203	0.997001706	0.999631286
154	0.000436207	0.002304351	0.005528038	0.013457347	0.986714265	0.994536372	0.997749113	0.99972054
155	0.000266745	0.002002368	0.005295615	0.015137353	0.98606775	0.994907287	0.997897523	0.999648711
156	0.000321019	0.002296138	0.005113031	0.012727188	0.985206772	0.993646542	0.997026205	0.999481425
157	0.000264449	0.001740393	0.004018128	0.011116982	0.985695653	0.994364029	0.997173349	0.999482607
158	0.000222005	0.002035885	0.004868957	0.011924517	0.986931489	0.995219628	0.997740469	0.999684971
159	0.000372083	0.002374201	0.005359776	0.011957969	0.987033391	0.995217934	0.99815356	0.999602126
160	0.000291207	0.002581667	0.005849476	0.013767608	0.986989578	0.995010018	0.997707202	0.999603604
161	0.000480859	0.00212553	0.005288256	0.011635912	0.987380398	0.995097644	0.997920641	0.999536983
162	0.000627608	0.002671087	0.005926795	0.013282024	0.988432461	0.995684555	0.99808459	0.999807238
163	0.00036807	0.002072533	0.005419205	0.013241682	0.987083124	0.994376062	0.997390368	0.999576623
164	0.000258309	0.002004053	0.005057248	0.01190319	0.986085989	0.994778775	0.997445271	0.999712442
165	0.00031855	0.002260855	0.005274689	0.01356814	0.98909418	0.995735766	0.998302241	0.999708642
166	0.000347984	0.001516536	0.004366642	0.012833512	0.988393353	0.995740934	0.997840815	0.999802398
167	0.000241717	0.001615966	0.005009631	0.012533991	0.987062724	0.994512663	0.997835476	0.999780398
168	0.000284665	0.002585498	0.004978404	0.013360584	0.989340369	0.995544058	0.99816864	0.999745631
169	0.000292937	0.001991781	0.004371377	0.011766938	0.989847979	0.995937277	0.998244976	0.999757547
170	0.000368737	0.002044987	0.004696333	0.013192439	0.986083861	0.994665859	0.998329993	0.999704536
171	0.000384153	0.002613929	0.00520357	0.0124102	0.987671812	0.994776048	0.99766707	0.999459995
172	0.000197853	0.002008705	0.004239165	0.011020231	0.985546177	0.994092818	0.997505188	0.999560923
173	0.000310823	0.001838322	0.005066721	0.012292926	0.986934903	0.995354745	0.997725201	0.999536979
174	0.000663912	0.002800397	0.00602073	0.014345843	0.989425636	0.995961292	0.998107943	0.999752334
175	0.000442722	0.002198764	0.004914897	0.012929605	0.986796125	0.994253533	0.99723937	0.999594291
176	0.000282618	0.002028842	0.004732956	0.012470569	0.98701296	0.994732782	0.998239384	0.999650327
177	0.000383892	0.002244313	0.005518606	0.012023814	0.989162766	0.99611901	0.998328242	0.999752954
178	0.000478727	0.001706572	0.004891133	0.012267048	0.987004307	0.99512016	0.997736663	0.999677223
179	0.00039094	0.002298292	0.005190632	0.012533068	0.987134491	0.994730595	0.998486741	0.999797162
180	0.000348857	0.002614971	0.005313458	0.012606714	0.986458689	0.994322806	0.997559151	0.999811781
181	0.000212882	0.001526695	0.003767602	0.010481688	0.987001009	0.99477701	0.997867383	0.999418022
182	0.000189839	0.002154853	0.004928801	0.011462153	0.986093254	0.993825865	0.997384406	0.999497291
183	0.000329237	0.001884528	0.005122288	0.011753421	0.988900653	0.994861177	0.997501743	0.999438214

184	0.000362134	0.001829844	0.004291953	0.011880352	0.988333613	0.994500567	0.997691986	0.999624782
185	0.000237555	0.001374863	0.003223828	0.010572117	0.987459248	0.994726175	0.997917177	0.999722955
186	0.000173062	0.002004158	0.005019359	0.011864639	0.986325458	0.995121414	0.997577365	0.999599653
187	0.00031989	0.002268443	0.004939664	0.012986209	0.986273808	0.993795326	0.998003098	0.999858666
188	0.000269819	0.002185691	0.005170211	0.011760054	0.988824439	0.995557159	0.997658157	0.99972937
189	0.000353294	0.001982308	0.004277506	0.010703235	0.987147947	0.994183023	0.997475738	0.999578431
190	0.000430245	0.002472087	0.005167188	0.012724879	0.986511505	0.993541884	0.997609099	0.999690918
191	0.000236465	0.002127249	0.004489755	0.012032073	0.987880374	0.994849479	0.998476186	0.99968103
192	0.000478588	0.002297743	0.004768201	0.01165628	0.988084895	0.994271208	0.997987984	0.999656554
193	0.000510177	0.00247232	0.005651682	0.012957349	0.989616125	0.996030102	0.998114221	0.999695243
194	0.000455141	0.001748061	0.00482381	0.011801988	0.988956325	0.99573558	0.998484459	0.999609067
195	0.000135404	0.002000843	0.004300858	0.011824337	0.988094663	0.994680507	0.99829951	0.999732142
196	0.000279715	0.002331049	0.00454011	0.011712863	0.985373371	0.993657015	0.996941015	0.9997457
197	0.000311634	0.00173886	0.004736863	0.011221726	0.989462354	0.995129424	0.998348532	0.999718411
198	0.000318254	0.001667391	0.004749075	0.01013499	0.987653276	0.994487473	0.998045179	0.999620933
199	0.000206805	0.002248437	0.005012959	0.011255363	0.988988313	0.996263753	0.998753021	0.999793711
200	0.000230161	0.001936901	0.003927824	0.011270664	0.989632143	0.996042493	0.99820919	0.999903721
201	0.000537383	0.002202163	0.005018093	0.013903117	0.988993855	0.995256401	0.997687169	0.999582777
202	0.000176038	0.001322855	0.003673271	0.011317916	0.987289958	0.995139941	0.998171252	0.999573497
203	0.000217027	0.002043083	0.004528242	0.011265731	0.985896262	0.994996072	0.998080463	0.999640181
204	0.000418302	0.002280531	0.004891498	0.012064689	0.987352002	0.994730882	0.998058733	0.999711473
205	0.000432936	0.00254	0.005512549	0.013772685	0.986718925	0.994281602	0.997424837	0.999342742
206	0.000455806	0.002292093	0.005756907	0.012989418	0.988156904	0.995012994	0.998390743	0.999793451
207	0.000181188	0.001627258	0.004490759	0.010900514	0.989056272	0.995272016	0.9979366	0.999640898
208	0.000208792	0.001872642	0.004914641	0.010566788	0.98603365	0.993836147	0.997643907	0.999558533
209	0.000401054	0.002811737	0.005283983	0.012076671	0.985064889	0.994381525	0.997488117	0.999549869
210	0.000322082	0.002235978	0.004869028	0.011234503	0.985374075	0.993769174	0.996968676	0.999644404
211	0.000165655	0.002322958	0.005984378	0.012760435	0.987388717	0.995119362	0.998098997	0.999717398
212	0.000236561	0.002034025	0.004974601	0.011726786	0.987878998	0.995758613	0.998091523	0.999650625
213	0.000237561	0.00184116	0.004617803	0.011828651	0.988449081	0.995235542	0.997796799	0.999821563
214	0.000336101	0.001935249	0.004995995	0.012549686	0.986700849	0.994759753	0.998086812	0.999460168
215	0.000282835	0.001714442	0.004330998	0.010751943	0.987888807	0.995415037	0.998111455	0.999690235
216	0.000424677	0.001765606	0.004773881	0.011250581	0.988320469	0.995711557	0.99841327	0.999814364
217	0.000340193	0.001586302	0.004453317	0.010099396	0.986545594	0.99478001	0.997842886	0.999652706
218	0.000360071	0.002123335	0.00426268	0.010618911	0.986666964	0.995066644	0.99760782	0.999538441
219	0.000373941	0.00179459	0.003846156	0.011476527	0.988906397	0.995856932	0.998315845	0.999692562
220	0.000442191	0.002164162	0.004556521	0.011070601	0.988695692	0.995404774	0.997608151	0.999635392
221	0.00017957	0.002078697	0.004343762	0.011147776	0.988836805	0.99547329	0.997667641	0.999543479
222	0.000153378	0.001481366	0.004268064	0.011562546	0.987128864	0.994568032	0.997104638	0.999547908
223	0.00033931	0.0014403	0.003411272	0.008828943	0.986388504	0.994286495	0.997548749	0.999795761
224	0.000362107	0.001808896	0.00397652	0.00938731	0.986888388	0.994532002	0.997257617	0.999574965

225	0.000146317	0.001721107	0.003755477	0.010665804	0.987110247	0.995325755	0.997875847	0.9998365
226	0.000367833	0.001834456	0.005051431	0.012038569	0.988821381	0.995552816	0.99824667	0.999684606
227	0.000192334	0.001509914	0.004722333	0.011688272	0.98712012	0.994667739	0.99767782	0.999690922
228	0.000145504	0.001750192	0.00385923	0.010973288	0.988242853	0.994472563	0.997834903	0.999492397
229	0.00055097	0.002290148	0.005953108	0.013027594	0.989590517	0.99592696	0.998303561	0.999830857
230	0.000143182	0.001389783	0.004150313	0.012214203	0.988633559	0.995257418	0.998077884	0.999640512
231	0.000483585	0.00236689	0.004973167	0.011210155	0.986738961	0.99450866	0.997539578	0.999515634
232	0.000300227	0.001731613	0.004097135	0.010409932	0.988633205	0.996004888	0.9985097	0.99973391
233	0.000257709	0.001764676	0.004220443	0.010653865	0.990338831	0.996078578	0.998283764	0.999785234
234	0.000264574	0.001734364	0.004085288	0.010519297	0.990461827	0.9956898	0.99765823	0.999737919
235	0.000417218	0.001543777	0.004525851	0.010375969	0.988852646	0.995136483	0.998424406	0.999789285
236	0.000171373	0.001839396	0.00448865	0.010811746	0.988603962	0.995127507	0.997821121	0.999807026
237	0.000438175	0.002172848	0.004811201	0.01195698	0.990234731	0.995804901	0.998382473	0.999621587
238	0.000165431	0.001486926	0.004555664	0.01245697	0.989305577	0.995639574	0.997818583	0.999731113
239	0.000226629	0.001580308	0.003697482	0.009366044	0.989803669	0.995733928	0.998136811	0.999752264
240	0.000343283	0.002506542	0.004987283	0.012048989	0.990121488	0.995868765	0.998552309	0.999827897
241	0.000218919	0.001408705	0.004023264	0.00989707	0.987686085	0.995226597	0.997881689	0.999772989
242	0.000260085	0.001749527	0.003996515	0.010583813	0.98799291	0.995392002	0.997729598	0.999280011
243	0.000420427	0.002110416	0.004443251	0.010291762	0.98975369	0.996034629	0.998411738	0.999751484
244	0.000221121	0.00140364	0.004449026	0.011070639	0.989557639	0.995507041	0.998038582	0.999826528
245	0.00025337	0.001759506	0.004125961	0.011520197	0.988287854	0.995382735	0.998436351	0.999844653
246	0.000318949	0.001563926	0.004445232	0.010557829	0.988603682	0.995205425	0.997904927	0.999570804
247	0.000249719	0.002208793	0.005561587	0.012242782	0.988626921	0.995959892	0.998412819	0.999745599
248	0.000188491	0.001924661	0.004091016	0.010426378	0.988114171	0.995537757	0.998526581	0.999884495
249	0.000249428	0.001745873	0.004495241	0.010774342	0.989896924	0.995516667	0.998482708	0.999813401
250	0.000209557	0.002059336	0.004297573	0.010950732	0.989339581	0.995040368	0.998006313	0.999839088
251	0.000267893	0.001847648	0.004171577	0.011374219	0.98844582	0.995419625	0.99819679	0.999755393
252	0.000263637	0.001687064	0.004578876	0.010111584	0.988699945	0.99562251	0.998129151	0.999645954
253	0.000295911	0.001818331	0.005023949	0.010727346	0.989227292	0.995665656	0.998424356	0.999740479
254	0.000592051	0.002036965	0.004556077	0.011404002	0.987768663	0.995277046	0.997963939	0.999769017
255	0.000220285	0.001107812	0.003413981	0.010145967	0.986843152	0.995022741	0.997687874	0.999596186
256	0.000409884	0.001772307	0.004491631	0.010689461	0.988399765	0.995249814	0.997959939	0.999733102
257	0.00026929	0.001989874	0.005174727	0.011633965	0.987555461	0.995307534	0.998228976	0.999689437
258	0.000164774	0.001840027	0.003781801	0.009864176	0.989644475	0.995807037	0.998198747	0.999711517
259	0.000251686	0.001466951	0.004917537	0.012388267	0.987700862	0.99473	0.998066719	0.999660196
260	0.000555684	0.001992212	0.004300455	0.009895284	0.989639225	0.995552913	0.998412068	0.999883656
261	9.71647E-05	0.001254543	0.003668236	0.009481846	0.988328886	0.995772813	0.998171727	0.999812539
262	0.000209659	0.001536376	0.004020971	0.011224419	0.988943561	0.996243881	0.998617488	0.999671079
263	0.000203768	0.002139751	0.005171403	0.011706729	0.989747129	0.995742196	0.99826464	0.999669786
264	0.00018911	0.001826662	0.003814227	0.009168041	0.987199535	0.994666658	0.997955188	0.999772596
265	0.00031036	0.001929072	0.004896771	0.010129007	0.988223575	0.994900872	0.997950846	0.999772675

266	0.000296477	0.001622849	0.003614933	0.009361899	0.989842333	0.995629428	0.998691949	0.999719142
267	0.000431248	0.001595598	0.004176801	0.01093363	0.989795044	0.995475025	0.998411706	0.999855872
268	0.000215429	0.001576656	0.003695787	0.009497306	0.988767994	0.995937751	0.998317283	0.999782134
269	0.000213748	0.001430777	0.003142554	0.008857877	0.990057729	0.995834372	0.998135868	0.999686735
270	0.000190546	0.002419762	0.004664268	0.011123372	0.98838934	0.994992655	0.997748789	0.999578946
271	0.000295185	0.001608045	0.004350666	0.010418708	0.988315587	0.994722024	0.997898949	0.999828402
272	0.000162773	0.001583927	0.004175077	0.010501611	0.988534251	0.994660997	0.99759964	0.999661434
273	0.000172452	0.001276829	0.004215119	0.011650497	0.98960702	0.995740542	0.998220345	0.999857773
274	0.000417964	0.001906854	0.004549514	0.011017421	0.990354204	0.995880483	0.998140926	0.999765062
275	9.57954E-05	0.001512587	0.003904737	0.010366752	0.990990479	0.995886669	0.998398203	0.999844351
276	0.000352001	0.002766455	0.005802692	0.012029747	0.988066024	0.994964721	0.99816507	0.999814351
277	0.000464388	0.002077834	0.004446764	0.010252315	0.988579175	0.995655583	0.998262676	0.999677924
278	0.000267651	0.001881078	0.004041211	0.010785668	0.990501572	0.996116451	0.997996743	0.999687686
279	0.000169341	0.001836603	0.004349013	0.010710338	0.988959817	0.995218718	0.997562041	0.999578341
280	0.000332574	0.001711705	0.003823912	0.010131619	0.989358592	0.995723364	0.998245048	0.999679363
281	0.000292902	0.002129765	0.005060973	0.011111612	0.990146522	0.995633183	0.998022736	0.999655512
282	0.000287829	0.00176523	0.003937678	0.008783941	0.988262542	0.9955911	0.99791677	0.99950868
283	0.000166556	0.001899531	0.004592836	0.011474715	0.990835999	0.996328021	0.998556265	0.999698589
284	0.000327367	0.002336127	0.004353837	0.009930637	0.988983315	0.996173122	0.998327618	0.99975697
285	0.00030302	0.001934949	0.005204218	0.012788412	0.990558792	0.995995474	0.998037753	0.999605056
286	0.000231882	0.001752509	0.004448941	0.010334761	0.989138673	0.995962762	0.998235459	0.999786748
287	0.000305088	0.00234468	0.005173454	0.011432041	0.990867951	0.995653037	0.998124161	0.99962985
288	0.000226755	0.001519492	0.004053657	0.010741754	0.987914852	0.995957295	0.998507269	0.999816317
289	0.000156069	0.001424048	0.004024029	0.010601652	0.989770448	0.995974533	0.998287644	0.999692254
290	0.000226876	0.001352639	0.003693397	0.00908116	0.990492837	0.996310848	0.998394489	0.99980442
291	0.000388319	0.002314076	0.004902452	0.012794079	0.989841709	0.996192272	0.998399772	0.999722123
292	0.000148388	0.000944864	0.003240066	0.008626964	0.989120574	0.995550161	0.998108452	0.999628721
293	0.0003686	0.001716222	0.00367346	0.009904873	0.990811647	0.996168507	0.998181309	0.99964272
294	0.00046505	0.001841093	0.003869711	0.009924552	0.989709716	0.995534066	0.998208259	0.999817762
295	0.000179897	0.00136005	0.004093266	0.010259884	0.987343039	0.994966258	0.998030636	0.99974179
296	0.000490915	0.001602322	0.003695175	0.008423328	0.989366286	0.996044579	0.998438963	0.999805211
297	0.000315353	0.001727465	0.003678713	0.009845055	0.989054491	0.996258879	0.99857308	0.999696589
298	0.000288471	0.001672668	0.003730065	0.009927703	0.988926513	0.995498748	0.998162122	0.999795879
299	0.000316677	0.001924234	0.003781763	0.009828358	0.989355431	0.995730318	0.998348373	0.999718823
300	0.00021502	0.001828114	0.004231395	0.010441044	0.990229229	0.996367781	0.998620507	0.999789966
301	0.000276837	0.001781869	0.004141599	0.010615963	0.990347165	0.995598042	0.997813345	0.999593897
302	0.000213789	0.001398179	0.003620004	0.00953995	0.990052582	0.995945363	0.998286143	0.999602751
303	0.000374642	0.001867262	0.003722418	0.009579334	0.988588115	0.995506531	0.998196716	0.999780822
304	0.000149151	0.001942703	0.004742842	0.01159285	0.989007228	0.995361907	0.998478393	0.999654792
305	0.000265795	0.001567024	0.003280002	0.009770594	0.989707408	0.996225096	0.998517593	0.999750783
306	0.000325476	0.002096184	0.00455348	0.011221794	0.990773972	0.996506517	0.998645993	0.999847756

307	0.000283175	0.001709573	0.004310128	0.010036369	0.989240037	0.995405262	0.998126532	0.999724587
308	0.000383521	0.001478081	0.003952487	0.008471153	0.989581179	0.995232822	0.998183172	0.999840427
309	0.00021861	0.002145037	0.004850995	0.01024544	0.989505109	0.995474909	0.998284498	0.99980827
310	0.000276987	0.001351835	0.003951298	0.009982712	0.989444846	0.995833879	0.998314726	0.999855059
311	0.000312584	0.001966108	0.00485772	0.009789683	0.990658729	0.996004351	0.998138576	0.99973991
312	0.000240912	0.001365087	0.003354041	0.010206974	0.989657895	0.996535674	0.998596713	0.999788005
313	0.00020135	0.002339288	0.004692267	0.011347891	0.990193772	0.996029142	0.998217621	0.999771948
314	0.000223289	0.002042095	0.004650113	0.010406989	0.990040428	0.995535546	0.998142651	0.999770525
315	0.000138073	0.001412155	0.0031216	0.008511064	0.987346386	0.995853729	0.998336055	0.999799661
316	0.000458548	0.001891167	0.004666518	0.011711909	0.989433961	0.996603556	0.998433513	0.999836249
317	0.000490089	0.002020853	0.005352334	0.01281496	0.990438848	0.996064653	0.998590115	0.999730134
318	0.00024289	0.001567912	0.003786571	0.010548314	0.990058002	0.996109249	0.998193572	0.999683548
319	0.00028018	0.001292069	0.003266	0.009123642	0.989212068	0.9954709	0.998220977	0.999673803
320	0.00032605	0.001859948	0.004065376	0.009330552	0.990033967	0.995982734	0.998119313	0.999625794
321	0.000270891	0.001843358	0.004567146	0.009207216	0.989191086	0.995974709	0.998288921	0.999630722
322	0.000322027	0.001947024	0.004916747	0.011543765	0.990062203	0.996478322	0.998478167	0.999612196
323	0.000265168	0.002316601	0.004077095	0.010335713	0.989190296	0.995530294	0.998107166	0.999802077
324	0.000391424	0.002164199	0.004520951	0.010313026	0.989950047	0.995873679	0.997659544	0.999733213
325	0.000217689	0.001461597	0.004099035	0.009782299	0.989304417	0.996548755	0.998668375	0.999740577
326	0.000239259	0.001737765	0.004150026	0.009203526	0.988285044	0.994773375	0.998232929	0.999672426
327	0.000261379	0.001566458	0.003481697	0.009178959	0.989814362	0.995979414	0.998300528	0.999757169
328	0.000252237	0.002012486	0.003731435	0.009515572	0.991184302	0.995964754	0.998242341	0.999696389
329	0.000158098	0.001580046	0.004063099	0.00995984	0.989938384	0.996165684	0.998214809	0.999786011
330	0.000245368	0.002192259	0.005133074	0.010443879	0.989236475	0.995442394	0.998178393	0.999791723
331	0.00026567	0.001934683	0.003951631	0.010508933	0.990759908	0.996080873	0.998129981	0.999813889
332	0.000110178	0.001458797	0.004223611	0.010026315	0.989595393	0.995253892	0.998257095	0.999703655
333	0.000329525	0.001163128	0.00396885	0.009703724	0.987794874	0.994231279	0.99802808	0.99958831
334	0.000301294	0.002044955	0.004321183	0.010603245	0.990964503	0.996847313	0.998528902	0.999837444
335	0.000357809	0.002030018	0.004472582	0.011116884	0.988817714	0.99564085	0.998029602	0.999578742
336	0.000287577	0.002441253	0.004889212	0.011510117	0.989790935	0.995229058	0.997970495	0.999791248
337	0.000250793	0.001617738	0.004152718	0.009584352	0.991297828	0.996212061	0.998485217	0.999715213
338	0.000254801	0.001322395	0.003525392	0.009237644	0.989800391	0.996363822	0.998495792	0.999826876
339	0.000350939	0.001680331	0.004004825	0.010121086	0.990897818	0.996266005	0.99861121	0.999759636
340	0.000265829	0.001326998	0.003487582	0.009426869	0.988440801	0.99593417	0.99828551	0.99969253
341	9.30124E-05	0.001466285	0.00376528	0.009248257	0.989803146	0.996515443	0.998588162	0.99959439
342	0.000358876	0.002182076	0.00438182	0.010388972	0.991134331	0.996324802	0.998636834	0.999752258
343	0.000398573	0.001465505	0.003884571	0.010019686	0.989935609	0.995933626	0.998237639	0.999753291
344	0.000139579	0.001090406	0.003121183	0.009175222	0.990216815	0.996581581	0.998415901	0.999707257
345	0.000269117	0.001522145	0.003548112	0.008795094	0.989348312	0.99537137	0.997895796	0.99971014
346	0.000219207	0.001696502	0.003559544	0.008505698	0.989189486	0.995453898	0.998209968	0.99967547
347	0.00022187	0.001575799	0.004036238	0.008939477	0.989140452	0.995014799	0.997646529	0.999409987

348	0.000216671	0.001587469	0.00426095	0.010109983	0.98804546	0.995289558	0.99818736	0.999654307
349	0.000140284	0.002173961	0.004159409	0.010037784	0.989944263	0.996784106	0.998740641	0.999787392
350	0.000357405	0.001441681	0.004126381	0.009859903	0.990404915	0.995669187	0.998130087	0.999811338
351	0.000161107	0.001858038	0.004676455	0.010094362	0.990169712	0.99587381	0.998189415	0.999633667
352	0.00019222	0.001562044	0.003950649	0.009754891	0.989092659	0.995953353	0.997987697	0.999727424
353	0.000305177	0.001599893	0.003751663	0.010063321	0.990546415	0.996796472	0.998484992	0.999818306
354	0.000165743	0.001748079	0.003614446	0.008822208	0.988413746	0.995525059	0.997754862	0.999542223
355	0.000444605	0.001071572	0.002925737	0.009542096	0.989197868	0.995907254	0.998132845	0.999688941
356	0.000218594	0.001681113	0.00402085	0.00842113	0.989280535	0.995094511	0.998268973	0.999540105
357	0.00029157	0.001433248	0.003836921	0.009867764	0.990482327	0.996390489	0.998717302	0.999855757
358	0.000240268	0.00179512	0.004009537	0.009739534	0.98965945	0.996213408	0.998656879	0.999713129
359	0.000461039	0.001690202	0.003920468	0.010041528	0.990097403	0.996100886	0.998363247	0.999704618
360	0.000335401	0.001693602	0.004127087	0.00959968	0.989458105	0.995412927	0.997636844	0.999617635
361	0.000223302	0.001359553	0.003843353	0.010001066	0.988374303	0.996208973	0.998367642	0.99970385
362	0.000124692	0.001550854	0.003909251	0.009205231	0.989387554	0.995706308	0.998321304	0.999744294
363	0.000184322	0.001512983	0.003482282	0.010725949	0.990094098	0.995756333	0.997972068	0.999656868
364	0.000191056	0.001423547	0.003476669	0.009693317	0.990679271	0.996369626	0.998495754	0.999749264
365	0.000276883	0.001848496	0.003947945	0.012254286	0.990901715	0.996860058	0.998553646	0.9999238
366	0.000440736	0.00182231	0.003805371	0.009289632	0.991665251	0.996143005	0.998710633	0.999784798
367	0.000370411	0.001890842	0.003780246	0.010226945	0.991887642	0.996697554	0.998361832	0.999741236
368	0.000297668	0.001654504	0.003516759	0.00878231	0.990288593	0.996353293	0.998529528	0.999710049
369	0.000232142	0.001487026	0.003981658	0.010022693	0.989255658	0.995780368	0.997852207	0.999664099
370	0.0002716	0.001390466	0.003208076	0.009688629	0.989430201	0.99567511	0.998542839	0.999691731
371	0.000212532	0.001875053	0.003782426	0.009112277	0.990094754	0.99634745	0.998279929	0.999795618
372	0.000177892	0.002199009	0.004591819	0.009254239	0.990144473	0.996748537	0.998678576	0.999786245
373	0.000223025	0.001434237	0.003760246	0.009523896	0.988793071	0.995947606	0.998332361	0.999670241
374	0.00028832	0.001376128	0.003663728	0.011177525	0.990866836	0.995919994	0.998094619	0.999655499
375	0.000434607	0.002147429	0.004561242	0.009835281	0.991013087	0.996084649	0.998412228	0.999767482
376	0.000178834	0.001626302	0.003805877	0.00903646	0.990260793	0.996208531	0.99805013	0.999555029
377	0.000201977	0.001448278	0.003876907	0.009426731	0.991319505	0.996106643	0.997987788	0.999783601
378	0.000312447	0.001642531	0.003786077	0.008617835	0.99009629	0.996336992	0.998718494	0.999860643
379	0.000172418	0.001437813	0.003923747	0.009623791	0.989373886	0.99596877	0.998420193	0.999634802
380	0.000181381	0.001587488	0.003958205	0.00908892	0.990153386	0.995917707	0.998164044	0.999667063
381	0.00032952	0.001611753	0.003663837	0.007812727	0.991952746	0.996444402	0.998611356	0.999807875
382	0.000253921	0.001658526	0.003537976	0.008449799	0.990187529	0.995885135	0.998435755	0.999813642
383	0.000327886	0.001461752	0.003427979	0.009086691	0.990103525	0.996274168	0.998559071	0.9998214
384	0.000133399	0.001158713	0.003521056	0.008370383	0.990894771	0.996349246	0.998750255	0.999676347
385	0.000164609	0.001253623	0.003273422	0.008938163	0.990608335	0.996451811	0.998551787	0.999819625
386	0.00022198	0.00126128	0.002667168	0.008285418	0.989739383	0.996459962	0.998410306	0.999795846
387	0.000197952	0.001638156	0.004045254	0.009116004	0.990084499	0.995878661	0.998446025	0.999727686
388	0.000152094	0.001506372	0.003773055	0.009285309	0.990588075	0.996294708	0.998453283	0.999803108

389	0.000325987	0.001427664	0.003530317	0.009177886	0.987231847	0.995570557	0.998316956	0.999767528
390	0.000174917	0.001354869	0.003658413	0.008972175	0.990756249	0.996547347	0.998488897	0.999726895
391	0.000316683	0.001451376	0.003820979	0.010690121	0.990890377	0.996381488	0.998261124	0.999644161
392	0.000396029	0.001407889	0.003755189	0.009174749	0.990082767	0.996268025	0.99868601	0.999779807
393	0.00011975	0.001514316	0.003659654	0.009587529	0.990458246	0.996092884	0.998464869	0.999742053
394	0.000265566	0.001916592	0.003668819	0.009548166	0.990131073	0.996074741	0.998302992	0.999781826
395	0.000294134	0.001765021	0.003681794	0.008274213	0.989585556	0.996143113	0.998571821	0.999826736
396	0.000152046	0.001367734	0.003218804	0.009934972	0.989828143	0.995253444	0.998484268	0.999836537
397	0.000219053	0.001754549	0.00353827	0.008542177	0.989417289	0.995412918	0.998279125	0.999601515
398	0.000203734	0.001245396	0.003135902	0.008726902	0.991501903	0.996881773	0.998557453	0.999907866
399	0.000250527	0.001702101	0.003589556	0.009393197	0.989399593	0.995194777	0.997595352	0.99982323
400	0.00012093	0.001255587	0.002942631	0.009150487	0.987892729	0.99528317	0.997757976	0.999780666
401	0.000232912	0.001735111	0.003941965	0.009517682	0.990516802	0.996649963	0.998651313	0.999809869
402	0.000419821	0.00193535	0.004279498	0.009982764	0.990968229	0.996125488	0.998468149	0.999748116
403	0.000226814	0.001520667	0.003379166	0.009135387	0.989630721	0.996141859	0.998102606	0.999790146
404	0.000172007	0.001619202	0.003216993	0.008184156	0.989590723	0.99537647	0.998100662	0.999535005
405	0.000138499	0.001655014	0.004136441	0.009389667	0.98979145	0.99598541	0.998322825	0.999838282
406	0.000311955	0.001563849	0.003216434	0.009200679	0.991869084	0.996593529	0.998312673	0.999768094
407	0.000134129	0.001089159	0.002493451	0.00789856	0.989956076	0.995778613	0.998221553	0.999734498
408	0.000126339	0.001748537	0.004094598	0.009352256	0.99016112	0.996183272	0.998025662	0.999639623
409	0.000216818	0.000965306	0.002761941	0.007964223	0.990541482	0.996563208	0.998437165	0.999836093
410	0.000217427	0.00159289	0.00400905	0.009467759	0.990141556	0.99663947	0.998676109	0.99989999
411	0.000153821	0.0018788	0.0038922	0.009489954	0.989821614	0.996326592	0.998455802	0.999758202
412	0.000200801	0.001495639	0.004130786	0.009831684	0.989771118	0.995473621	0.998325343	0.999800231
413	0.000243053	0.001773559	0.003564273	0.007995547	0.990080003	0.996119272	0.998246401	0.999786498
414	0.000154429	0.001830039	0.004008545	0.009684259	0.991458732	0.996354714	0.998350495	0.999786546
415	0.000227344	0.001419337	0.003664854	0.009291607	0.990120703	0.996285385	0.998399845	0.999752126
416	0.000282638	0.002074117	0.00438166	0.010336195	0.991448614	0.996001635	0.99847472	0.99974304
417	0.000306997	0.001582199	0.004188919	0.01014688	0.991760317	0.996902343	0.9988651	0.999766196
418	0.000114959	0.001175069	0.003050701	0.007845317	0.989455763	0.996227997	0.998126963	0.999821673
419	0.000183751	0.001320088	0.003817253	0.010839712	0.990767284	0.996516933	0.99862931	0.999781782
420	0.000330123	0.001892555	0.004656462	0.010320216	0.990873801	0.99529913	0.997862375	0.999713821
421	0.000259154	0.001053795	0.003002395	0.007741465	0.99019236	0.995761181	0.997978495	0.99968975
422	0.000227547	0.001888093	0.003584851	0.008497502	0.991208575	0.996781594	0.99867841	0.999871749
423	0.000354862	0.001552575	0.004109736	0.009967771	0.992201951	0.996601314	0.998732439	0.999660471
424	0.000128747	0.001758685	0.003565115	0.009539231	0.990354168	0.995700831	0.997926343	0.99954652
425	0.000425218	0.002531376	0.005270701	0.010790427	0.990793561	0.996169942	0.998421558	0.999630765
426	0.00019846	0.00153703	0.003819494	0.008765269	0.99143093	0.996699818	0.99880562	0.999892065
427	0.000481714	0.0024667	0.004166068	0.009610737	0.989217031	0.995959306	0.998064263	0.999616929
428	0.000239832	0.001687141	0.004010027	0.010282956	0.99007221	0.995986277	0.998311695	0.999767491
429	0.000317204	0.002237918	0.003783467	0.009911943	0.990385873	0.99660723	0.998759241	0.999845923

430	0.000322293	0.001947131	0.003756068	0.008598828	0.990285376	0.996125283	0.998245476	0.999754593
431	0.000315867	0.00153381	0.003985355	0.009219217	0.989911522	0.995694899	0.998645527	0.999844865
432	0.000248153	0.00160218	0.004291154	0.010605559	0.991709916	0.996781719	0.998449686	0.999753226
433	0.000204534	0.001345185	0.003989222	0.00939231	0.991332849	0.996589145	0.998603133	0.99982137
434	0.000326581	0.001780023	0.00421219	0.009683407	0.99081561	0.995800052	0.997976479	0.999574088
435	0.000277661	0.001677368	0.003791554	0.010056305	0.9903495	0.996348462	0.998041941	0.999735403
436	0.000235672	0.001223697	0.003144821	0.007160729	0.989884057	0.995778242	0.99850024	0.999778266
437	0.000244208	0.001631242	0.003547539	0.008574544	0.991199945	0.996654668	0.99868287	0.999807926
438	0.000406337	0.001921692	0.005043452	0.009936233	0.991098517	0.996945324	0.998597452	0.999834723
439	0.000246284	0.00152234	0.00333641	0.007489236	0.990993339	0.996494608	0.998455428	0.999809883
440	0.000198794	0.001140072	0.003235622	0.008445115	0.990879431	0.99678369	0.998678173	0.999907406
441	0.000214434	0.001336581	0.003399146	0.008693095	0.990389383	0.995751641	0.998656422	0.999771252
442	0.000383331	0.002130773	0.004326972	0.010746293	0.990826496	0.996474113	0.998359986	0.999704969
443	0.000317582	0.001389012	0.003368718	0.009116796	0.991206437	0.996522417	0.998430593	0.999766768
444	0.000203371	0.001360002	0.003632114	0.010162414	0.989371737	0.995724643	0.998155186	0.999873339
445	0.000295773	0.001673545	0.003936555	0.010672845	0.990981008	0.996314827	0.998399489	0.999816912
446	0.000491468	0.001890831	0.003770916	0.008717515	0.991185028	0.995743668	0.998579336	0.9997266
447	0.000253167	0.001606864	0.003576923	0.008255087	0.992048923	0.997226717	0.998714974	0.999842842
448	0.000196925	0.001827553	0.003660429	0.008524499	0.990554829	0.996199126	0.99843179	0.999426684
449	0.000257296	0.001449608	0.003715001	0.009611737	0.991350908	0.995894108	0.998064936	0.999800186
450	0.000321993	0.001831872	0.004339727	0.008931542	0.99249742	0.99703974	0.998623231	0.999772725
451	0.000269961	0.001910384	0.004100507	0.009655646	0.991431726	0.996412358	0.998257423	0.999732644
452	0.000358152	0.001606315	0.003397191	0.009250827	0.99136135	0.99611684	0.998169342	0.999701994
453	0.000293479	0.001614532	0.003413505	0.010654902	0.991278399	0.996240876	0.998235831	0.999623272
454	0.00020534	0.001701822	0.00390772	0.00900017	0.991474466	0.996761271	0.998523222	0.99982714
455	0.000285976	0.001417091	0.003310302	0.008287035	0.991248105	0.996697857	0.998546869	0.999736751
456	0.000168106	0.001685107	0.003739855	0.008296168	0.990345787	0.995980505	0.998178194	0.999746547
457	0.000217187	0.001244826	0.003856033	0.009112698	0.991899013	0.996777094	0.998737558	0.999865493
458	0.000235325	0.00125632	0.003305601	0.008620974	0.990423298	0.996293941	0.998760339	0.999787999
459	0.000178177	0.001497386	0.003486683	0.008252343	0.991128274	0.996620603	0.998678713	0.99979248
460	0.000195061	0.00135681	0.003582562	0.00901814	0.990897549	0.996642283	0.998567264	0.999668552
461	0.000257549	0.001274849	0.003206989	0.008471405	0.991454395	0.996592795	0.998643495	0.999776621
462	0.000299697	0.001537936	0.003350322	0.008296846	0.990958396	0.996597502	0.998484284	0.999842459
463	0.000257677	0.00132814	0.00311252	0.008629486	0.991899307	0.99641602	0.998022213	0.999633252
464	0.0001524	0.001501242	0.00336075	0.008607274	0.989038361	0.995180337	0.997893377	0.999729864
465	0.000140826	0.001181743	0.003416041	0.009719507	0.990052623	0.996303792	0.998479659	0.999815784
466	0.000165387	0.001839924	0.004229226	0.00946231	0.990821641	0.996787847	0.998531303	0.999677825
467	0.000210439	0.001485562	0.00322183	0.008477202	0.99085247	0.996580169	0.998372503	0.999730596
468	0.000346611	0.001483955	0.003368688	0.009360862	0.990583199	0.995622603	0.998007956	0.999818747
469	0.000122461	0.001657771	0.003712936	0.008051372	0.991359305	0.996155697	0.998506217	0.99981631
470	0.000270087	0.002183827	0.003604587	0.009286661	0.990103553	0.995616924	0.998034215	0.999842081

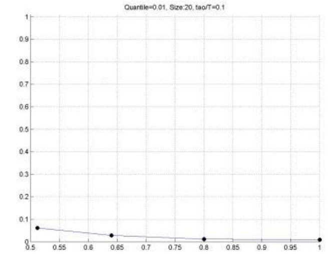
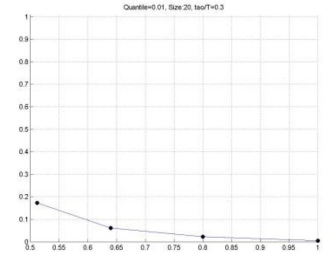
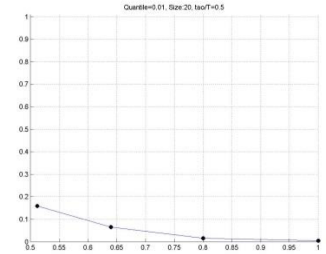
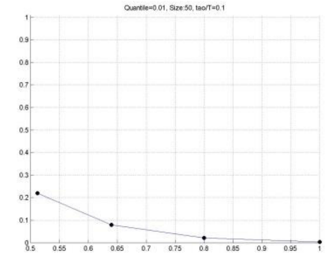
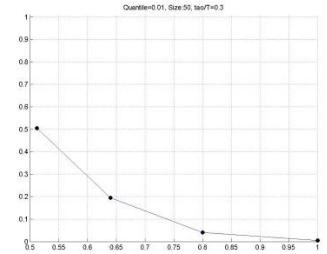
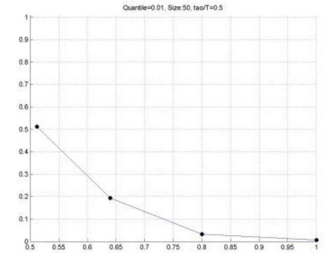
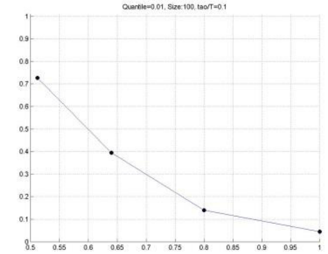
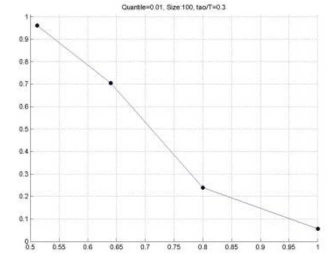
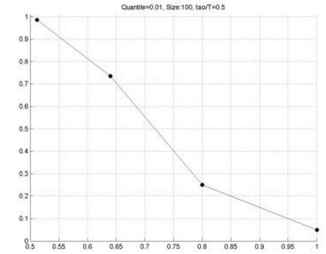
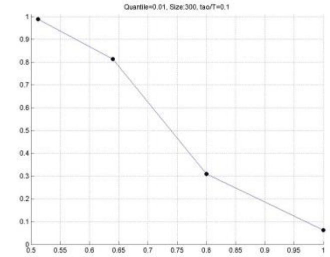
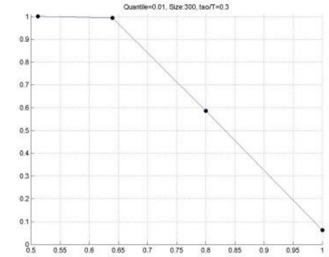
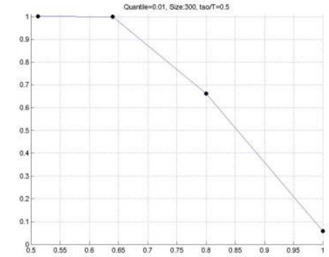
471	0.000339056	0.001485345	0.003818168	0.008356132	0.991378445	0.995990085	0.998170605	0.999918992
472	0.000249993	0.001309808	0.003470289	0.009552964	0.992680314	0.997250432	0.998850189	0.999794468
473	8.73137E-05	0.001588529	0.003865895	0.009951356	0.9896371	0.99611513	0.998441594	0.999734121
474	0.000290071	0.001705285	0.004253913	0.010620294	0.991575228	0.996403586	0.998472452	0.999794363
475	0.000239102	0.001748503	0.005231113	0.010094764	0.989505928	0.995664926	0.998157898	0.999842281
476	0.000235506	0.001628241	0.003177749	0.008153966	0.989945404	0.996202906	0.998118352	0.99955287
477	0.000278466	0.001694373	0.003865757	0.010692142	0.990253841	0.995935592	0.998584352	0.999753194
478	0.000307977	0.002165689	0.004344598	0.009018021	0.990394583	0.996666522	0.998712435	0.99982409
479	0.000169858	0.001534047	0.003486087	0.00915392	0.991595122	0.99648737	0.998441241	0.999729028
480	0.000313879	0.001662545	0.003366114	0.008967209	0.99108481	0.996445265	0.998764019	0.999762075
481	0.00019102	0.001212263	0.002903627	0.007877022	0.990920867	0.995702346	0.998274128	0.99966632
482	0.000235142	0.001628504	0.003377922	0.008473774	0.992078131	0.996792535	0.998531746	0.999866226
483	0.000286551	0.001532437	0.003356553	0.00843929	0.990435408	0.996708107	0.998728305	0.99982733
484	0.000255119	0.00154608	0.004287688	0.010379908	0.991235945	0.996553863	0.998546306	0.999780929
485	0.000183007	0.001421818	0.003170386	0.007196498	0.990714647	0.995957872	0.998302129	0.99982494
486	0.00015644	0.001352361	0.003286322	0.008246181	0.990801784	0.996121898	0.99822302	0.999862135
487	0.000176786	0.001932838	0.004747782	0.011870643	0.990785031	0.995939229	0.998103443	0.999809334
488	0.000224888	0.001464382	0.003766283	0.008950357	0.991526851	0.996691875	0.998667395	0.999841602
489	0.000477756	0.002054075	0.004652053	0.010359872	0.990250435	0.995860542	0.998122924	0.999769702
490	0.000211255	0.001341472	0.003582599	0.008797496	0.990454255	0.99596327	0.998346308	0.99982239
491	0.000157055	0.001876536	0.003896197	0.009016001	0.990549351	0.995967518	0.998147761	0.999648553
492	0.000328804	0.001781779	0.004401046	0.008770435	0.990883596	0.996281546	0.998445223	0.999757999
493	0.000207493	0.001261063	0.002965912	0.00844858	0.991346768	0.996819536	0.99851085	0.999772453
494	0.000221466	0.001424534	0.003477027	0.009530355	0.991698149	0.996680203	0.998415631	0.999819253
495	0.000239302	0.001596668	0.003276787	0.009047338	0.991471963	0.997074114	0.998736854	0.999850894
496	0.000232754	0.001953951	0.00481398	0.009826969	0.990234702	0.995980831	0.998365828	0.999711706
497	0.000227299	0.001311805	0.002931281	0.007308908	0.991202407	0.996552368	0.998693881	0.999759744
498	0.000333871	0.001484681	0.003488396	0.00788519	0.990931447	0.996402231	0.998406544	0.999776131
499	0.000366294	0.001894744	0.004296421	0.010531073	0.99051937	0.996243242	0.998531547	0.999878911
500	0.000245751	0.001467289	0.0032862	0.008689101	0.990868729	0.996390863	0.9987614	0.999908893

Appendix 3. Plots.

A.3.1 Power of Samuel´s test.

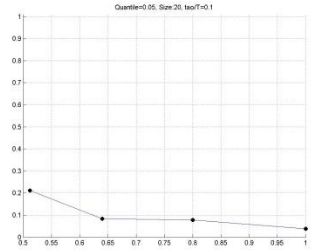
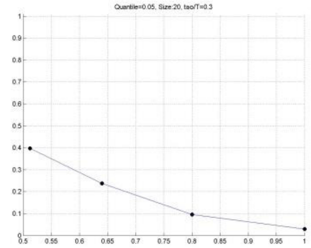
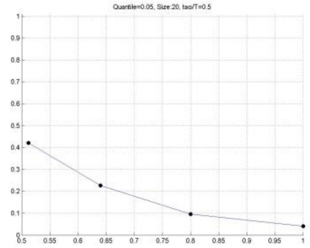
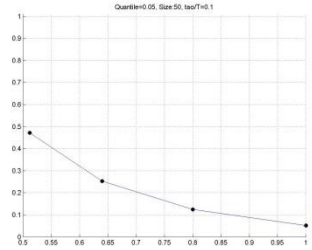
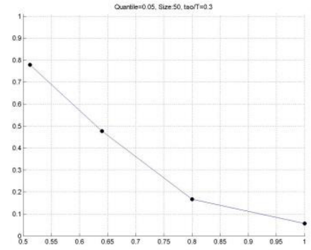
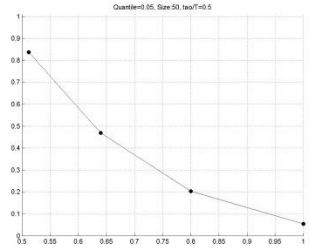
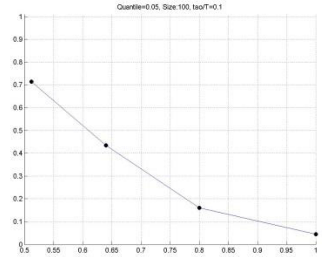
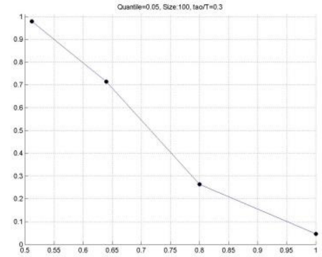
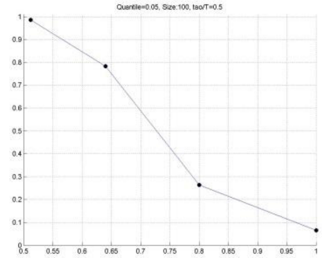
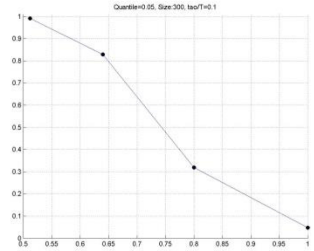
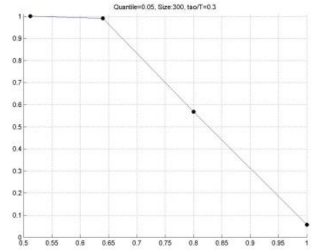
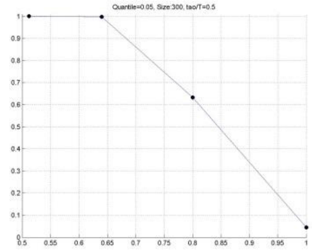
$$\boldsymbol{S_{.01}}$$

T	$\frac{\tau}{T} = .1$	$\frac{\tau}{T} = .3$	$\frac{\tau}{T} = .5$

20			
50			
100			
300			

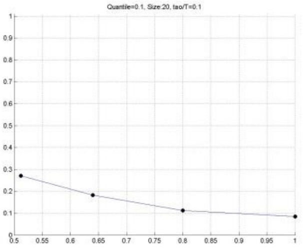
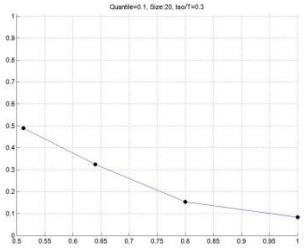
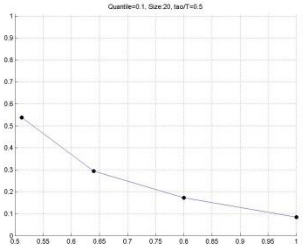
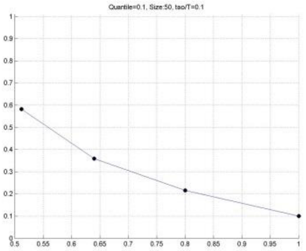
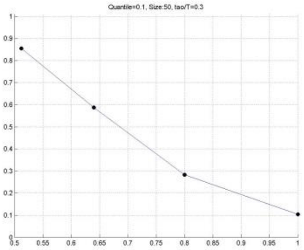
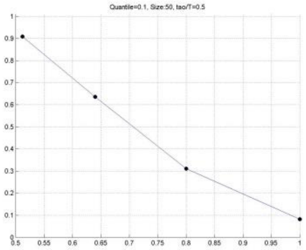
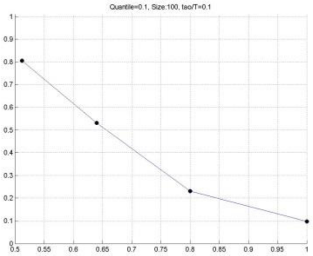
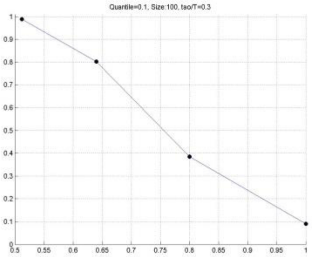
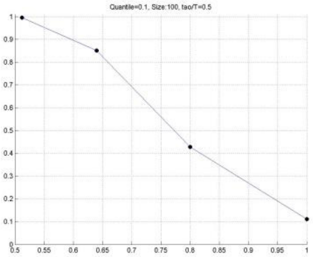
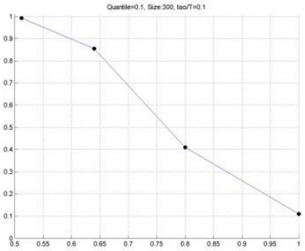
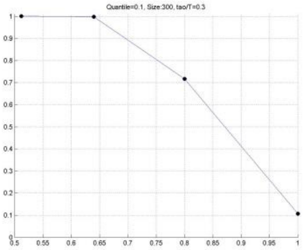
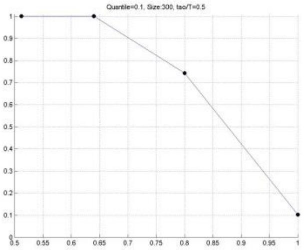
$\boldsymbol{S_{.05}}$

T	$\frac{\tau}{T} = .1$	$\frac{\tau}{T} = .3$	$\frac{\tau}{T} = .5$

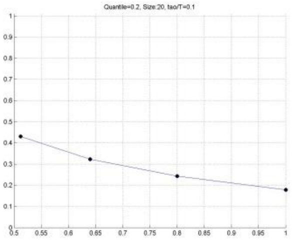
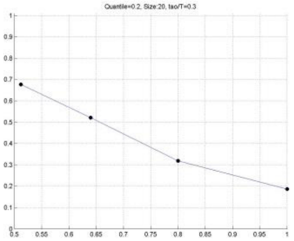
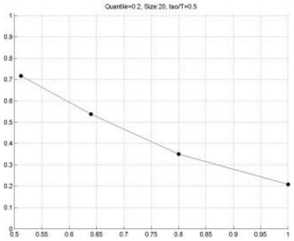
20	 Quantile=0.05, Size:20, tao/T=0.1 	 Quantile=0.05, Size:20, tao/T=0.3 	 Quantile=0.05, Size:20, tao/T=0.5
50	 Quantile=0.05, Size:50, tao/T=0.1 	 Quantile=0.05, Size:50, tao/T=0.3 	 Quantile=0.05, Size:50, tao/T=0.5
100	 Quantile=0.05, Size:100, tao/T=0.1 	 Quantile=0.05, Size:100, tao/T=0.3 	 Quantile=0.05, Size:100, tao/T=0.5
300	 Quantile=0.05, Size:300, tao/T=0.1 	 Quantile=0.05, Size:300, tao/T=0.3 	 Quantile=0.05, Size:300, tao/T=0.5

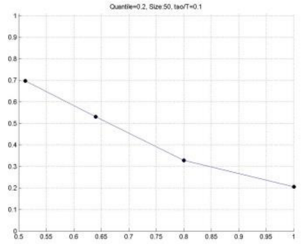
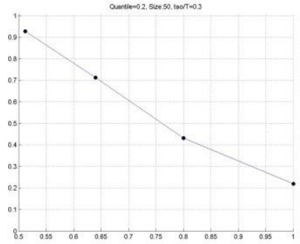
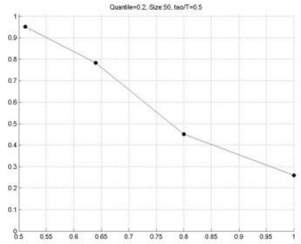
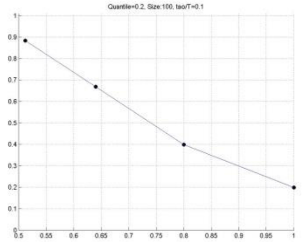
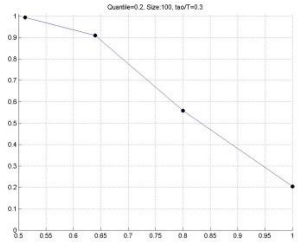
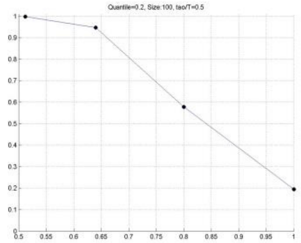
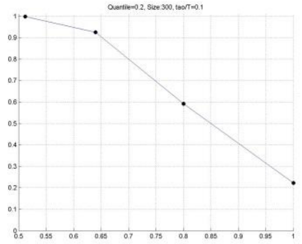
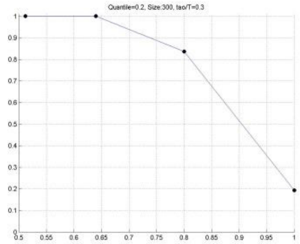
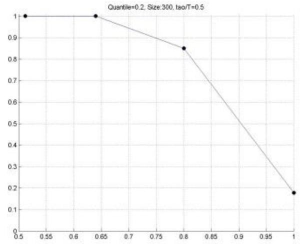
$$\boldsymbol{S_{.1}}$$

T	$\frac{\tau}{T} = .1$	$\frac{\tau}{T} = .3$	$\frac{\tau}{T} = .5$

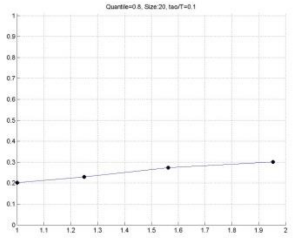
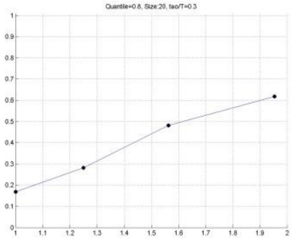
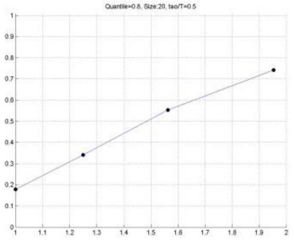
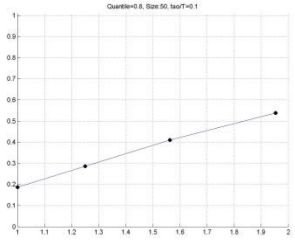
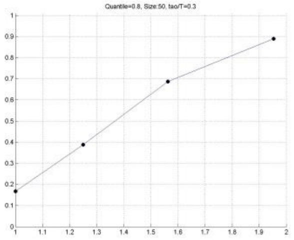
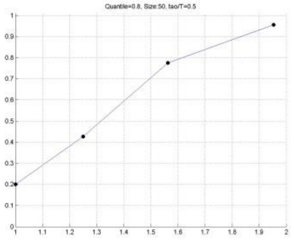
20			
50			
100			
300			

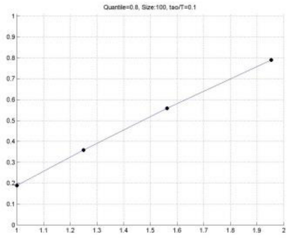
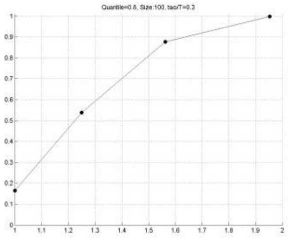
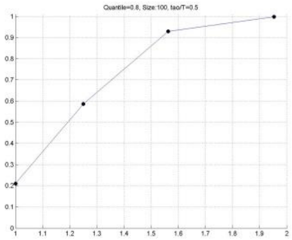
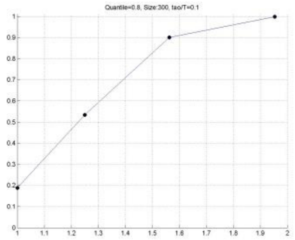
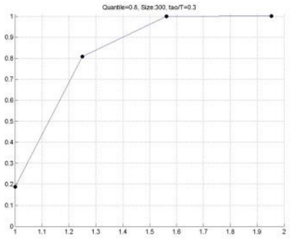
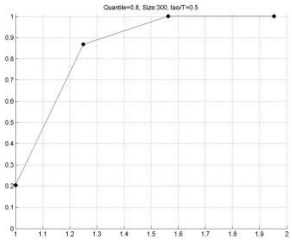
$\boldsymbol{s}_{.2}$

T	$\frac{\tau}{T} = .1$	$\frac{\tau}{T} = .3$	$\frac{\tau}{T} = .5$
20			

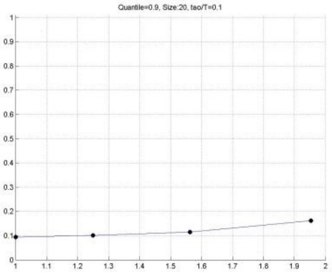
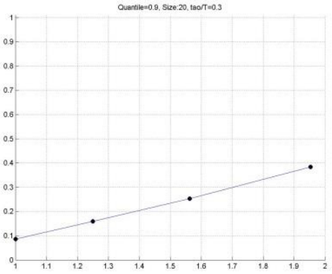
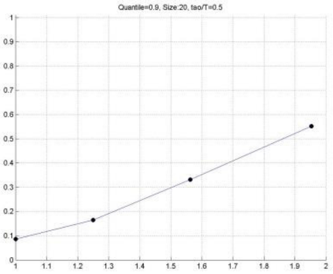
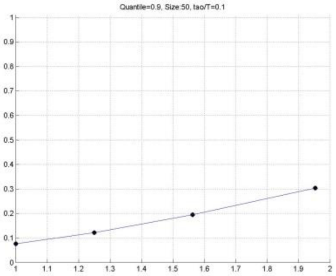
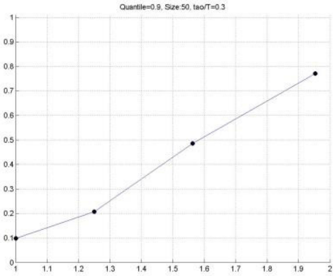
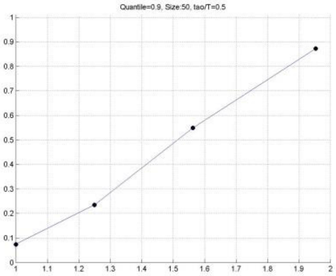
50			
100			
300			

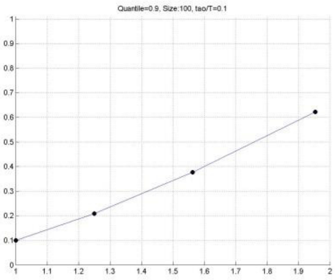
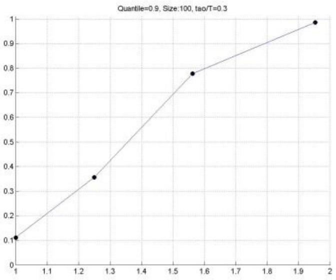
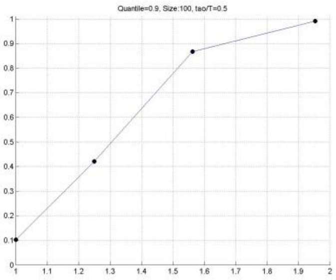
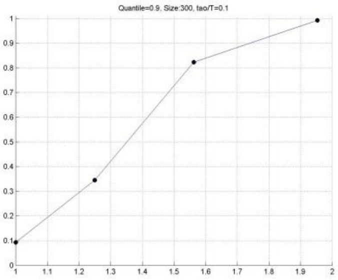
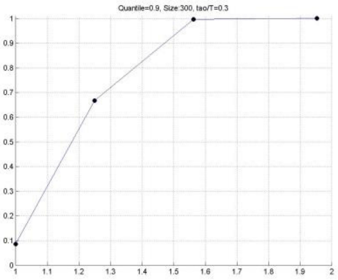
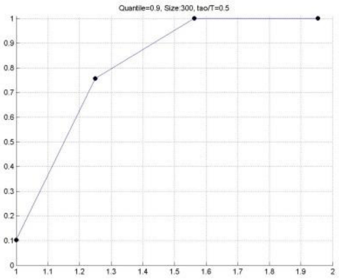
$$s_{.8}$$

T	$\frac{\tau}{T} = .1$	$\frac{\tau}{T} = .3$	$\frac{\tau}{T} = .5$
20			
50			

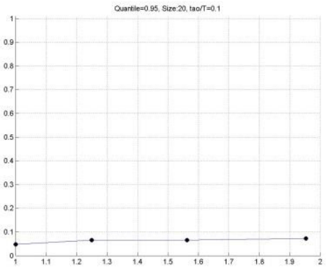
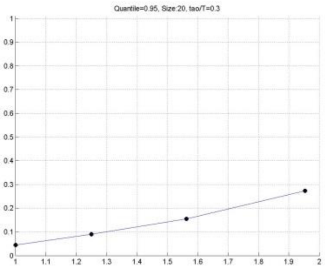
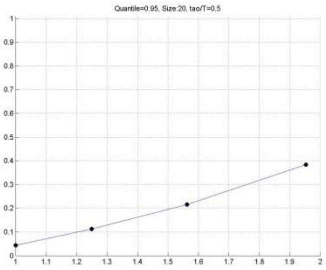
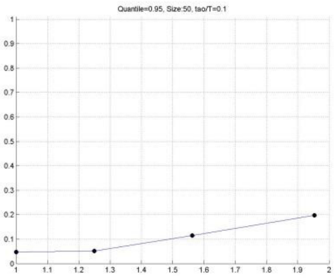
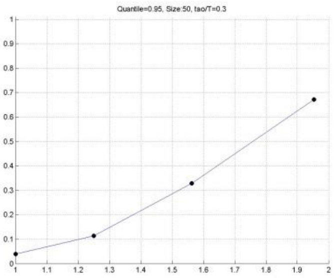
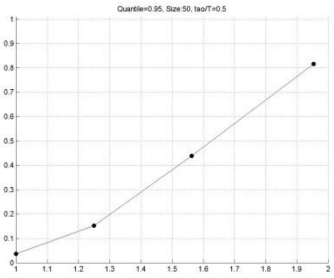
100			
300			

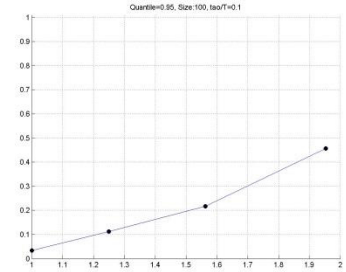
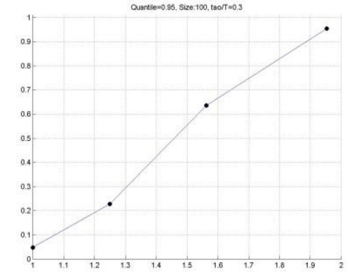
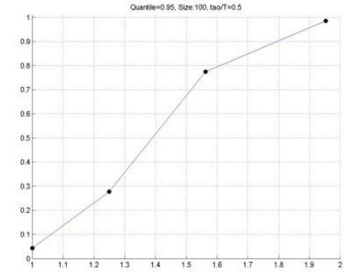
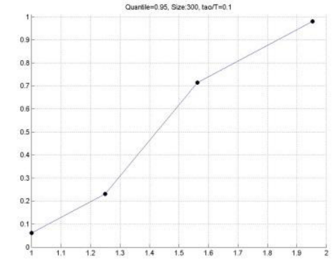
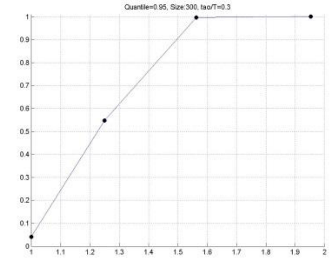
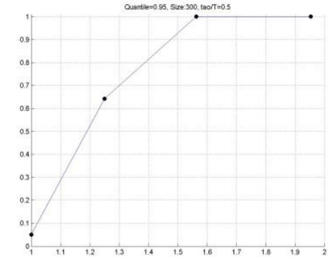
$s_{.9}$

T	$\frac{\tau}{T} = .1$	$\frac{\tau}{T} = .3$	$\frac{\tau}{T} = .5$
20			
50			

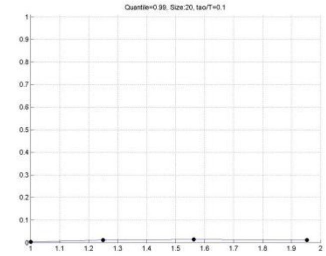
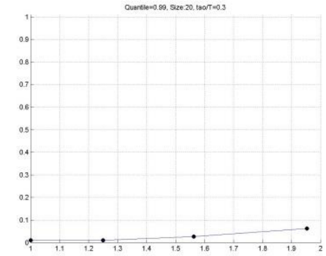
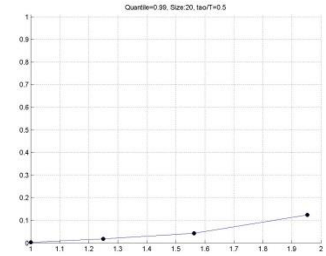
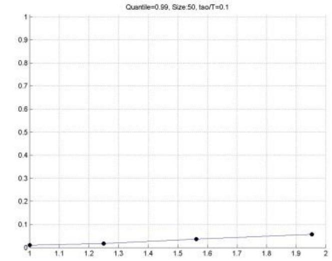
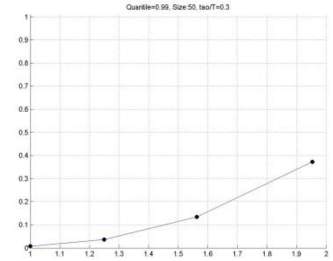
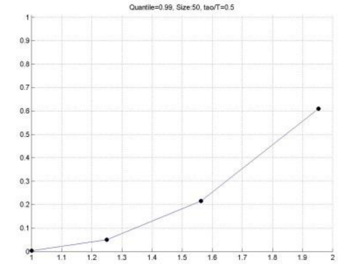
100			
300			

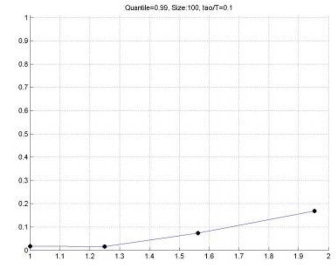
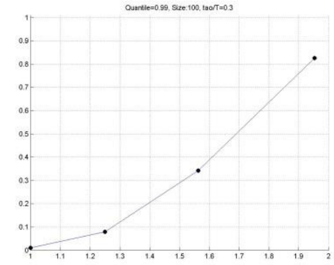
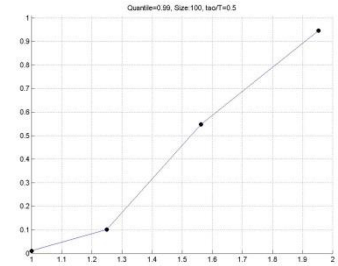
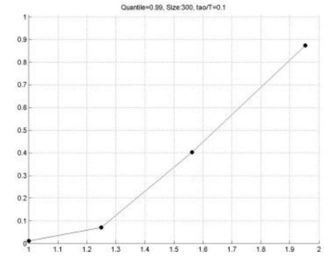
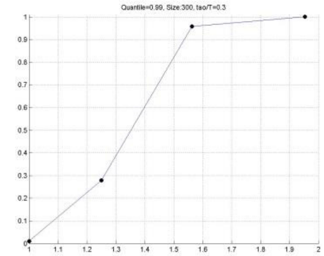
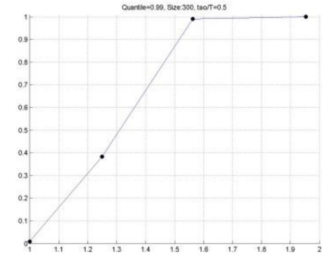
$\boldsymbol{s_{.95}}$

T	$\frac{\tau}{T} = .1$	$\frac{\tau}{T} = .3$	$\frac{\tau}{T} = .5$
20			
50			

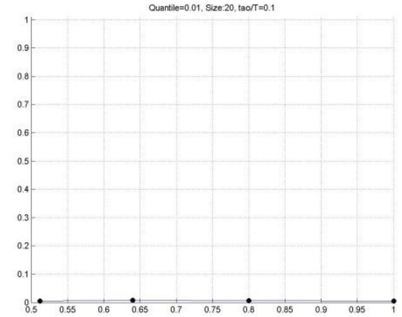
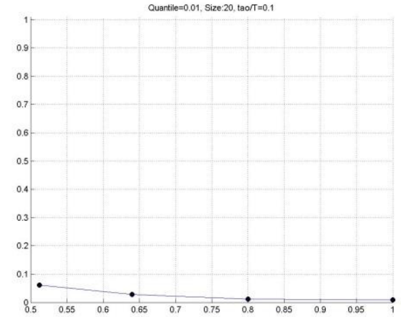
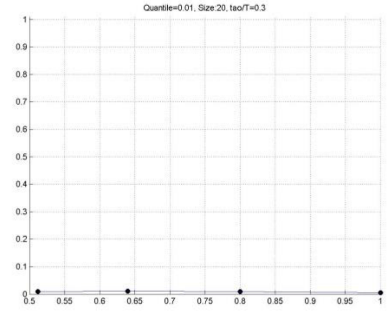
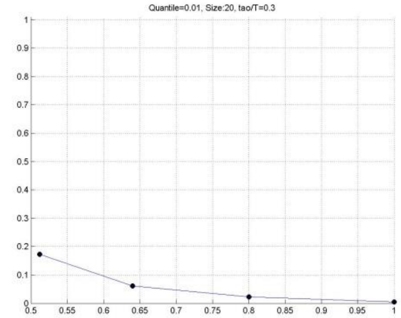
100			
300			

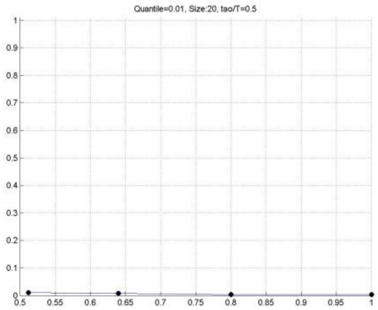
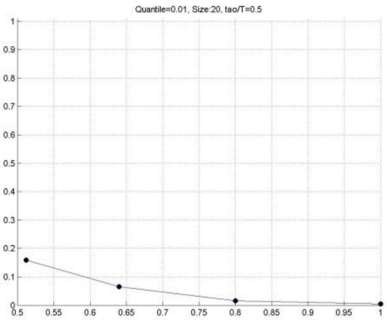
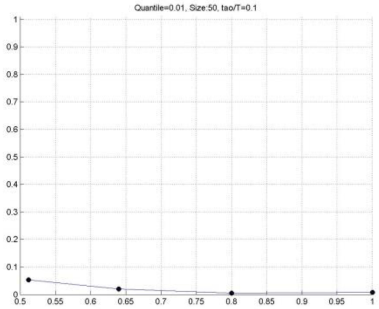
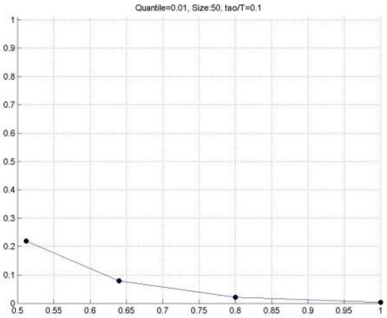
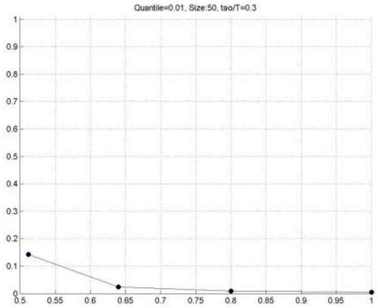
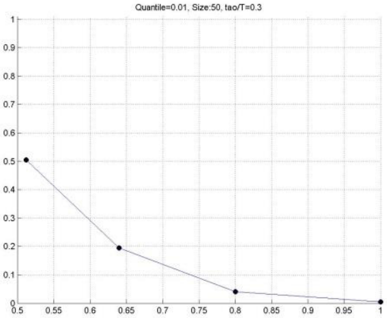
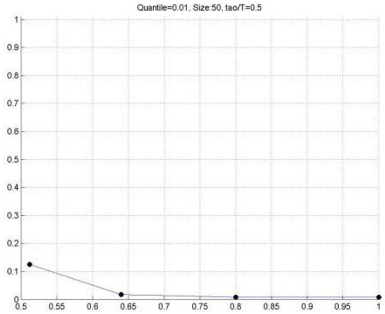
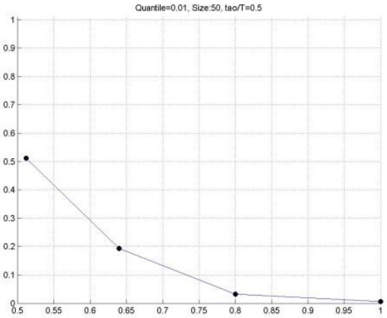
$$S_{.99}$$

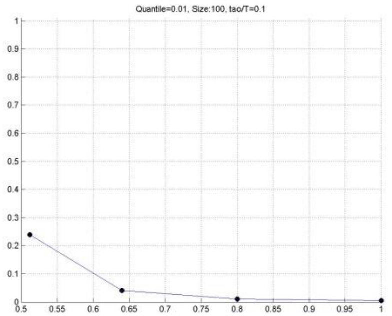
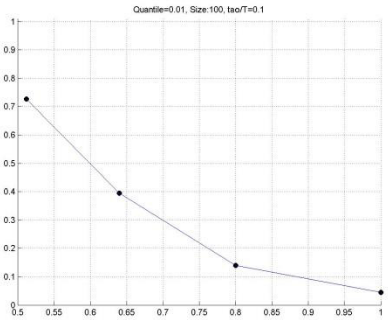
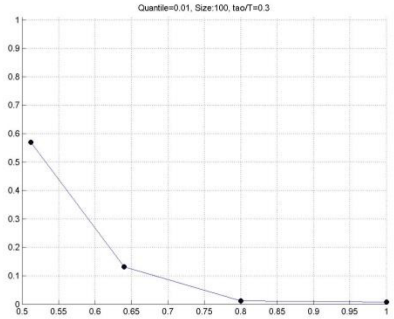
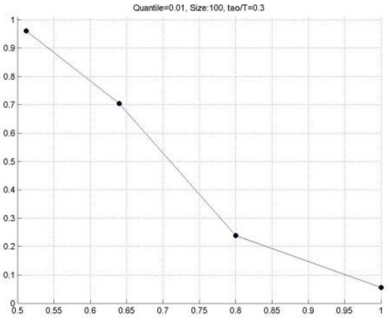
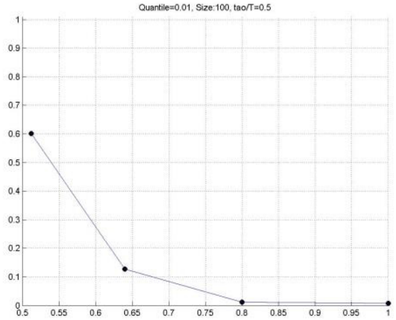
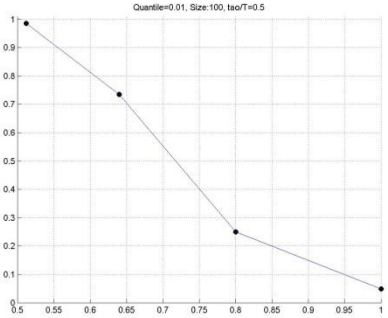
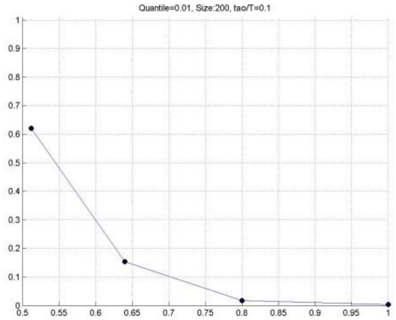
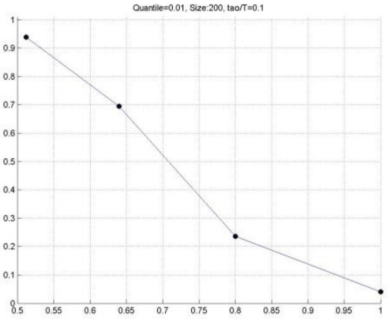
T	$\frac{\tau}{T} = .1$	$\frac{\tau}{T} = .3$	$\frac{\tau}{T} = .5$
20			
50			

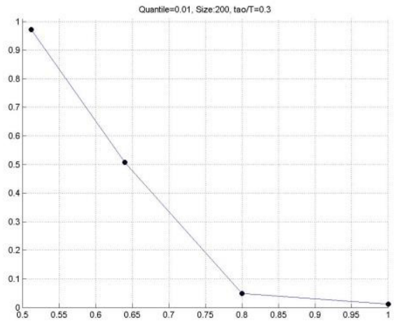
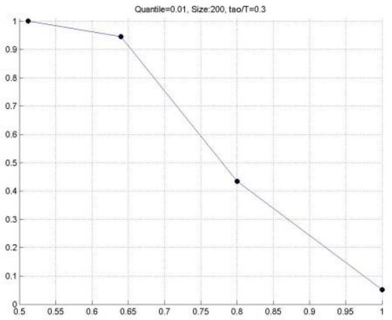
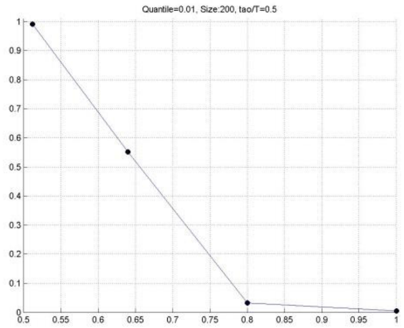
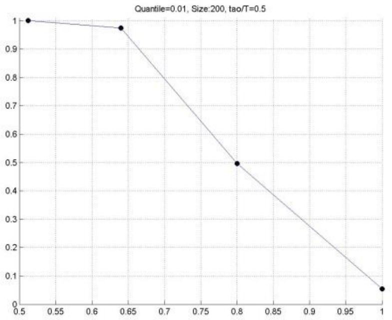
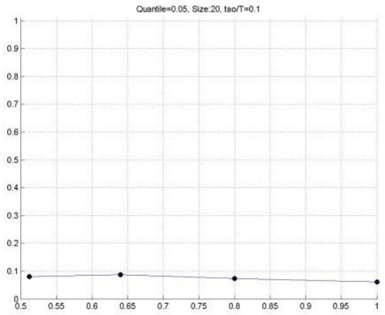
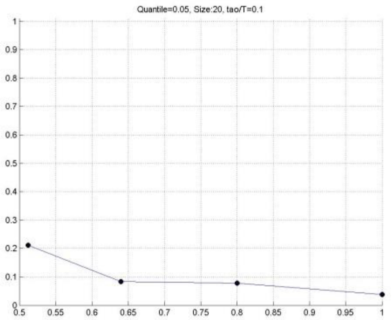
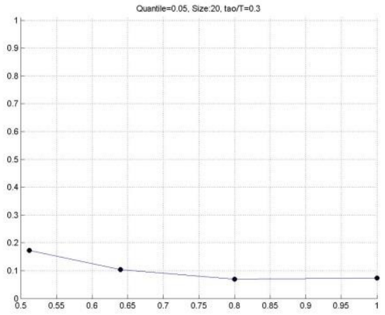
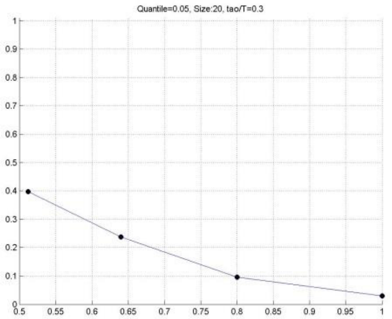
100			
300			

A.3.2 Comparison with Chen's test.

Parameters (Quantile, size, $\frac{\tau}{T}$)	Chen	Samuel
(0.01, 20, .1)		
(0.01, 20, .3)		

(0.01, 20, .5)		
(0.01, 50, .1)		
(0.01, 50, .3)		
(0.01, 50, .5)		

(0.01, 100, .1)		
(0.01, 100, .3)		
(0.01, 100, .5)		
(0.01, 200, .1)		

(0.01, 200, .3)	 Quantile=0.01, Size:200, tao/T=0.3	 Quantile=0.01, Size:200, tao/T=0.3
(0.01, 200, .5)	 Quantile=0.01, Size:200, tao/T=0.5	 Quantile=0.01, Size:200, tao/T=0.5
(0.05, 20, .1)	 Quantile=0.05, Size:20, tao/T=0.1	 Quantile=0.05, Size:20, tao/T=0.1
(0.05, 20, .3)	 Quantile=0.05, Size:20, tao/T=0.3	 Quantile=0.05, Size:20, tao/T=0.3

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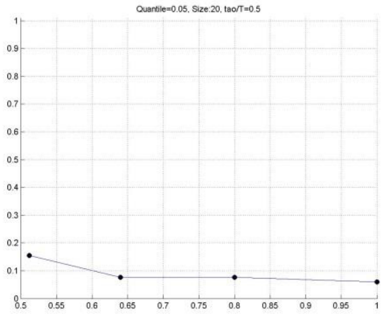
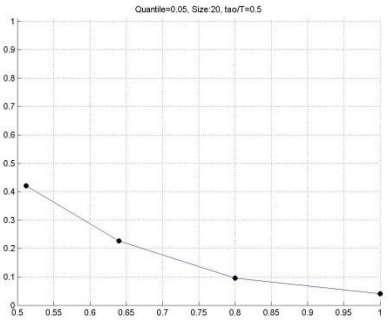
<td> </td>
<td> </td>
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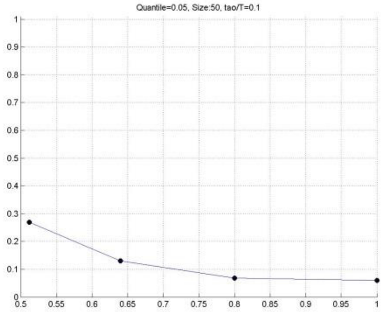
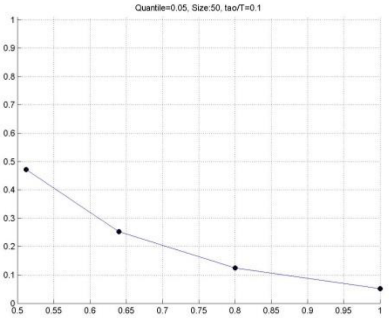
<td> </td>
<td> </td>
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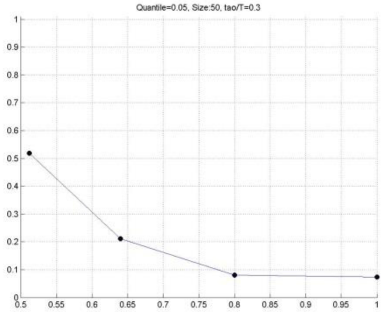
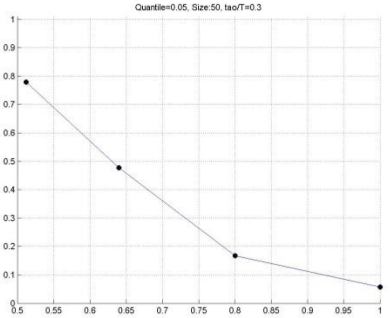
<td> </td>
<td> </td>
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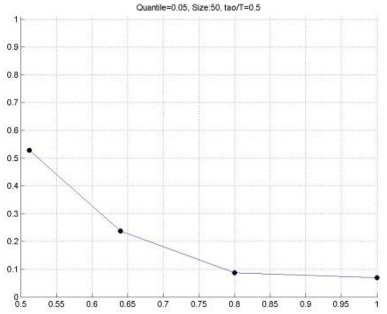
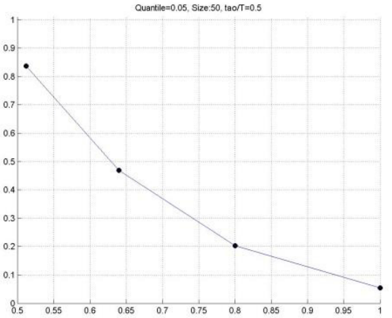
<td> </td>
<td> </td>
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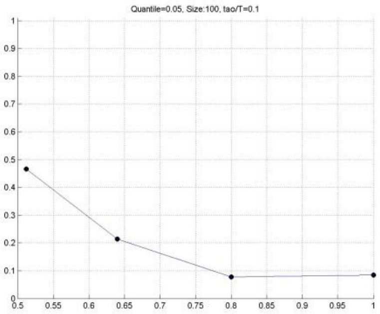
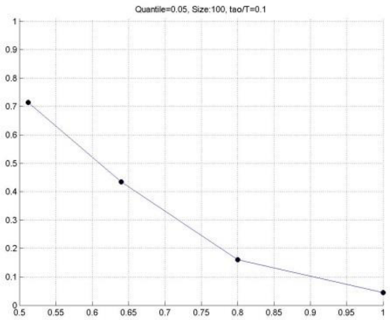
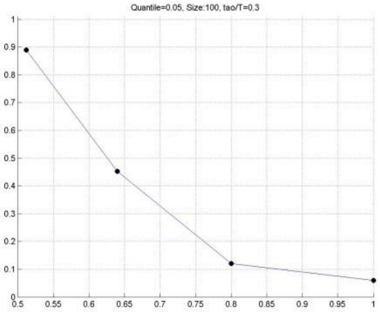
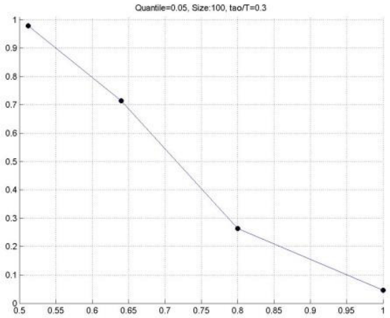
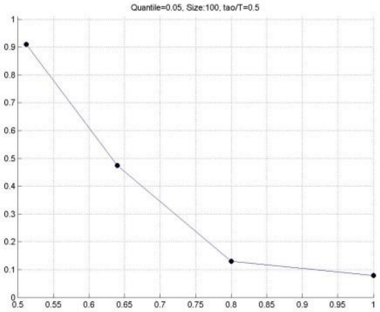
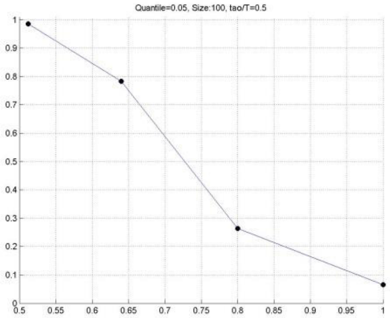
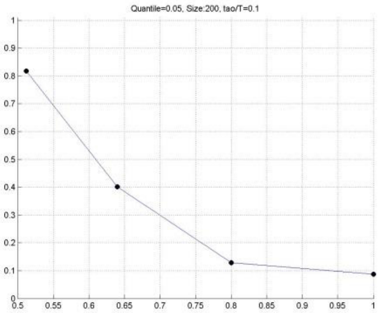
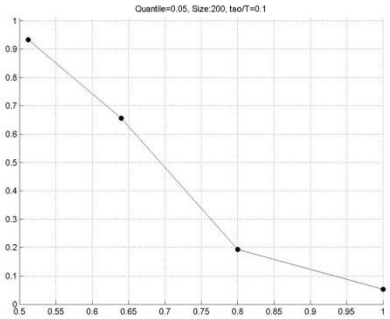
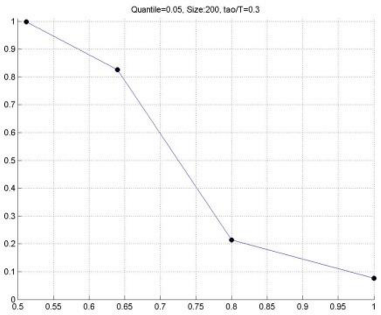
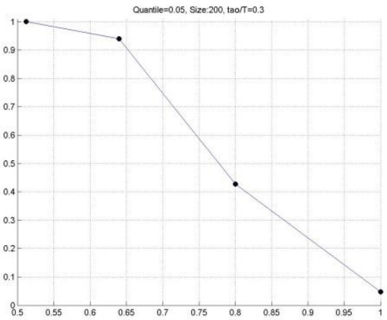
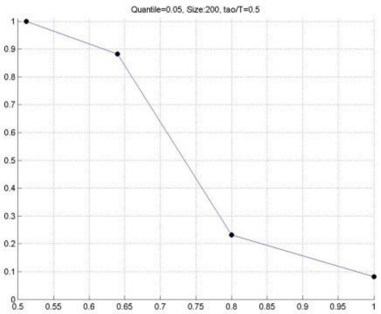
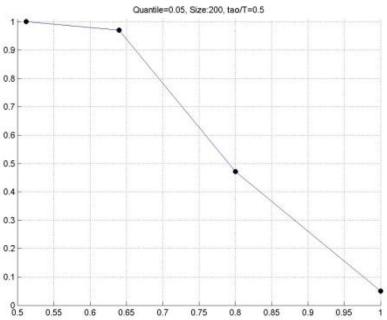
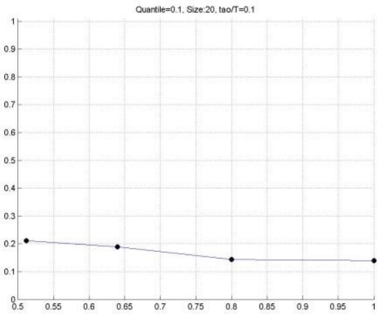
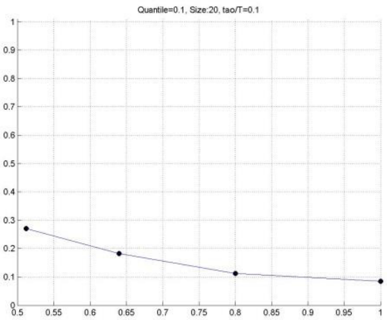
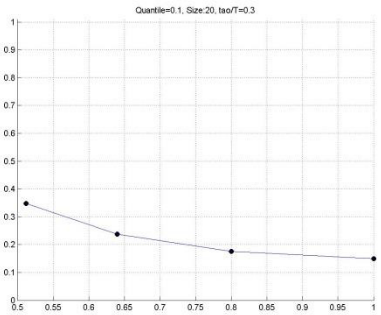
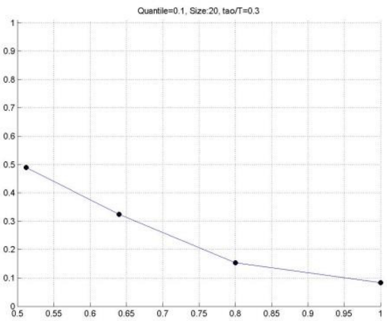
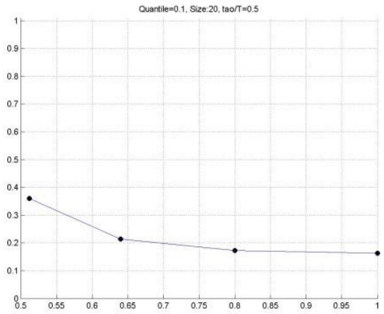
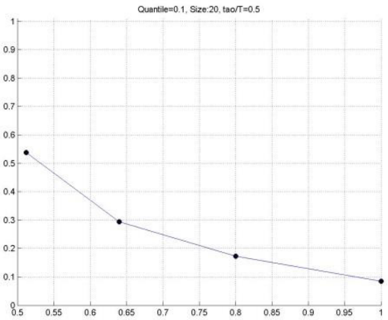
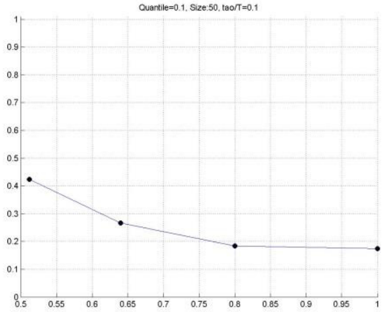
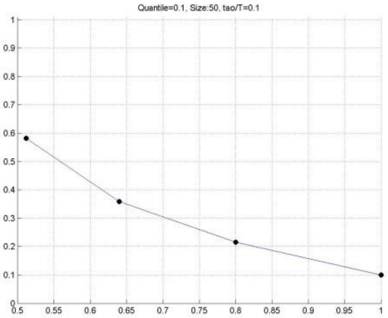
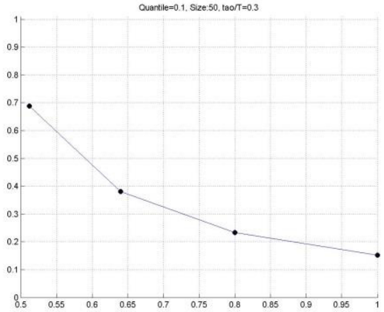
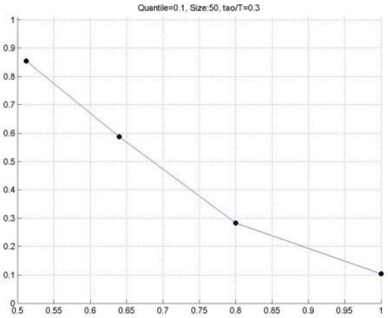
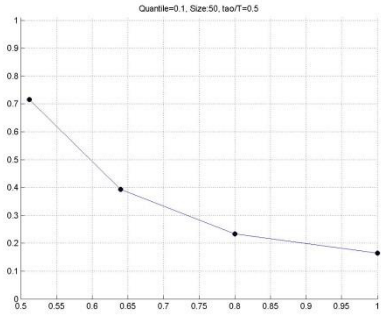
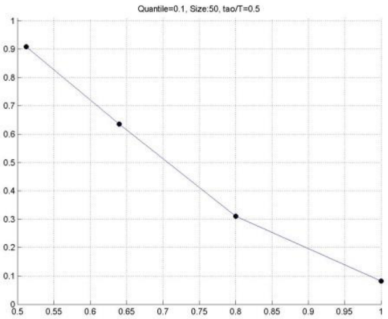
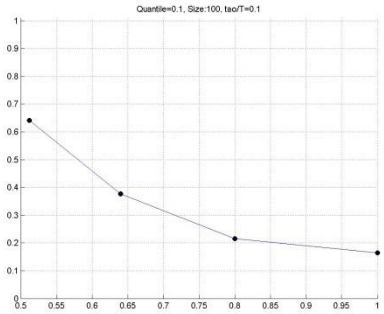
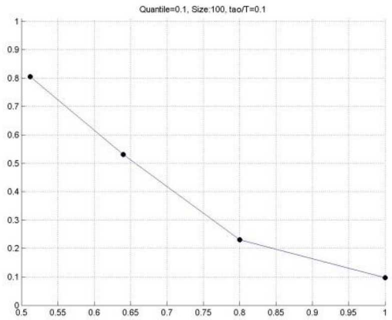
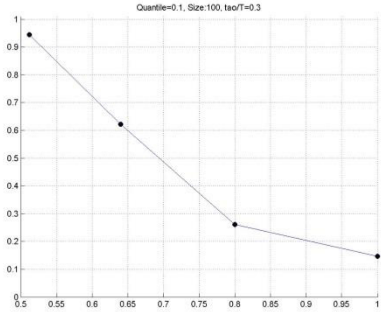
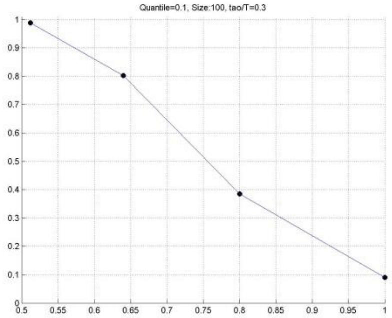
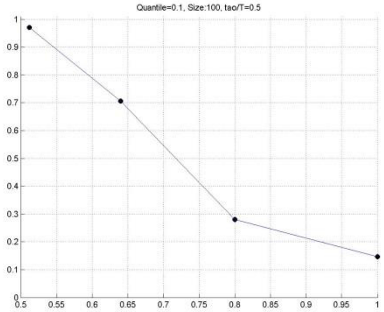
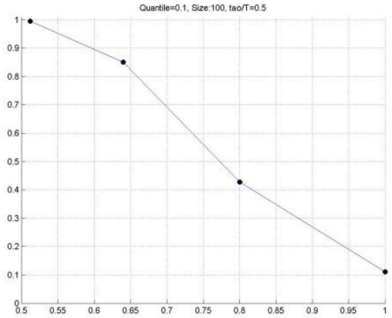
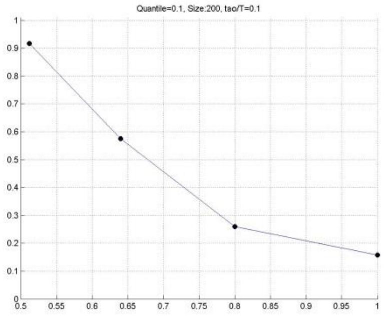
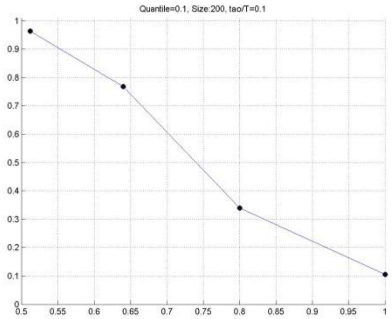
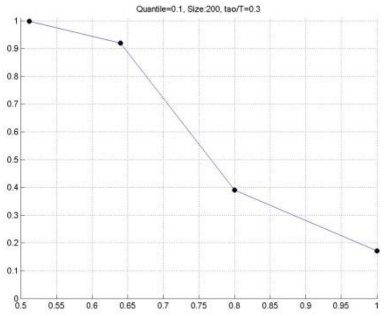
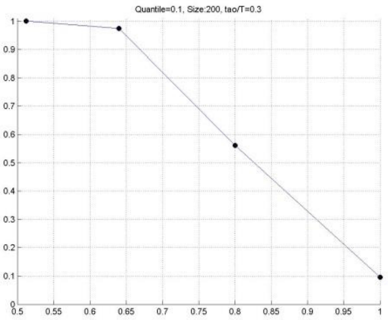
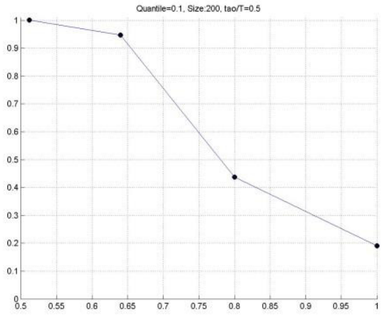
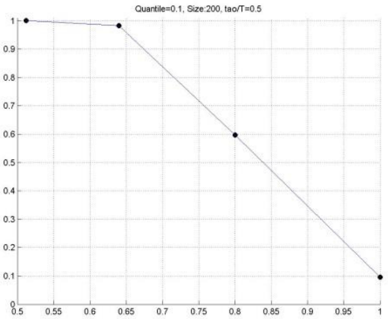
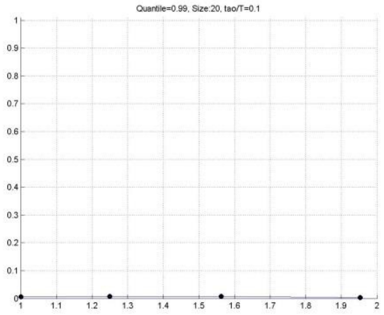
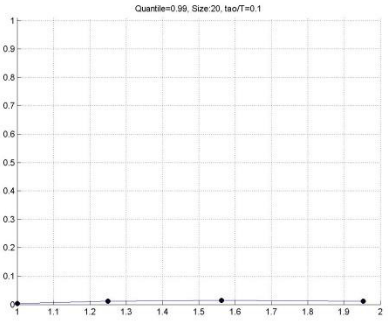
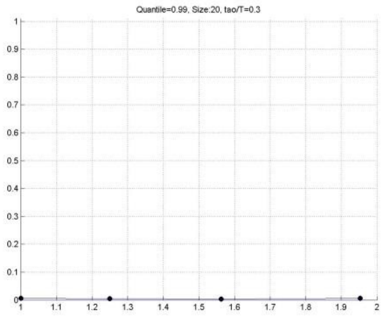
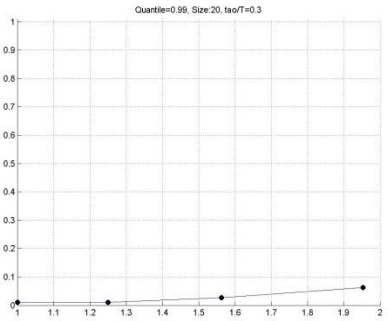
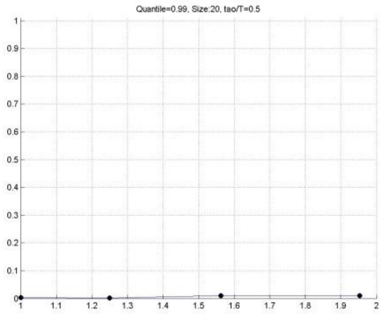
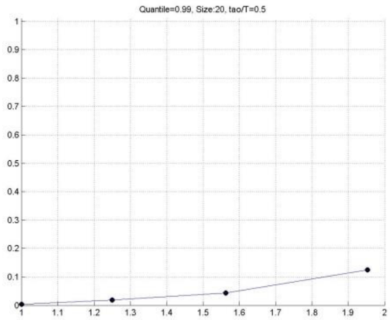
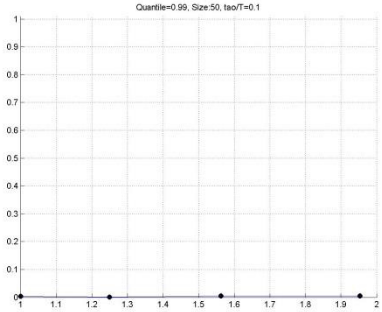
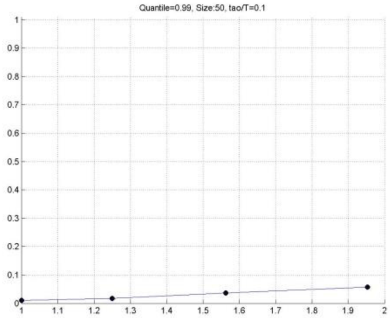
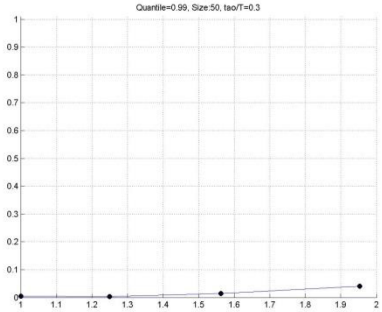
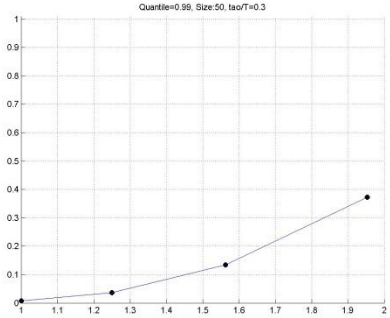
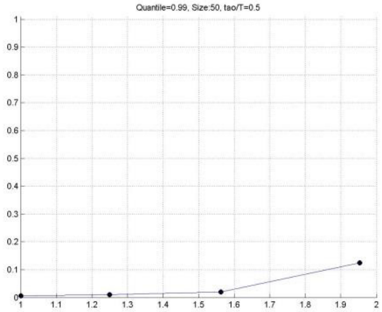
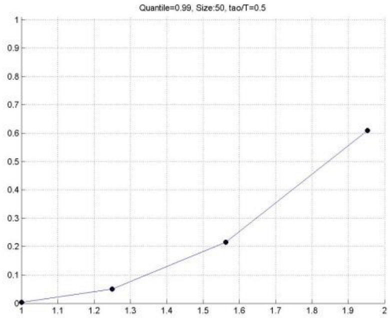
(0.05, 100, .1)		
(0.05, 100, .3)		
(0.05, 100, .5)		
(0.05, 200, .1)		

(0.05, 200, .3)		
(0.05, 200, .5)		
(0.1, 20, .1)		
(0.1, 20, .3)		

(0.1, 20, .5)		
(0.1, 50, .1)		
(0.1, 50, .3)		
(0.1, 50, .5)		

(0.1, 100, .1)		
(0.1, 100, .3)		
(0.1, 100, .5)		
(0.1, 200, .1)		

(0.1, 200, .3)		
(0.1, 200, .5)		
(0.99, 20, .1)		
(0.99, 20, .3)		

(0.99, 20, .5)		
(0.99, 50, .1)		
(0.99, 50, .3)		
(0.99, 50, .5)		

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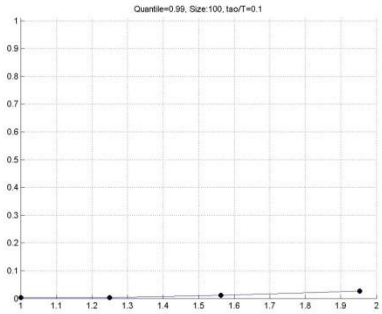
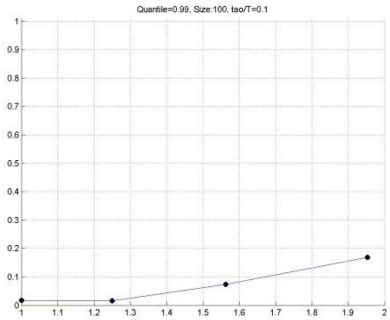
<td> </td>
<td> </td>
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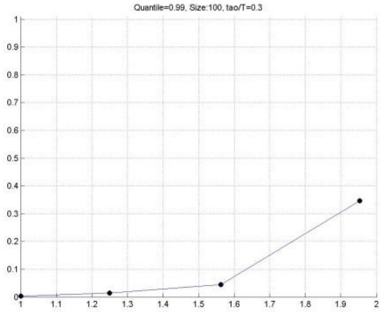
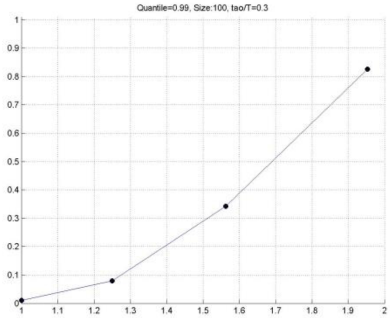
<td> </td>
<td> </td>
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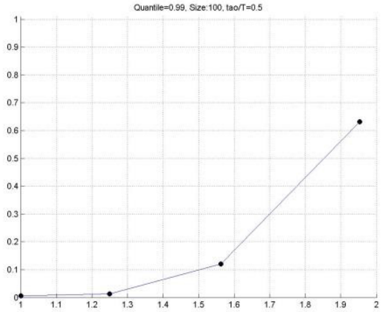
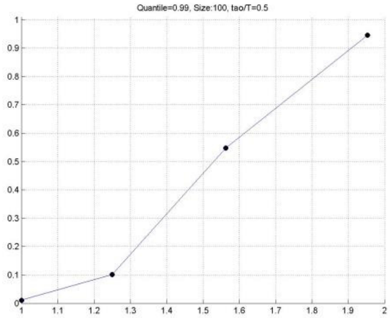
<td> </td>
<td> </td>
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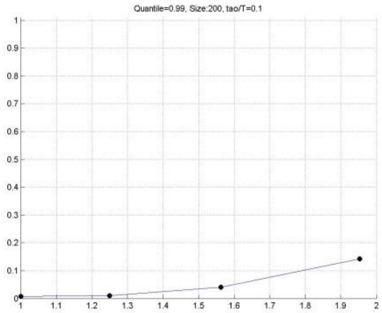
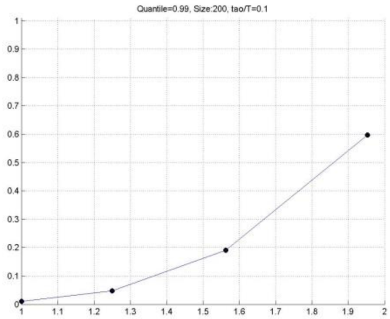
<td> </td>
<td> </td>
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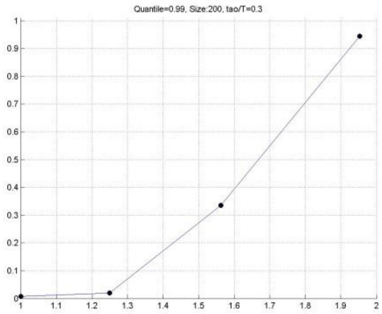
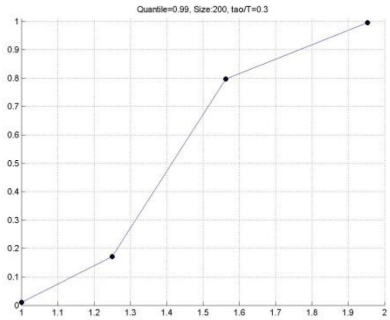
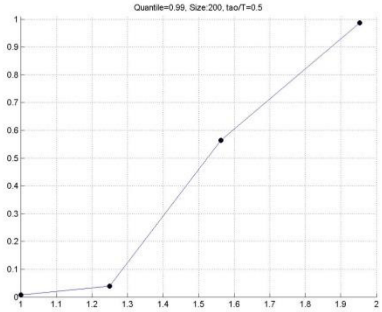
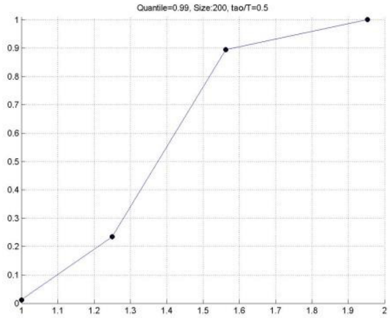
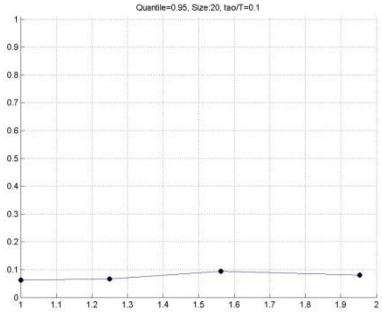
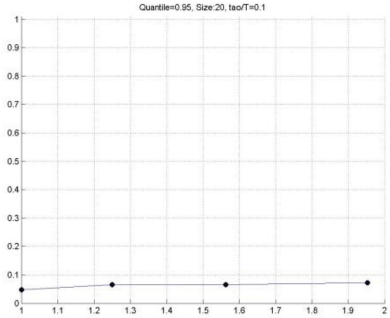
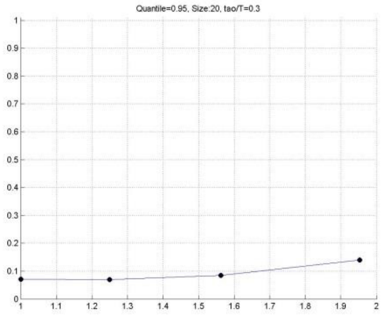
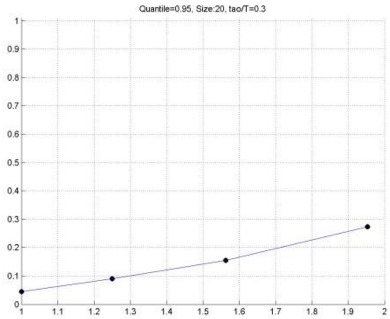
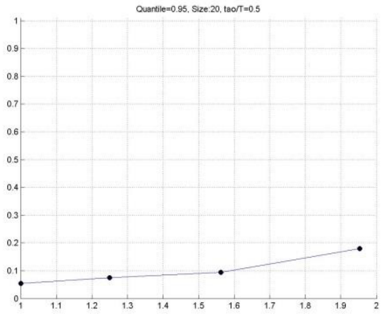
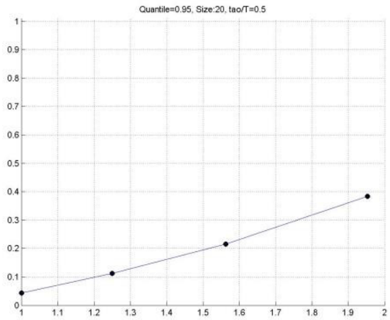
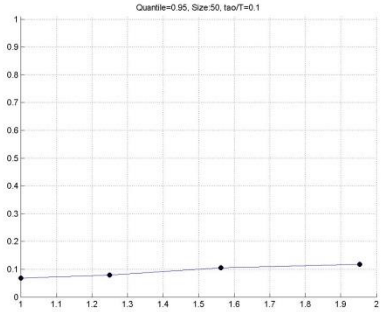
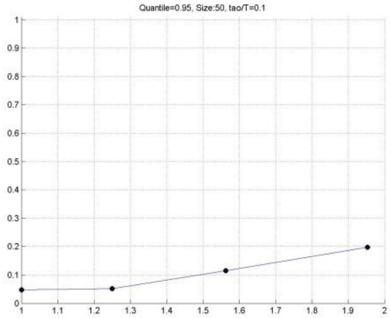
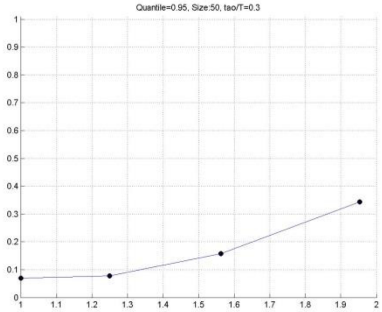
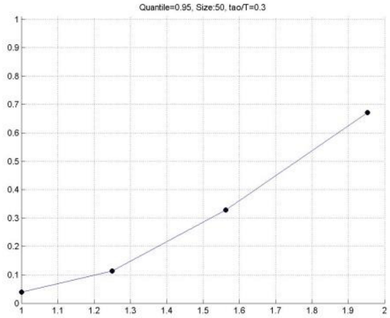
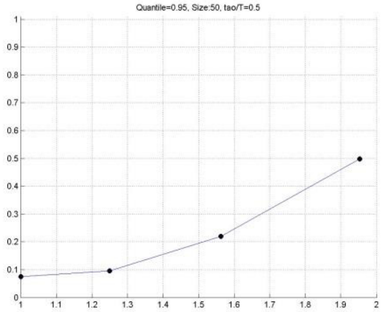
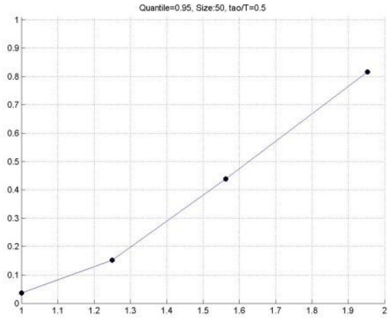
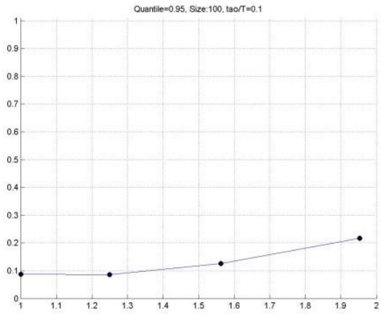
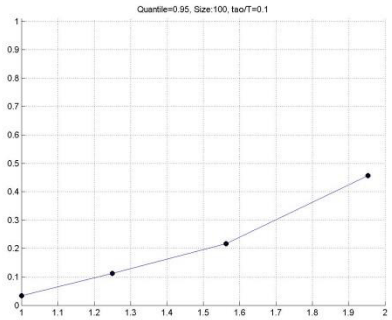
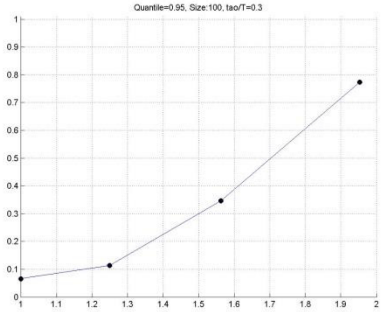
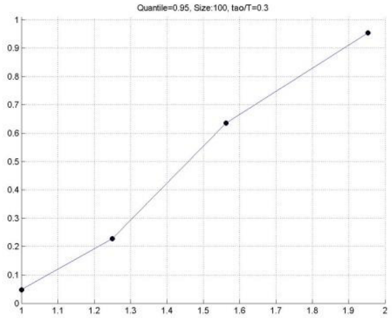
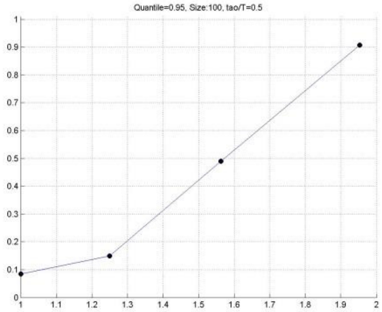
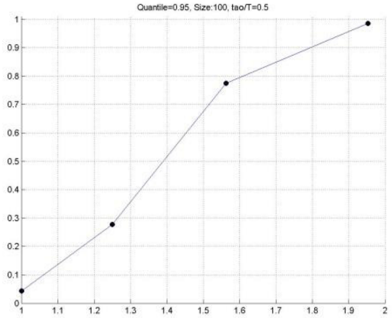
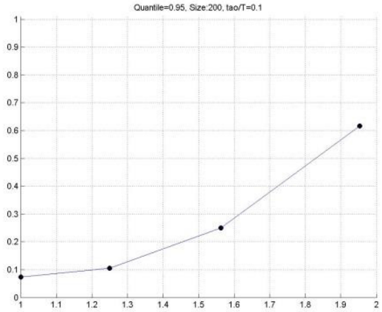
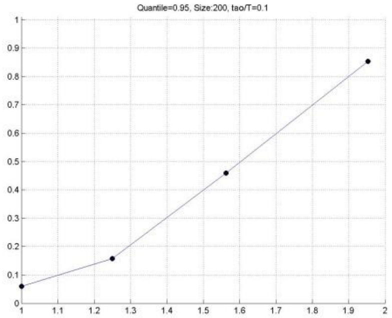
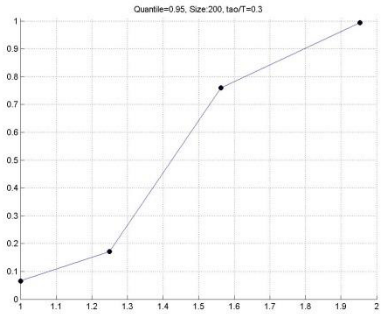
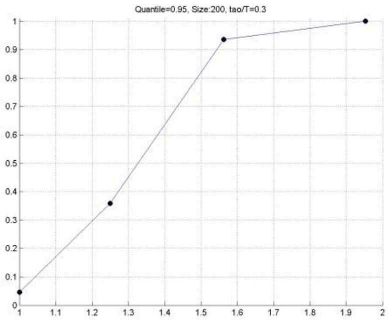
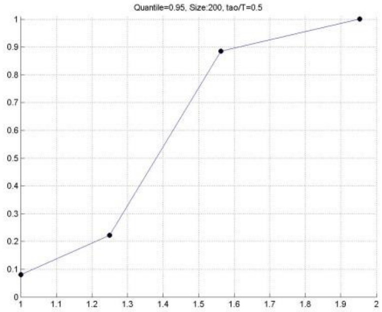
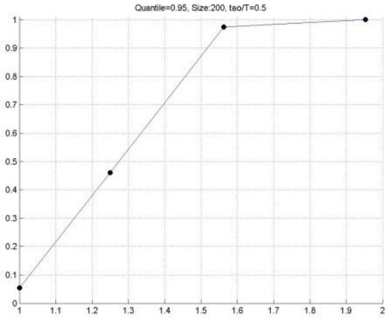
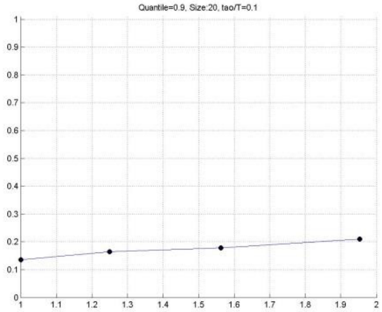
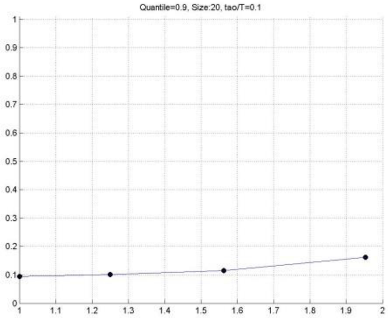
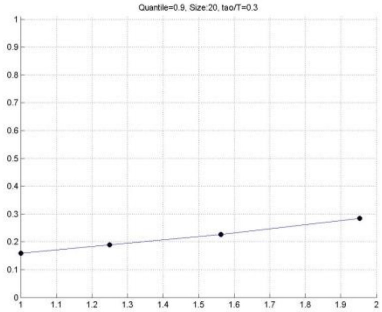
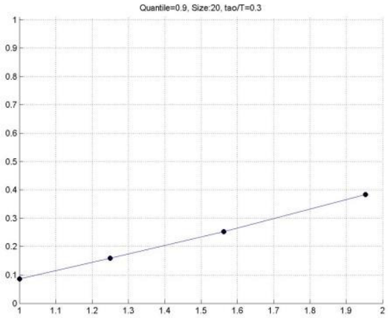
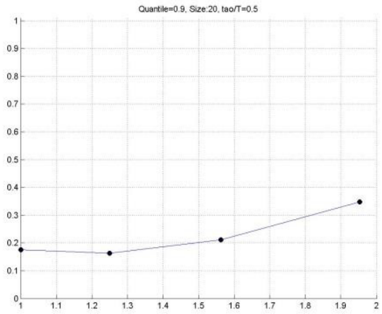
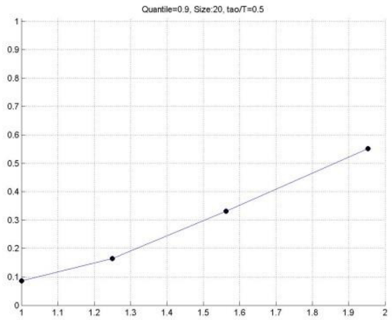
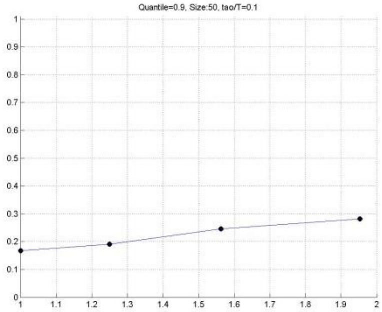
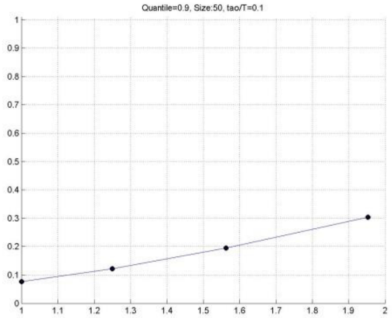
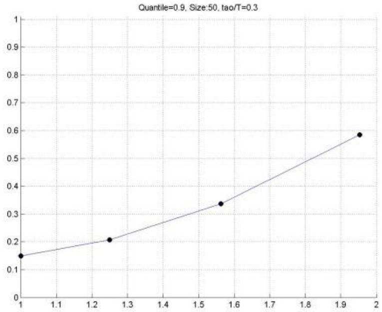
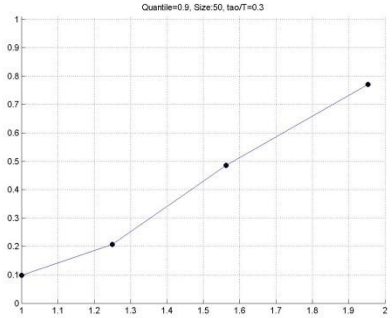
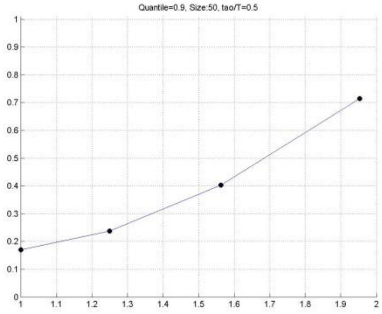
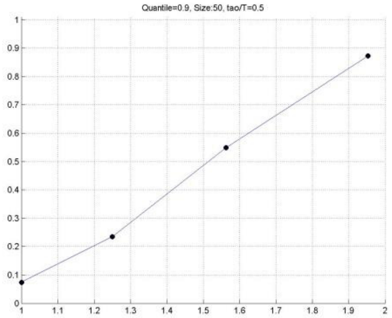
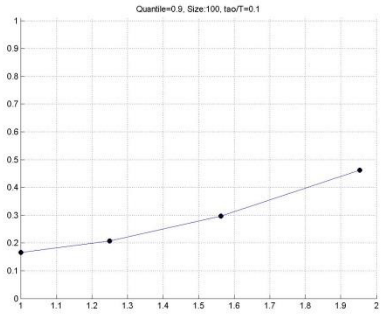
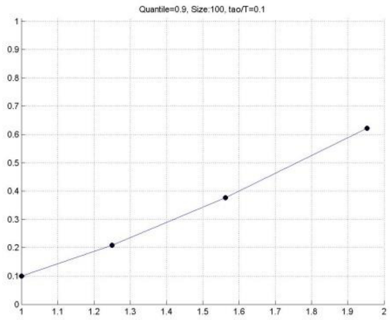
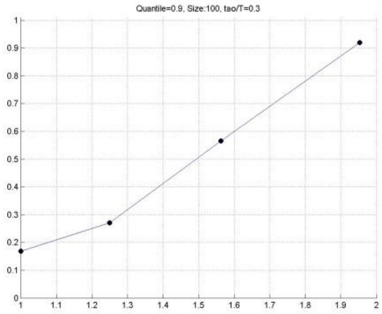
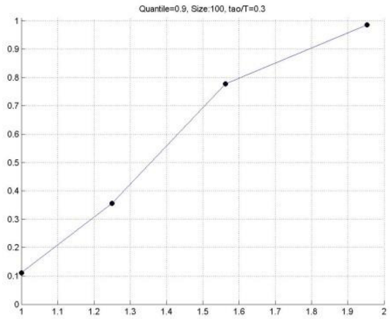
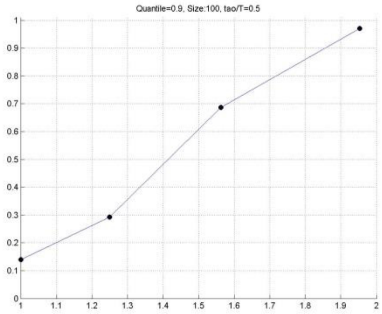
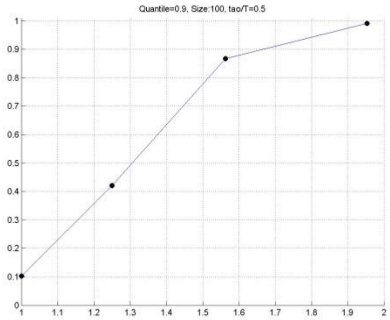
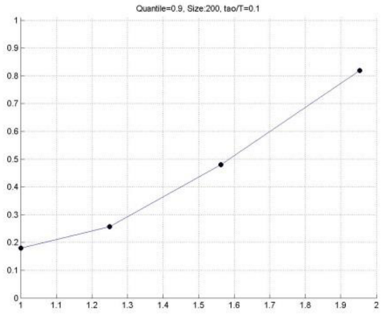
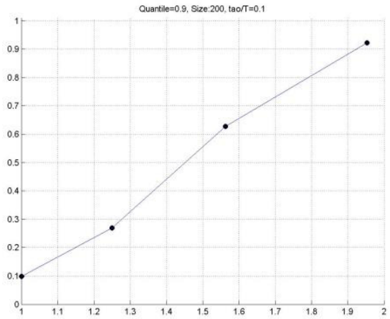
(0.99, 200, .3)		
(0.99, 200, .5)		
(0.95, 20, .1)		
(0.95, 20, .3)		

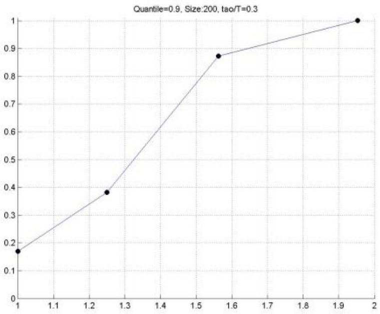
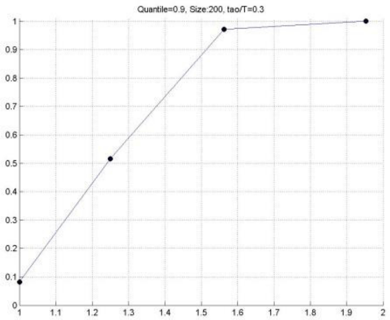
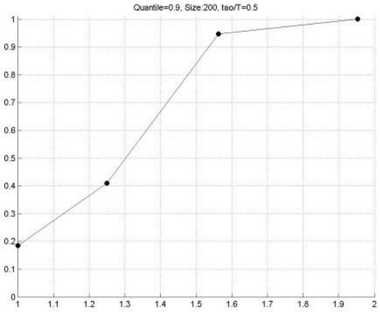
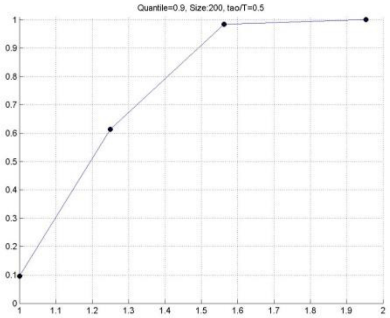
(0.95, 20, .5)		
(0.95, 50, .1)		
(0.95, 50, .3)		
(0.95, 50, .5)		

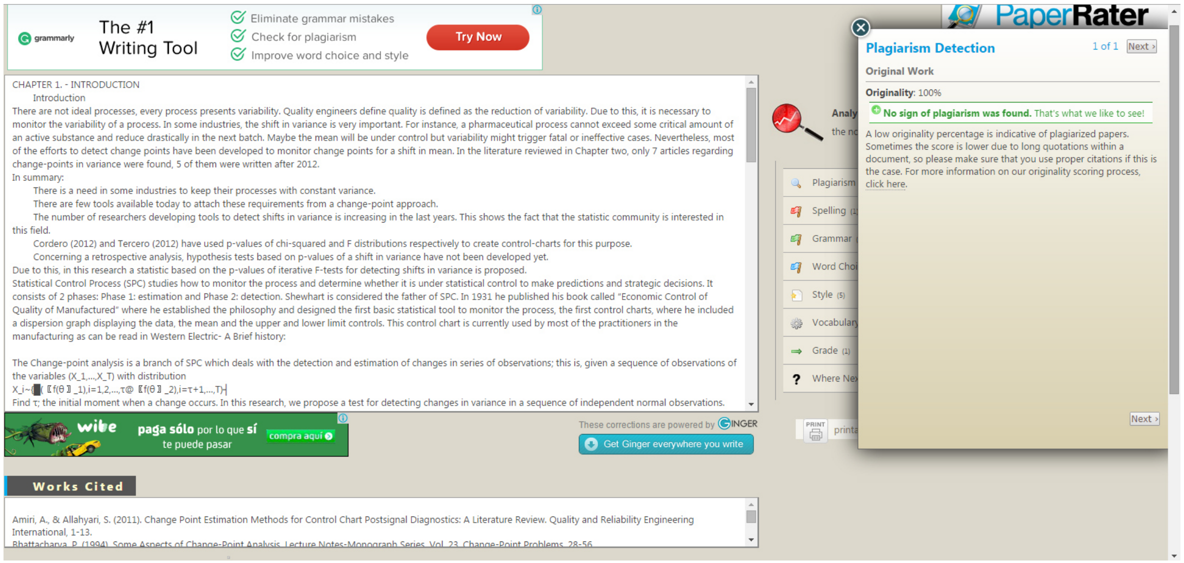
(0.95, 100, .1)		
(0.95, 100, .3)		
(0.95, 100, .5)		
(0.95, 200, .1)		

(0.95, 200, .3)		
(0.95, 200, .5)		
(0.9, 20, .1)		
(0.9, 20, .3)		

(0.9, 20, .5)		
(0.9, 50, .1)		
(0.9, 50, .3)		
(0.9, 50, .5)		

(0.9, 100, .1)		
(0.9, 100, .3)		
(0.9, 100, .5)		
(0.9, 200, .1)		

(0.9, 200, .3)		
(0.9, 200, .5)		



CHAPTER 1. - INTRODUCTION

Introduction

There are not ideal processes, every process presents variability. Quality engineers define quality is defined as the reduction of variability. Due to this, it is necessary to monitor the variability of a process. In some industries, the shift in variance is very important. For instance, a pharmaceutical process cannot exceed some critical amount of an active substance and reduce drastically in the next batch. Maybe the mean will be under control but variability might trigger fatal or ineffective cases. Nevertheless, most of the efforts to detect change points have been developed to monitor change points for a shift in mean. In the literature reviewed in Chapter two, only 7 articles regarding change-points in variance were found, 5 of them were written after 2012.

In summary:

There is a need in some industries to keep their processes with constant variance.

There are few tools available today to attach these requirements from a change-point approach.

The number of researchers developing tools to detect shifts in variance is increasing in the last years. This shows the fact that the statistic community is interested in this field.

Cordero (2012) and Tercero (2012) have used p-values of chi-squared and F distributions respectively to create control-charts for this purpose.

Concerning a retrospective analysis, hypothesis tests based on p-values of a shift in variance have not been developed yet.

Due to this, in this research a statistic based on the p-values of iterative F-tests for detecting shifts in variance is proposed.

Statistical Control Process (SPC) studies how to monitor the process and determine whether it is under statistical control to make predictions and strategic decisions. It consists of 2 phases: Phase 1: estimation and Phase 2: detection. Shewhart is considered the father of SPC. In 1931 he published his book called "Economic Control of Quality of Manufactured" where he established the philosophy and designed the first basic statistical tool to monitor the process, the first control charts, where he included a dispersion graph displaying the data, the mean and the upper and lower limit controls. This control chart is currently used by most of the practitioners in the manufacturing as can be read in Western Electric- A Brief history:

The Change-point analysis is a branch of SPC which deals with the detection and estimation of changes in series of observations; this is, given a sequence of observations of the variables (X_1,...,X_T) with distribution

X_i~{█(〖f(θ〗 _1),i=1,2,...,τ@ 〖f(θ〗 _2),i=τ+1,...,T)┤

Find τ; the initial moment when a change occurs. In this research, we propose a test for detecting changes in variance in a sequence of independent normal observations.

Amiri, A., & Allahyari, S. (2011). Change Point Estimation Methods for Control Chart Postsignal Diagnostics: A Literature Review. Quality and Reliability Engineering International, 1-13.

Bhattacharya, P. (1994). Some Aspects of Change-Point Analysis. Lecture Notes-Monograph Series, Vol. 23, Change-Point Problems, 28-56.

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